We are going to spend the rest of this lecture by looking into an interesting subclass of Markov chains for which steady-state convergence exists-- the class of birth and death processes.

So what is a birth and death process?

It's a Markov chain whose diagram looks like this-- the states of the Markov chain start from zero and go to some finite integer number m.

And if you are at a typical state in the middle, say i.

Then next you will either go to the right-- or up by one unit-- or you will go to the left-- or down by one unit-- or you will stay where you are.

And this will happen with the following transition probabilities-- here p i-- i function of the state-- here q i.

And this one the remaining 1 minus p i minus q i.

So it's like keeping track of some animal population.

Animals get born.

They die.

The assumption here is that at any point in time, either one animal gets born, or one dies, or nothing happens.

There are no multiple deaths, no births happening at the same time.

There are many practical applications where this structure provides a basic first-level model.

An example of a chain of this kind of was the supermarket counter example that we discussed before, where the states represented the number of customers in a queue.

A customer arrives and the queue increases by one.

A customer leaves and the queue decreases by one.

Or nothing happens and the queue stays as it is.

Now, in this supermarket example, the p's and the q's were all taken to be the same across the states.

But we can generalize.

For example, the departure rate, q, could be different from state to state.

For example, with lots of customers in the queue, perhaps the clerk will work faster.

Such a chain can also be used to model the spread of disease in a population.

For example, the states could represent the number of people in a given population that have the flu.

One more person gets the flu.

The count goes up.

One more person gets healed.

The count goes down.

These probabilities, in such an epidemic model, will certainly depend on the current state.

If lots of people already have the flu, the probability that another person catches it would be pretty high.

But if no one has the flu, then the probability that one gets a transition-- where someone catches the flu-- would be pretty small.

The rate at which new people get the disease definitely depends on how many people already have it.

And that motivates cases where those p's, here, depend on the state of the chain.

There are lots of other applications for which these special Markov chains are useful.

So how do we study them?

Such a chain consists of a single aperiodic recurrent class, and so has a well-defined steady-state behavior.

To calculate the steady-state probability pi of i of a state i, you can write the system of m linear equations in the pi's that we had discussed before.

But this may be cumbersome, and in fact, more work than one actually needs to do.

There is a very clever shortcut that applies to birth and death processes.

And it's based on the frequency interpretation that we discussed in a recent clip.

To see this, let's draw a line somewhere in the middle of this chain, and focus on the transitions between this left part and this right part.

Assume that the line cuts through the middle of two adjacent states, i and i plus one, like here.

And so you zoom here and this is the picture of what you have here.

So what is the chain going to do?

Let's say it starts on the left.

It's going to move around.

And at some point, it makes a transition to the other side, going here.

And that is a transition from i to i plus 1.

And now, once it's on the right side, it's going to move around as well.

And then at one point, it will come back to that part again.

And this is a transition from i plus one to i, and so on and so forth.

Now, there is a certain balance that must be obeyed here.

The number of upward transitions through this line cannot be very different from the number of downward transitions from this line.

Because if we cross this way, then in order to cross again this way, you will have first to cross down the other way.

You can not go up 100 times here, in that direction, and go down here 50 times.

If you have gone up 100 times, it means that you have gone down 99, 100 or 101, but nothing different from that.

So the long-term frequency with which transitions of these kind occur has to be the same as the long-term frequency that transitions of that kind occur.

Or, in this diagram, the frequency of that kind has to be the same as the frequency of that kind.

So what are these frequencies?

We discussed that before.

The fraction of times at which transitions of this kind are observed is the fraction of time that we happen to be at that state, which is pi of i.

And out of the times that we are in that state, the fraction of time that transitions of that time happen is p of i.

So the overall frequency will be pi i times p of i.

And with the same argument, this is the frequency with which transitions of that kind are observed-- pi i plus 1 times q i plus 1.

Since these two frequencies are the same, we get an equation that relates pi of i to pi of i plus 1, like this.

So this is the frequency that we observed here, of this transition.

And these are the frequencies of these transitions.

And they have to be equal.

This has a nice form, because it gives us a recursion.

If we knew pi i, we can calculate pi i plus 1 as such.

So it's a system of equations that's easier to solve than the original system of linear equations which we presented before for a general Markov chain.

But how do we get started?

If we knew pi of 0, then we could use it to find pi of 1.

Then from pi of 1, you can get pi of 2, pi of 3, etc.

But we don't know pi of 0.

It's one more unknown.

It's an unknown and we need to actually use the extra normalization conditions that the sum of all the pi j, has to be equal to 1.

After we use that normalization condition, then we can find all of the pi by first solving for pi of 0.

How?

This can be returned pi 0 plus pi 1, which is pi 0 times p0 over q1 plus pi of 2, which is pi of 1 times p1 over q2, pi of 0 times p0 times p1 over q1 times q2, which is pi 2, plus et cetera equals 1.

This equation allows us to find pi of 0.

And then we use this recursion to find pi of 1, pi of 2, pi 3, et cetera.