LECTURE 13: Conditional expectation and variance revisited;

Application: Sum of a random number of independent r.v.'s

- A more abstract version of the conditional expectation: $\mathbf{E}[X \mid Y]$
  - view it as a random variable
  - the law of iterated expectations
- A more abstract version of the conditional variance
  - view it as a random variable
  - the law of total variance
- Sum of a random number of independent r.v.'s
  - mean
  - variance

Conditional expectation as a random variable

- Function h

  e.g., $h(x) = x^2$, for all x

- Random variable X; what is $h(X)$?

- $h(X)$ is the r.v. that takes the value x^2, if X happens to take the value x

- $g(y) = \mathbf{E}[X \mid Y = y] = \sum_x x p_{X|Y}(x \mid y)$
  (integral in continuous case)

- $g(Y)$: is the r.v. that takes the value $\mathbf{E}[X \mid Y = y]$, if Y happens to take the value y

- Remarks:
  - It is a function of Y
  - It is a random variable
  - Has a distribution, mean, variance, etc.

Definition: $\mathbf{E}[X|Y] = g(Y)$

The mean of $\mathbf{E}[X \mid Y]$: Law of iterated expectations

- $g(y) = \mathbf{E}[X \mid Y = y]$

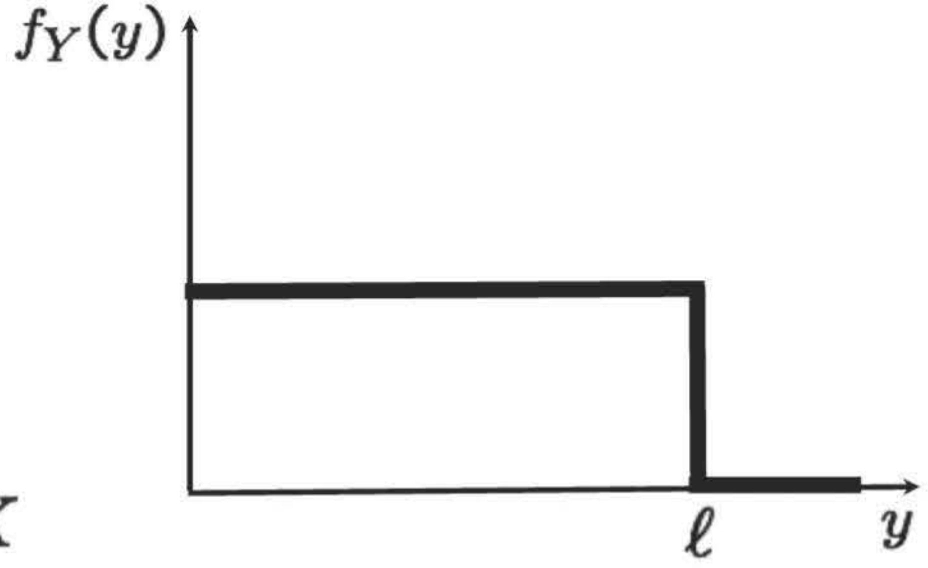
$$\mathbf{E}\big[\mathbf{E}[X \mid Y]\big] = \mathbf{E}[X]$$

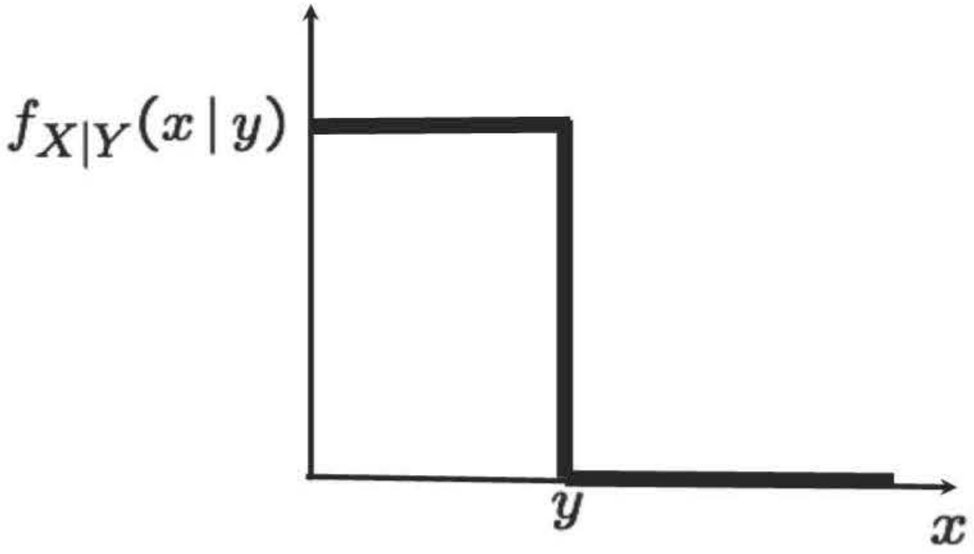
$$\mathbf{E}\big[\mathbf{E}[X \mid Y]\big]$$

Stick-breaking example

- Stick example: stick of length ℓ
  break at uniformly chosen point Y

  break what is left at uniformly chosen point X





$$\mathbf{E}[X \mid Y = y] =$$

$$\mathbf{E}[X \mid Y] =$$

$$\mathbf{E}[X] =$$

Forecast revisions

$$\mathbf{E}\big[\mathbf{E}[X \mid Y]\big] = \mathbf{E}[X]$$

- Suppose forecasts are made by calculating expected value, given any available information

- X: February sales

- Forecast in the beginning of the year:

- End of January: will get new information, value y of Y

  Revised forecast:

- Law of iterated expectations:

The conditional variance as a random variable

$$\mathrm{var}(X) = \mathbf{E}\big[(X - \mathbf{E}[X])^2\big]$$

$$\mathrm{var}(X \mid Y = y) = \mathbf{E}\big[(X - \mathbf{E}[X \mid Y = y])^2 \mid Y = y\big]$$

$\mathrm{var}(X \mid Y)$ is the r.v. that takes the value $\mathrm{var}(X \mid Y = y)$, when $Y = y$

- Example: X uniform on $[0, Y]$

$\mathrm{var}(X \mid Y = y) =$

$\mathrm{var}(X \mid Y) =$

Law of total variance: $\mathrm{var}(X) = \mathbf{E}\big[\mathrm{var}(X \mid Y)\big] + \mathrm{var}\big(\mathbf{E}[X \mid Y]\big)$

Derivation of the law of total variance

$$\text{var}(X) = \mathbf{E}\big[\text{var}(X \mid Y)\big] + \text{var}\big(\mathbf{E}[X \mid Y]\big)$$

- $\text{var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2$

$\text{var}(X \mid Y = y) =$

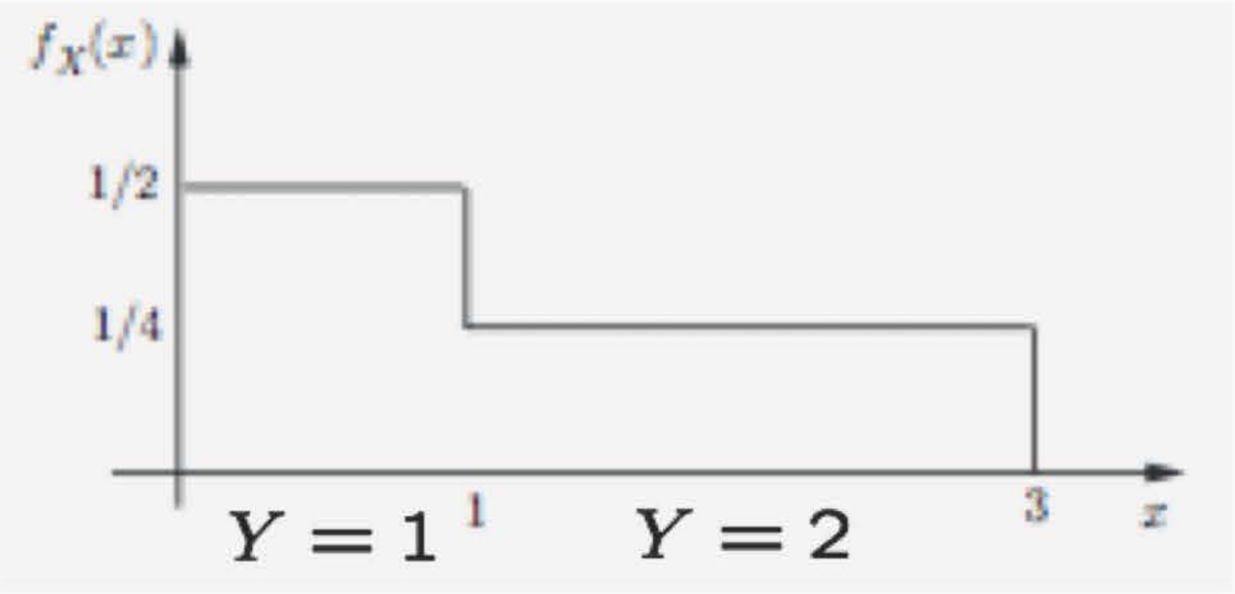
$\text{var}(X \mid Y) =$

$\mathbf{E}\big[\text{var}(X \mid Y)\big] =$

$\text{var}\big(\mathbf{E}[X \mid Y]\big) =$

A simple example

$$\text{var}(X) = \mathbf{E}\big[\text{var}(X \mid Y)\big] + \text{var}\big(\mathbf{E}[X \mid Y]\big)$$



$$\text{var}(X \mid Y) = \begin{cases} \text{var}(X \mid Y = 1) = \\ \text{var}(X \mid Y = 2) = \end{cases}$$

$$\mathbf{E}\big[\text{var}(X \mid Y)\big] =$$

$$\mathbf{E}[X \mid Y] = \begin{cases} \mathbf{E}[X \mid Y = 1] = \\ \mathbf{E}[X \mid Y = 2] = \end{cases}$$

$$\mathbf{E}\big[\mathbf{E}[X \mid Y]\big] =$$

$$\text{var}\big(\mathbf{E}[X \mid Y]\big) =$$

Section means and variances

- Two sections of a class: $y = 1$ (10 students); $y = 2$ (20 students)

  x_i: score of student i

- Experiment: pick a student at random (uniformly)

  random variables: X and Y

- Data: $y = 1: \ \frac{1}{10}\sum_{i=1}^{10} x_i = 90$ $\quad y = 2: \ \frac{1}{20}\sum_{i=11}^{30} x_i = 60$

$\mathbf{E}[X] =$

$\mathbf{E}[X \mid Y = 1] =$

$\mathbf{E}[X \mid Y] =$

$\mathbf{E}[X \mid Y = 2] =$

$\mathbf{E}\big[\mathbf{E}[X \mid Y]\big] =$

Section means and variances (ctd.)

$$\mathbf{E}[X \mid Y] = \begin{cases} 90, & \text{w.p. } 1/3 \\ 60, & \text{w.p. } 2/3 \end{cases} \qquad \mathbf{E}\big[\mathbf{E}[X \mid Y]\big] = 70 = \mathbf{E}[X]$$

$$\text{var}\big(\mathbf{E}[X \mid Y]\big) =$$

- More data: $\frac{1}{10}\sum_{i=1}^{10}(x_i - 90)^2 = 10 \qquad \frac{1}{20}\sum_{i=11}^{30}(x_i - 60)^2 = 20$

$\text{var}(X \mid Y = 1) =$

$\text{var}(X \mid Y) =$

$\text{var}(X \mid Y = 2) =$

$\mathbf{E}\big[\text{var}(X \mid Y)\big] =$

$$\text{var}(X) = \mathbf{E}\big[\text{var}(X \mid Y)\big] + \text{var}\big(\mathbf{E}[X \mid Y]\big)$$

var(X) = (average variability **within** sections) + (variability **between** sections)

Sum of a random number of independent r.v.'s

$$\mathbf{E}[Y] = \mathbf{E}[N] \cdot \mathbf{E}[X]$$

- N: number of stores visited
  (N is a nonnegative integer r.v.)
- Let $Y = X_1 + \cdots + X_N$

- X_i: money spent in store i
  - X_i independent, identically distributed
  - independent of N

$$\mathbf{E}[Y \mid N = n] \;=$$

- Total expectation theorem:

$$\mathbf{E}[Y] = \sum_n p_N(n)\,\mathbf{E}[Y \mid N = n]$$

- Law of iterated expectations:

$$\mathbf{E}[Y] = \mathbf{E}\big[\mathbf{E}[Y \mid N]\big]$$

Variance of sum of a random number of independent r.v.'s

$Y = X_1 + \cdots + X_N$

$$\mathrm{var}(Y) = \mathbf{E}\big[\mathrm{var}(Y \mid N)\big] + \mathrm{var}\big(\mathbf{E}[Y \mid N]\big)$$

$$\mathrm{var}(Y) \;=\; \mathbf{E}[N]\,\mathrm{var}(X) + (\mathbf{E}[X])^2\,\mathrm{var}(N)$$

- $\mathbf{E}[Y \mid N] = N\,\mathbf{E}[X]$

  $\mathrm{var}\big(\mathbf{E}[Y \mid N]\big) \;=$

- $\mathrm{var}(Y \mid N = n) \;=$

  $\mathrm{var}(Y \mid N) \;=$

  $\mathbf{E}\big[\mathrm{var}(Y \mid N)\big] \;=$