RES.TLL-004 STEM Concept Videos, Fall 2013

Transcript – Divergence

A soldier's helmet can be designed to help protect against the shock waves from explosions. To achieve the best design, researchers need to understand how those shock waves cause damage. In this video you'll see how they can model shock waves using three differential equations, all of which involve a concept called divergence.

This video is part of the Differential Equations video series.
Laws that govern a system's properties can be modeled using differential equations.

Hi, my name is Raul Radovitzky, and I am a professor of Aeronautics and Astronautics at MIT. Today, I want to talk to you about divergence. Whenever you are modeling a system that involves a vector field quantity, knowing whether or not the vector field is divergent often provides valuable information. This concept is applicable to a number of topics, including fluid dynamics and electromagnetism, and is fundamental to the mathematical description of exciting physical phenomena such as black holes, traffic flow, and explosive detonation.

Before watching this video you should know the definition of divergence, and how to compute the divergence of a vector field.

After watching this video, you should be able to determine whether or not a vector field has divergence. You will do this by imagining the vector field to be a velocity field and considering what happens to the area of a differential volume element in the vector field.

My expertise is in understanding how materials respond to extreme loads---such as in turbines, shuttles, and planes moving at hypersonic speed. When I arrived at MIT one month after 9/11, I was motivated to use this expertise in order to help protect people.

Explosions create a lot of damage. In particular, the shock wave produced in the blast is presumed to be a cause of brain damage.

One of the things we are interested in is developing helmets that protect people from such blasts.

In order to do this, we need to understand how injury to the head occurs, and what material properties and helmet geometries can mitigate the blast damage.

The approach has been to put together multidisciplinary teams--involving neurologists, cell biologists, biomechanical engineers, computational modelers, and material scientists to create complex mathematical models.

We design experiments that investigate the effects of a shock wave as it damages an object. We use data from those experiments to validate our mathematical models.

We of course need to model the blast wave. This is a strong pressure or shock wave that results from the sudden release of chemical energy from the explosive material.

And this shock wave is described by 3 equations: conservation of mass, conservation of momentum, and conservation of energy. All three differential equations involve divergence. So the very start of our model involves modeling solutions to these 3 equations. And this is the simple part.

In the rest of this video, my graduate student Michelle Nyein will help you begin to understand how to use divergence to describe physical phenomena. You will come back to the example of detonation at the end of the video to see if you can predict whether or not a blast wave has positive divergence, negative divergence, or zero divergence.

My name is Michelle Nyein, and I am a graduate student in the department of aeronautics and astronautics, working with the Institute for Soldier Nanotechnologies.

Divergence is a local property of vector fields.

For two dimensional vector fields, the divergence is a measure of the net flux per area of a vector field through a differential area element.

Or, in simple terms, divergence measures how much the flow expands or compresses at each point.

In two dimensions, divergence is the net flux of the vector field over a very small area with side lengths 2 Delta x and 2 Delta y.

By making an approximation for the vector field on each side of this differential element, determine a formula for the divergence.   [pause]

Because the component tangent to an edge does not affect the flux through the edge, we need only considering the normal components of the vector field at each edge.

We approximate the flux through a side to be the value of the vector in the middle times the length of the side, divided by the area of the box. So adding the fluxes through each side with appropriate signs, we obtain the following expression.

We simplify the expression, and take the limit as Delta x and Delta y approach zero. We end up with an expression for the divergence in terms of partial derivatives of the components of the vector field.

In 3 dimensions, we can take a differential volume element with side lengths 2 Delta x, 2 Delta y, and 2 Delta z in Cartesian coordinates.

We look at our vector field at points on the interior of each face. Because flux only depends on the components of a vector field perpendicular to the surface, we can decompose each vector into its x, y, and z components.

And we only need to consider the component that is perpendicular to each face.

We find an analogous formula for the net flux per unit volume through a differential volume element.

Okay, so that is divergence. But what does it mean? And how are we supposed to use this to describe physical phenomena in the world around us?

For the moment we will describe how to think of divergence for 2-dimensional vector fields because it is simpler to draw. However, all of the ideas we will discuss are true in 3-dimensions as well by replacing the word area with volume.

One handy way to imagine divergence is to think of the vector field as being a velocity field. Suppose a small region is outlined by a number of tracer particles, each moving in the velocity field.

If the divergence of the field is positive, then the net flux is out of the region, the particles move outwards according to the magnitude and direction of the velocity vector field. Here the particles are moving with constant x-velocity, but different y-velocities, increasing the area by expanding in the y-direction.

If the divergence of the field is less than zero, then the area outlined by the particles decreases as the particles flows according to the velocity field. Here the particles are moving with zero y-velocity, but with x-velocities that decrease as the particles move to the right.

And if the divergence is zero, the area doesn't change. Let's see if we can use this model to recognize positive, negative, and zero divergence.

Let's start by considering a vector field we encounter in the world.

Fluid flow has a velocity vector field. Here you see a clip of a classic MIT fluid flow video. The region of flow is outlined. Square regions of hydrogen bubbles have been added to the flow field. These bubbles are analogous to tracer particles.

Make a prediction as to whether or not this vector field is divergent. [pause]

You may have noticed that the squares turn into parallelograms, because due to the flow geometry, the horizontal component of the velocity near the top wall is larger than the horizontal velocity component near the bottom wall.

What you may not have noticed is that while the shape changes, the area of each element does not change. Thus the velocity field of this fluid flow regime is divergence-free.

In fluid dynamics, when the velocity field of a fluid is divergence-free, the fluid is said to be incompressible. No divergence means that there is zero net flux through any volume element.

When the flow field of a fluid is divergence-free, it means that it is impossible to compress or expand the average separation between the molecules in the fluid.

What do you think it means to be compressible? [pause]

Gasses are a familiar substance that we understand to be compressible. You are probably familiar with the ideal gas law: PV=nRT.

So you know that increasing pressure allows the average space between molecules to be compressed, which increases the average density.

During compression, the divergence of the average velocity field is negative.

Decreasing pressure allows the average space between molecules to expand, decreasing density. During expansion, the divergence of the average velocity field is positive.

Now let's look at common 3-dimensional vector field, an electric field generated by a positive point charge.

Since it is common to use spherical coordinates when considering electric fields, we will consider differential volume elements that are in spherical coordinates for convenience.

If we imagine this spherical volume element is made up of tracer particles, and the field is a velocity field, the volume clearly increases. So this electric field appears to have positive divergence.

However, any differential volume element that does not contain the point charge actually will not change in volume. So the divergence is only non-zero at the point charge generating the field.

Consider the electric field generated by a negative point charge.   What is the divergence of the vector field at the point charge? [pause]

That's right, the volume of a differential sphere surrounding the point charge would decrease, so the divergence is negative.

The electric field generated by a positive charge is said to have a source, while an electric field generated by a negative charge is said to have a sink.

The vector field at any source has positive divergence, while the vector field at any sink has negative divergence.

Let's look at a rotational vector field in the plane: V = y i-hat minus x j-hat. The magnitudes of the vectors increase as they move radially outwards from the center. Do you think this field has positive divergence, negative divergence, or zero divergence? [pause]

 A square of tracer particles placed about the origin simply rotates. This means that this field cannot have a source or a sink at the origin.   If we add a square of tracer particles at any other point in the field, you see that it revolves about the origin without changing area. This happens because the particles further from the center move faster than particles near the origin.   A quick computation confirms that this field has zero divergence everywhere.

By looking at these common vector fields that you will encounter in your studies, we can see that divergence can tell us something important about the behavior of each field.

Given what you just learned about divergence, what can you say about the divergence of a blast wave? [pause]

Blast waves are NOT divergence-free.   At the simplest level of understanding, the blast wave is created by highly compressed air and chemicals, which do not experience any resistance to expansion. So they expand very quickly, with positive divergence.

To review:

Detonations produce blast waves, which are described by solutions to a system of 3 partial differential equations, each containing divergence terms.

The divergence is a local property of vector fields that describes the net flux per volume through an infinitesimal volume element.

Understanding the divergence of a fluid flow tells us if the fluid is compressible or not.

The divergence of a vector field is positive at a source, and negative at a sink.

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