MITOCW | stability_analysis

When you put a cold drink on the kitchen counter the counter surface temperature will decrease. But, if the cold drink is removed, the counter will eventually return to room temperature. If instead we place a cup of tea on the counter, the counter temperature rises; but if we remove the cup of tea, the counter top eventually returns to room temperature. We say that the counter at room temperature is a stable equilibrium. In this video, we discuss the world from the perspective of equilibrium and stability, and in particular linear stability. This video is part of the Linearity video series. Many complex systems are modeled or approximated as linear because of the mathematical advantages. All the world is an initial value problem, and the matter merely state variables. However, and less poetically, there are alternative interpretations of physical, and indeed social, systems that can prove very enlightening. The purpose of this video is to introduce you to the framework of equilibrium and stability analysis. We hope this motivates you to study the topic in greater depth. To appreciate the material you should be familiar with elementary mechanics, ordinary differential equations, and eigenproblems. Let's look at the iced drink and hot teacup example from the perspective of equilibrium and stability. In this example, the governing partial differential equation is the heat equation shown here. At equilibrium, the solution of the governing equations is time-independent, that is, the partial time derivative is zero. This tells us that the del square temperature term must also be zero, which is only possible, given the boundary conditions, if the entire counter is at room temperature. Stability refers to the behavior of the system when perturbed from a particular equilibrium near the uniform counter temperature; a stable system returns to the equilibrium state; an unstable system departs from the equilibrium state. We say that this equilibrium is stable because if we perturb the temperature by increasing or decreasing the temperature slightly, it will return to room temperature, the equilibrium state, after enough time. Here we intuitively understand that the equilibrium is stable from our own experiences. But for other situations, we need mathematical methods for determining whether or not equilibria are stable. One way to determine if the equilibrium of this partial differential equation is stable is to apply an energy argument. Manipulation of this heat equation permits us to derive a relationship that describes how the "mean square departure" of the counter temperature from room temperature evolves over time, as shown here. Note u(x) is the deviation of the temperature in the counter from room temperature, Î© is the counter region, and Î“ is the counter surface; d1 and d2 are positive constantsâ€”determined by the thermal properties of the counter. Because the right--hand side of this equation is negative, it drives the temperature fluctuationsâ€”the integral of u(x) squared over the counter regionâ€”to zero. This energy argument is the mathematical prediction of the behavior we observe physically. Linear stability theory refers to the case in which we limit our attention to initially small perturbations. This allows us to model the evolution with linear equation! Linearizing the governing equations has many mathematical advantages. Let's consider the following framework for linear stability analysis. choose a physical system of interest; develop a (typically nonlinear) mathematical model; identify equilibria; linearize the governing equations about these equilibria; convert the initial value problem to an eigenproblem; inspect the eigenvalues and associated eigenmodes (or eigenvectors) Let's see how to use this framework as we proceed through the example of the real, physical pendulum seen here. You see a large bob, the rod, and a flexural hinge, which is designed to reduce friction losses. Let's develop the mathematical model. We show here the simple pendulum consisting of a bob connected to a massless rod; we denote the (angular) position of the bob by theta(t), and the angular velocity of the bob by Ï‰(t); g is the acceleration due to gravity; and L is the effective rod length for our simple pendulum. The effective length L is chosen such that the simple pendulum replicates the dynamics of the real, physical pendulum; L is calculated from the center of mass, the moment of inertia, and the mass of the compound pendulum. The dynamics may be expressed as a coupled system of ordinary differential equations that describe how the angular displacement and angular velocity evolve over time. These equations are nonlinear due to the presence of sin(Î¸) and the drag function fdrag(Ï‰). The drag function is quite complicated. For large Ï‰â€”it is equal to c|omega|omega, where c is a negative constant. But for very small angular velocities, near points on the trajectory where the pendulum isn't moving, or at least not moving fast, drag is given by to b omega, where b is a negative constant. Next, let's explore the validity of this model. Here you see a comparison between a numerical simulation, and an experiment, courtesy of Drs. Yano and Penn, respectively. The numerical simulation is created by calibrating the drag function to the experimental data. The agreement is quite good for both small and large initial displacement angles. However, because we are fitting the dissipation to the data, this comparison does not truly validate the mathematical model. To validate the mathematical model, we must focus on a system property that is largely independent of the here, small, dissipation, such as the natural frequency, or period, of the pendulum motion. And a comparison of the natural frequency of the physical pendulum to that predicted by the numerical model shows that the natural frequencies agree quite well, for any initial displacement angle, even large. We now look for equilibrium states, solutions that are independent of time. To find these equilibria we set the left--hand side of the equations to zero and solve for Î¸ and Ï‰. We can readily conclude that there are two equilibria: (Î¸, Ï‰) = (0, 0), which we denote the "bottom"equilibrium; and (Î¸, Ï‰) = (Ï€, 0), which we denote the "top"equilibrium. The mathematical model actually has infinitely many equilibria corresponding to theta values that are integral multiples of pi. But for our stability analysis, two will suffice. Are these equilibrium solutions stable or unstable for the physical pendulum? Pause the video here and discuss. They are stable if a small nudge will result in commensurately small bob motion; They are unstable, if a small nudge will result in incommensurately large bob motion. So we can predict that the bottom equilibrium is stable; the top equilibrium, unstable. If our mathematical model is good, it should predict the same behavior as the physical system. But how do we mathematically analyze stability of our model? Let's work through the linear stability analysis framework for the bottom equilibrium, Î¸ = 0 and Ï‰ = 0. First, we linearize the equations about the equilibrium. The linearized equations are only valid near the equilibrium, theta = 0 and omega =0, i.e. for small displacements, theta prime, with small angular velocities, omega prime. The first equation of our system is already linear. So we only need to worry about linearizing the sin(0+Î¸') term and the dissipation term of the second equation for small theta prime and omega prime. What is the linear approximation of sin(theta') about theta=0. Hint: use a Taylor series. Pause the video and write down your answer. If we assume that theta prime is small, we can approximate sin(Î¸') by theta' â€" the deviation of Î¸ from 0â€" by ignoring the higher order terms in the Taylor series expansion of sin theta' about 0. Because we are linearizing near omega prime sufficiently close to zero, the dissipation term is asymptotic to b omega prime, as mentioned previously. We then substitute these expressions into our dynamical equations to obtain the, linear equations indicated. We must supply initial conditions, and the initial angle and angular velocity must be small in order for this linearized system of equations to be applicable. We now write the linear equations in matrix form, in order to prepare for the next step â€" formulation as an eigenproblem. To do this, we assume temporal behavior of the form eÎ»t. This yields an eigenvalue problem for Î». Note that our matrix is 2Ã—2 and hence there will be two eigenvalues and two associated eigenvectors, or eigenmodes. Once we obtain the eigenvalues and eigenvectors we may reconstruct the solution to the linearized equations, as shown here; the constants c1 and c2 are determined by the initial conditions. However, to determine stability, a simple inspection of the eigenvalues suffices: What happens to e to the lambda t when each of the eigenvalues has negative real part? Pause the video. We observe that eÎ»t decays in this case, and the system is stable. What happens if any of the eigenvalues has positive real part? Pause the video. If an eigenvalue has positive real part â€" in which case eÎ»t corresponds to exponential growth away from the equilibrium â€" the system is unstable; note that even one eigenvalue with a positive real part is sufficient to deem the system unstable, since sooner or later, no matter how small initially, the growing exponential term will dominate. Lastly, if the real part of the eigenvalue with largest real part is zero â€" in which case eÎ»t is of constant magnitude â€" the system is marginally stable and requires further analysis. We now proceed for our particular system. For this problem it is easy to find the eigenvalues analytically. We observe that both Î»1 and Î»2 have a small negative real part - due to our negative damping coefficient, b. Thus, the system is stable as we predicted based on the physical pendulum. For our experimental system, b L over g is much less than 1. And thus the damping does indeed have little effect on the period of motion. If we recall the connection between the complex exponential and sine and cosine, we may conclude that the linearized system response is a very slowly decaying oscillation. This linear approximation to our governing equations can predict not just the stability of the system, but also because the system is stable, the evolution of the systemâ€"assuming of course small initial conditions Indeed, comparing our original non-linear numerical solution to the numerical solution obtained from the linear model, we find that the agreement between the two is very good for the low-amplitude case. As advertised, we obtain a solution to the linear model with which we can predict the motion resulting from small perturbations away from equilibrium. But as you see here, the accuracy of the linear theory is indeed limited to small perturbations â€" for even moderately larger angles, the linear model no longer adequately represents the behavior of the pendulum. But this is to be expected since the linear approximation of these governing equations is only valid when theta prime and omega prime are very close to zero. To Review We have thus linearized the governing equations for the pendulum near the equilibrium theta = 0, omega = 0. By solving an eigenvalue problem, we showed that the equilibrium was stable at this point. The linear equations even predict the behavior of the pendulum near this stable equilibrium. This video has focused primarily on the modal approach to stability analysis, which is the simplest and arguably the most relevant approach in many applications. However, there are other important approaches too, such as the "energy"approach, briefly mentioned at the outset of the video. We shall leave to you to analyze the case of the "top"equilibrium, which we predicted to be unstable. You'll find that there's an unstable mode AND a stable mode. The unstable mode will ultimately dominate, but the stable mode is still surprising. Our intuition suggests than any perturbation should grow. The stable mode requires a precise specification of both initial angular displacement AND initial angular velocity. This explains the apparent contradiction between the mathematics and our expectations.