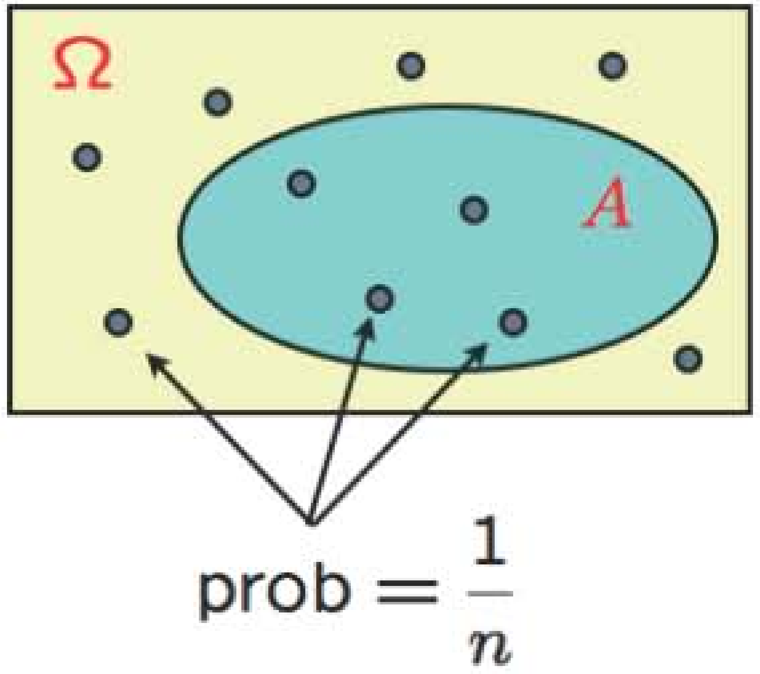
LECTURE 4: Counting



Discrete uniform law

– Assume Ω consists of n equally likely elements
– Assume A consists of k elements

Then : $$\mathbf{P}(A) = \frac{\text{number of elements of } A}{\text{number of elements of } \Omega} = \frac{k}{n}$$

- Basic counting principle

- Applications

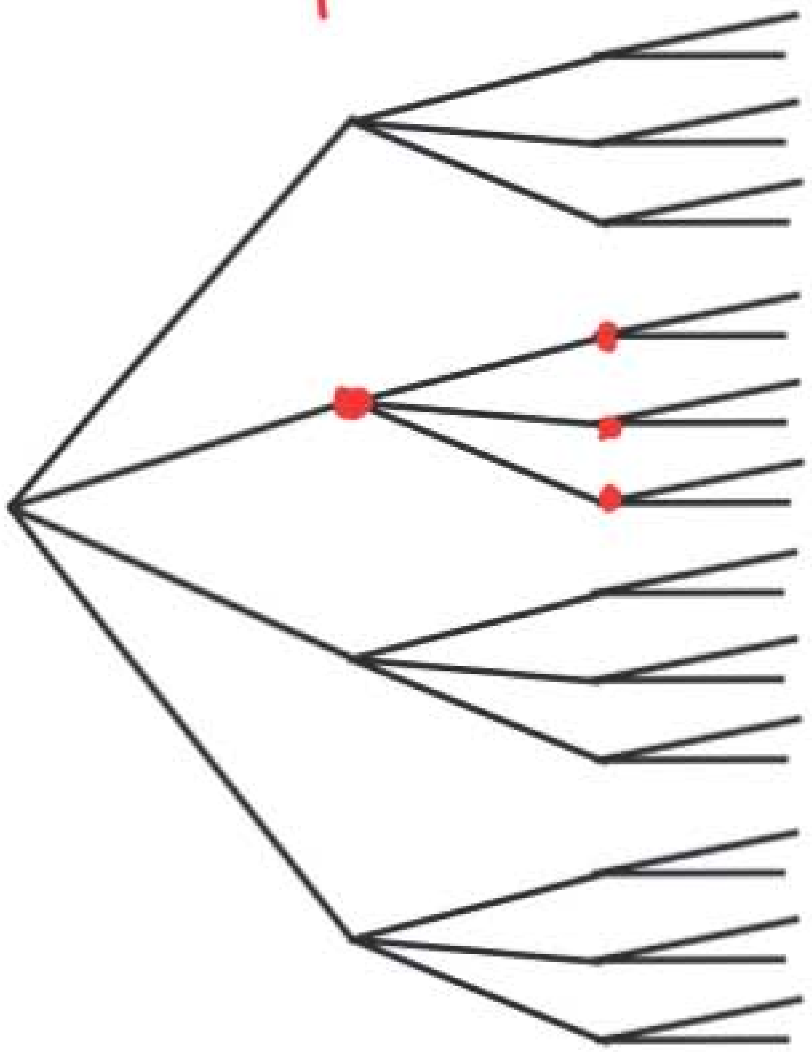
permutations	number of subsets
combinations	binomial probabilities
partitions	

Basic counting principle

4 shirts
3 ties
2 jackets

Number of possible attires?

4 12 $24 = 4 \cdot 3 \cdot 2$



$r = 3$

$n_1 = 4$

$n_2 = 3$

$n_3 = 2$

- r stages
- n_i choices at stage i

Number of choices is: $n_1 \cdot n_2 \cdots n_r$.

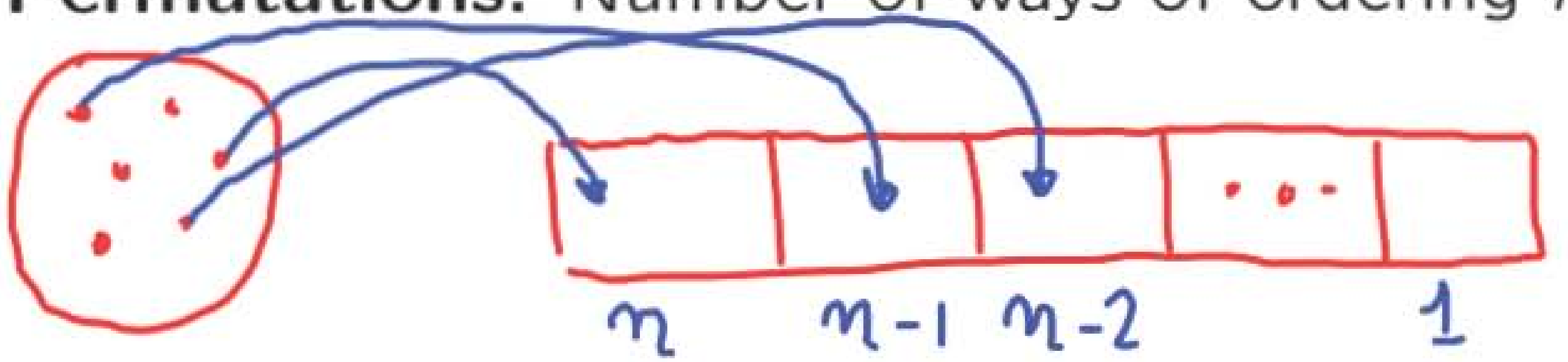
Basic counting principle examples

- Number of license plates with 2 letters followed by 3 digits:

$$26 \cdot 26 \cdot 10 \cdot 10 \cdot 10$$

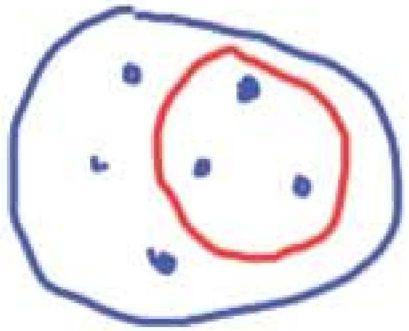
  - ... if repetition is prohibited: $26 \cdot 25 \cdot 10 \cdot 9 \cdot 8$

- **Permutations:** Number of ways of ordering n elements:



$$n \cdot (n-1)(n-2) \cdots 1 = n!$$

- Number of subsets of $\{1, \dots, n\}$:



$$2 \cdot 2 \cdots 2 = 2^n$$

$n=1 \quad \{1\} \qquad 2^1 = 2$

$\{1\} \quad \emptyset$

Example

- Find the probability that:
  $\underbrace{\text{six rolls of a (six-sided) die}}$ $\underbrace{\text{all give different numbers.}}$ A

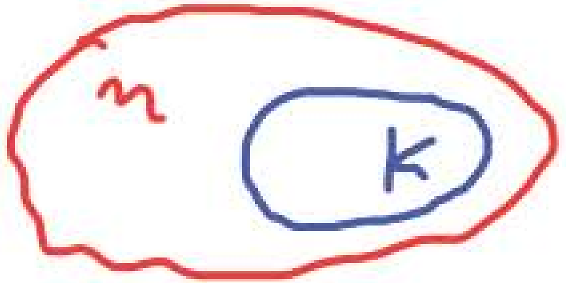
  (Assume all outcomes equally likely.)

typical outcome $\quad P(2,3,4,3,6,2) = 1/6^6$

" element of A: $\quad (2,3,4,1,6,5) = 6!$

$$P(A) = \frac{\#\text{ in } A}{\#\text{ possible outcomes}} = \frac{6!}{6^6}.$$

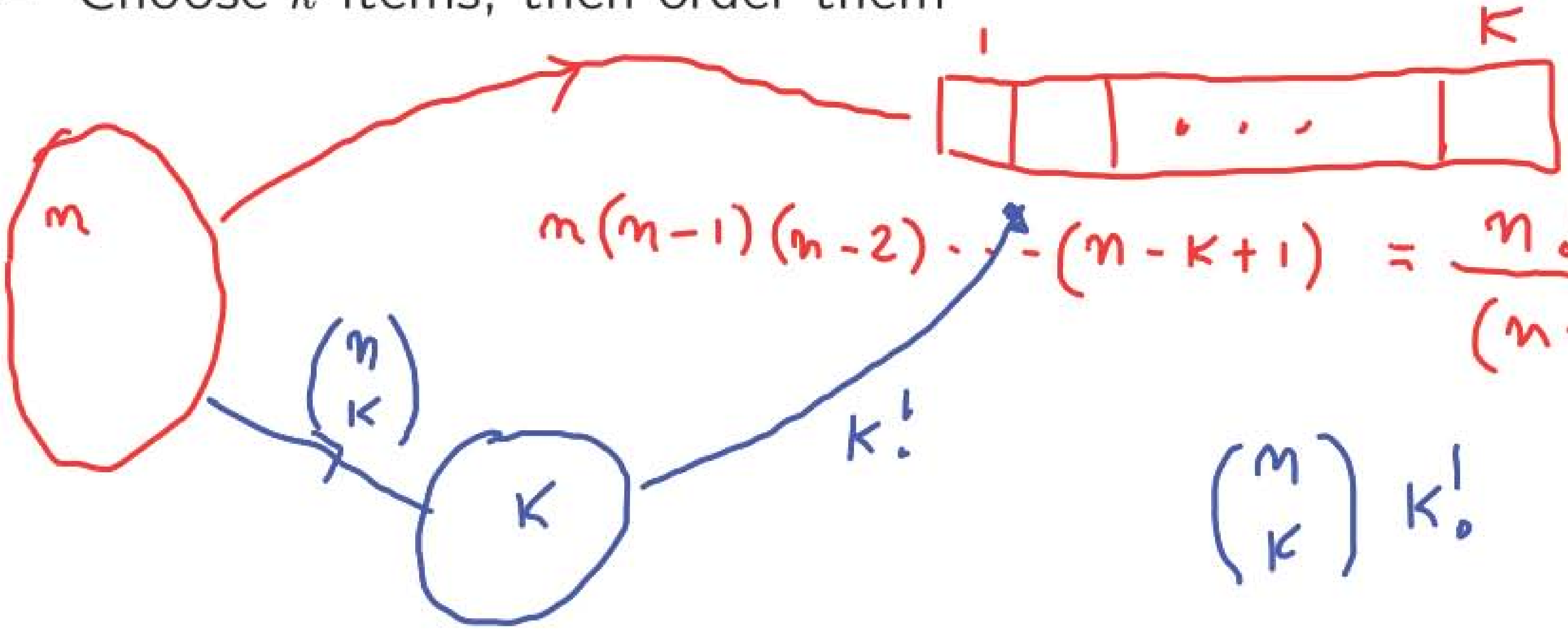
Combinations



- Definition: $\binom{n}{k}$: number of k-element subsets of a given n-element set $\boxed{= \frac{n!}{k!(n-k)!}}$

$n = 0, 1, 2, \ldots$

- Two ways of constructing an **ordered** sequence of k **distinct** items: $k = 0, 1, \ldots, n$
  - Choose the k items one at a time
  - Choose k items, then order them



$$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!} = \binom{n}{k}k!$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$\binom{n}{n} = 1 \qquad \frac{n!}{n!\,0!}$$

$0! = 1$ convention

$$\binom{n}{0} = \frac{n!}{0!\,n!} = 1$$

$\emptyset$

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \cdots \binom{n}{n} = \#\text{all subsets} = 2^n$$

Binomial coefficient $\binom{n}{k} \longrightarrow$ Binomial probabilities

$$\boxed{\mathbf{P}(k \text{ heads}) = \binom{n}{k} p^k (1-p)^{n-k}}$$

- $n \geq 1$ independent coin tosses; $\mathbf{P}(H) = p$

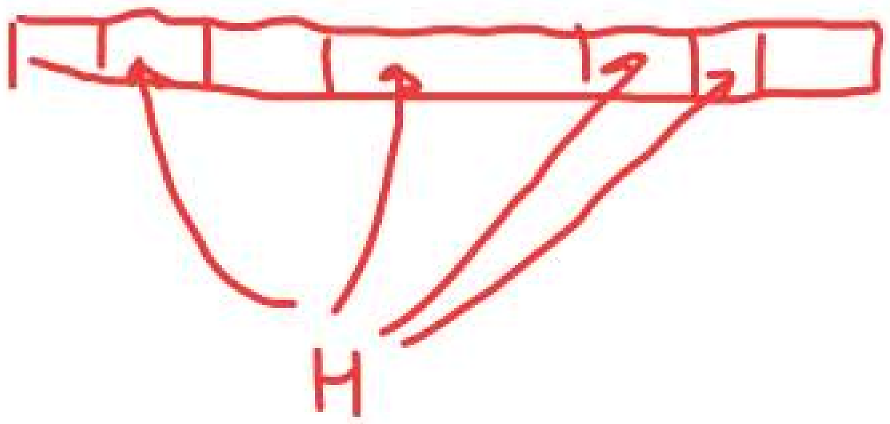
  $n=6$

- $\mathbf{P}(HTTHHH) = p(1-p)(1-p)ppp = p^4(1-p)^2$

- $\mathbf{P}(\text{particular sequence}) = p^{\#\text{heads}}(1-p)^{\#\text{tails}}$

- $\mathbf{P}(\text{particular } k\text{–head sequence}) = p^k(1-p)^{n-k}$

$\mathbf{P}(k \text{ heads}) = p^k(1-p)^{n-k} \cdot (\# \ k\text{-head sequences})$



$\binom{n}{k}$

A coin tossing problem

- Given that there were 3 heads in 10 tosses, what is the probability that the first two tosses were heads?
  - event A: the first 2 tosses were heads
  - event B: 3 out of 10 tosses were heads

Assumptions:

- independence
- $\mathbf{P}(H) = p$

$$\mathbf{P}(k \text{ heads}) = \binom{n}{k} p^k (1-p)^{n-k}$$

- First solution:

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)} = \frac{\mathbf{P}(H_1 H_2 \text{ and one } H \text{ in tosses } 3, \ldots, 10)}{\mathbf{P}(B)}$$

$$= \frac{p^2 \cdot \binom{8}{1} p^1 \cdot (1-p)^7}{\binom{10}{3} p^3 (1-p)^7} = \frac{\binom{8}{1}}{\binom{10}{3}} = \frac{8}{\binom{10}{3}}.$$

A coin tossing problem

- Given that there were 3 heads in 10 tosses,
  what is the probability that the first two tosses were heads?
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Assumptions:
- independence
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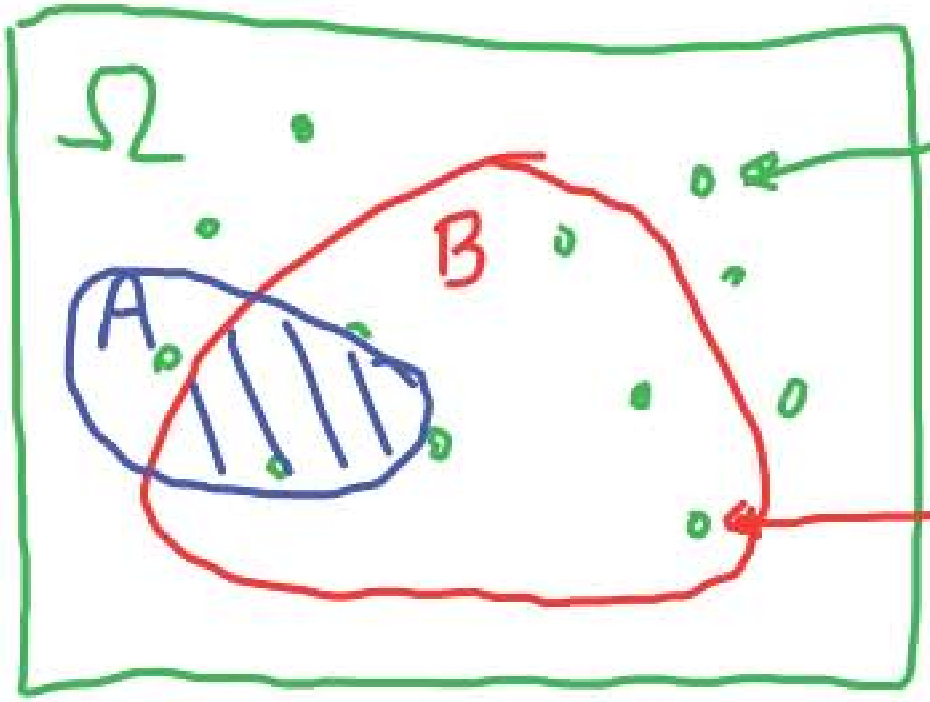
$$\mathbf{P}(k \text{ heads}) = \binom{n}{k} p^k (1-p)^{n-k}$$

- Second solution: Conditional probability law (on B) is uniform



length 10 seq.

3-head seqs

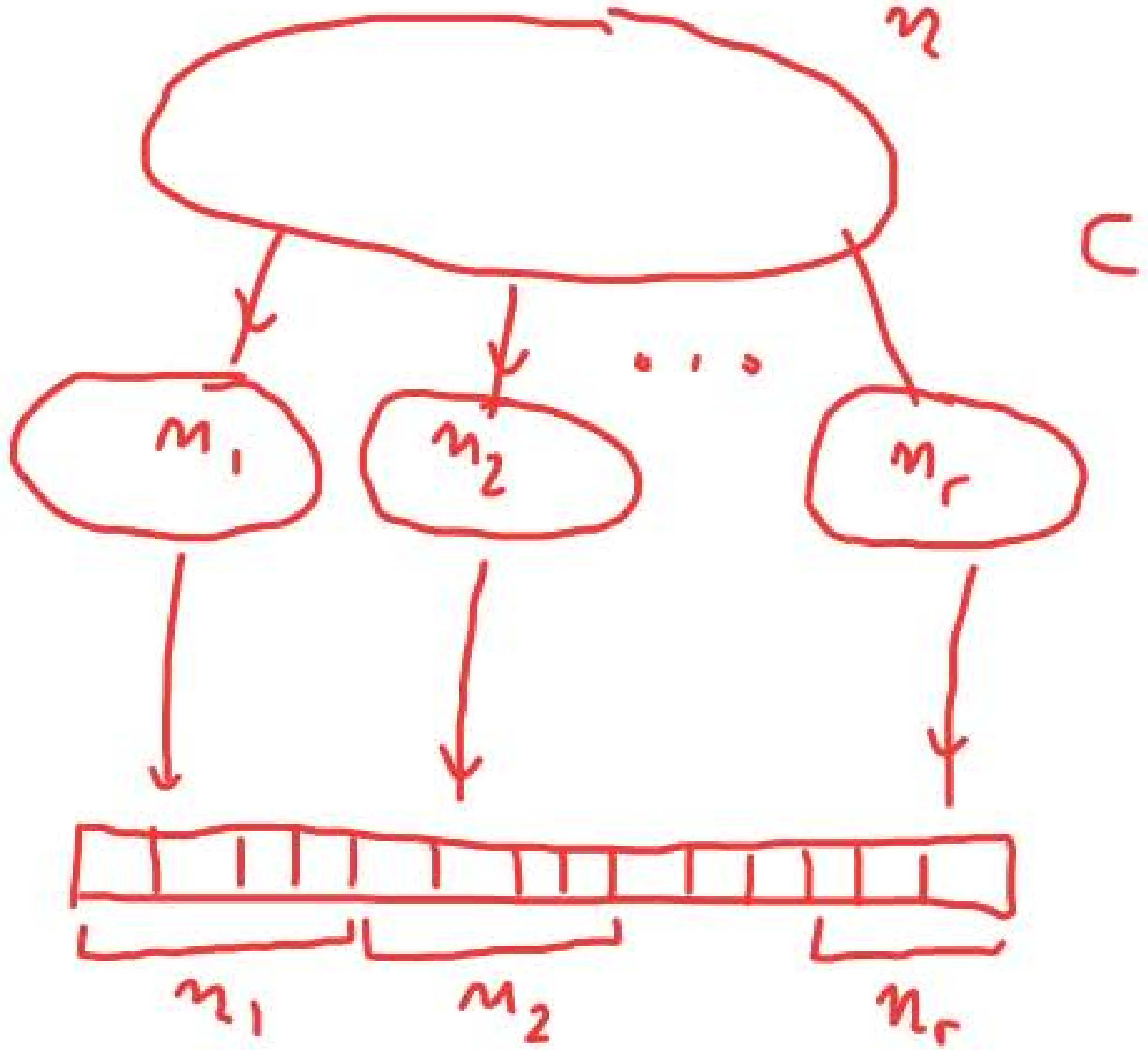
$p^3(1-p)^7$

$$\frac{\# \text{ in } (A \cap B)}{\# \text{ in } B} = \frac{8}{\binom{10}{3}}.$$

Partitions

- $n \geq 1$ distinct items; $r \geq 1$ persons
  give n_i items to person i
  - here $n_1, \ldots, n_r$ are given nonnegative integers
  - with $n_1 + \cdots + n_r = n$
- Ordering n items: $n!$
  - Deal n_i to each person i, and then order

$$c \; n_1! \, n_2! \cdots n_r! = n!$$



$r=2 \qquad n_1 = k \qquad n_2 = n-k$

$$\text{number of partitions} = \frac{n!}{n_1!\, n_2! \cdots n_r!} \qquad \text{(multinomial coefficient)}$$

Example: 52-card deck, dealt (fairly) to four players.
Find $\mathbf{P}$(each player gets an ace)

- Outcomes are: partition equally likely
  - number of outcomes: $\dfrac{52!}{13!\,13!\,13!\,13!}$

- Constructing an outcome with one ace for each person:
  - distribute the aces $4 \cdot 3 \cdot 2 \cdot 1$
  - distribute the remaining 48 cards $\dfrac{48!}{12!\,12!\,12!\,12!}$

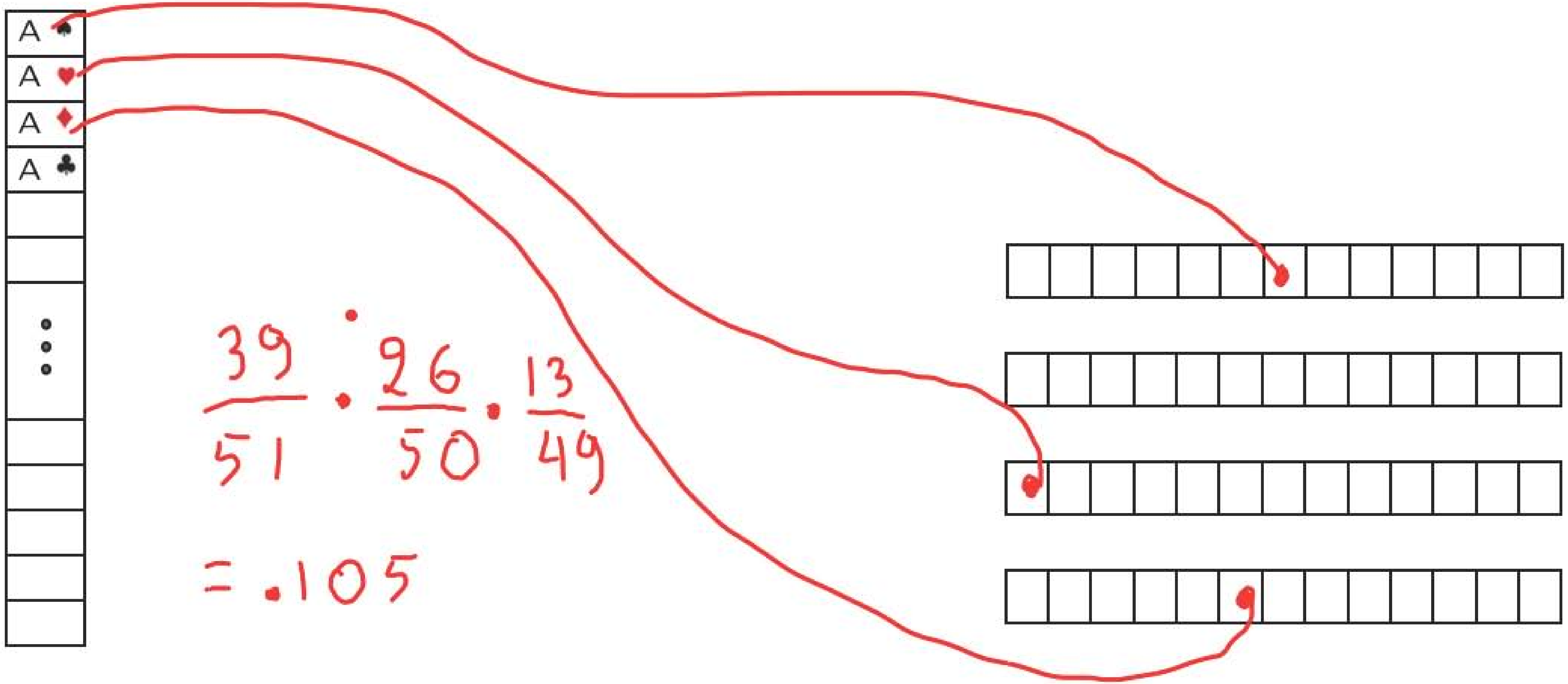
- Answer:

$$\frac{4 \cdot 3 \cdot 2 \cdot \dfrac{48!}{12!\,12!\,12!\,12!}}{\dfrac{52!}{13!\,13!\,13!\,13!}}$$

Example: 52-card deck, dealt (fairly) to four players. Find $\mathbf{P}$(each player gets an ace)

A smart solution

Stack the deck, aces on top



$$\frac{39}{51} \cdot \frac{26}{50} \cdot \frac{13}{49}$$

$= .105$