

## LECTURE 2: Conditioning and Bayes' rule

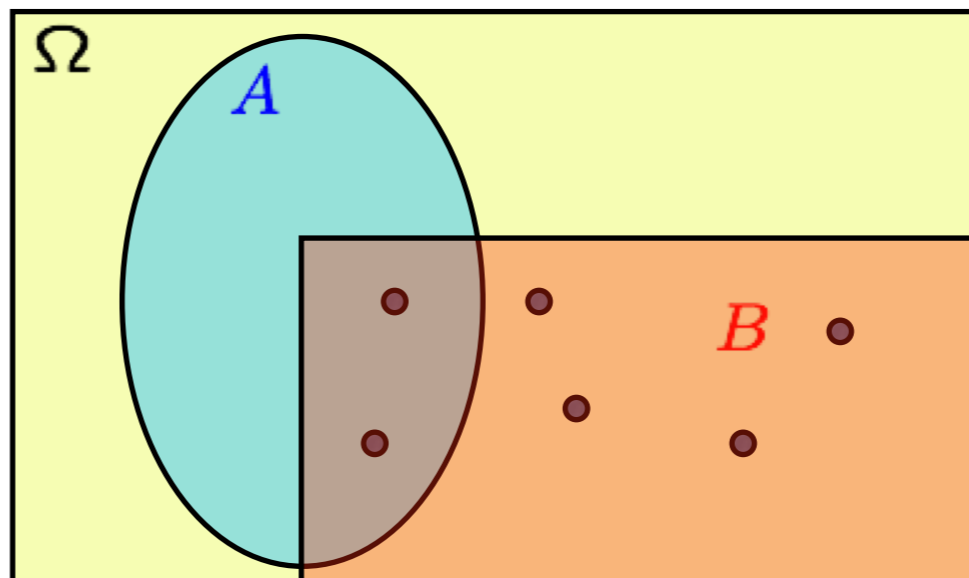
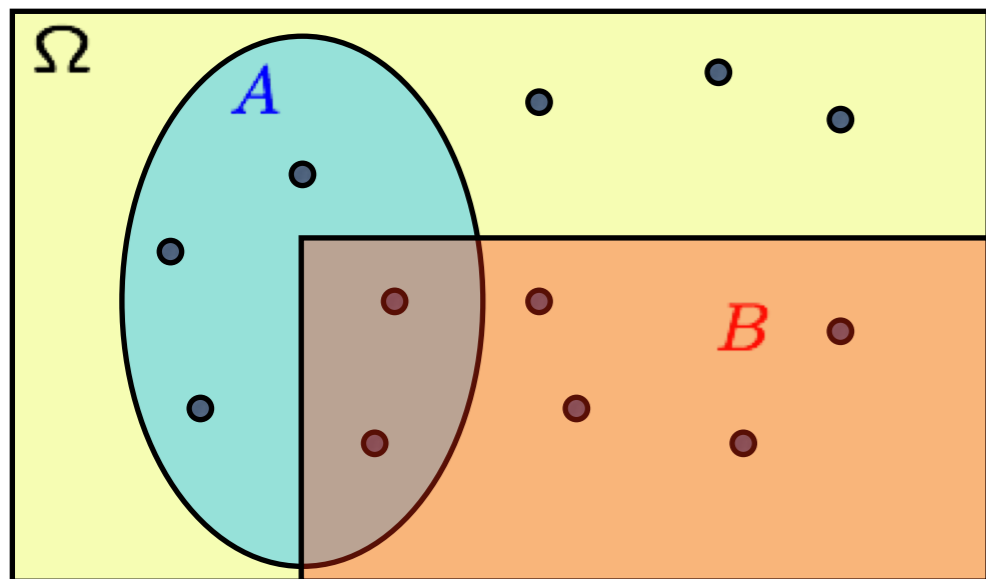
- Conditional probability
- Three **important** tools:
  - Multiplication rule
  - Total probability theorem
  - Bayes' rule ( $\longrightarrow$  inference)

# The idea of conditioning

# Use new information to revise a model

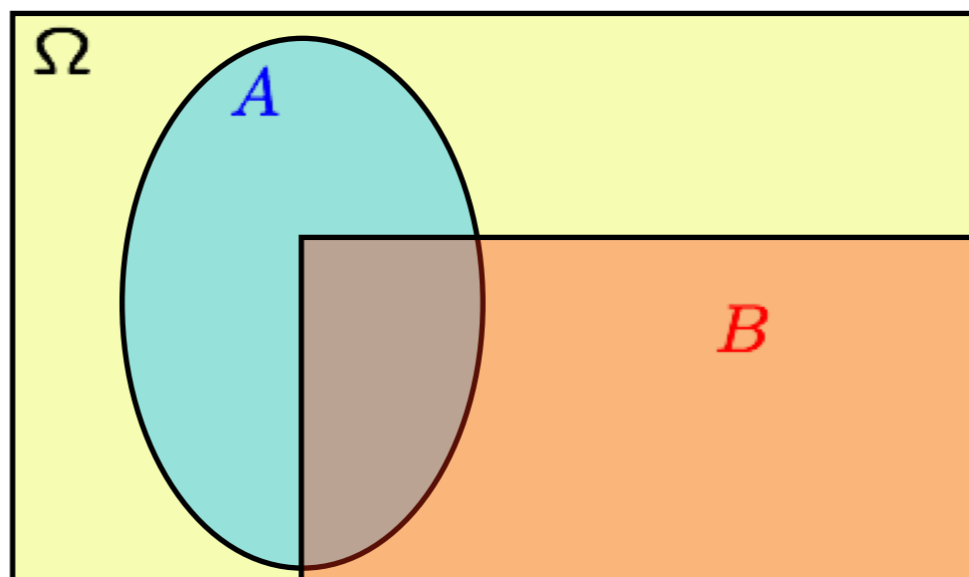
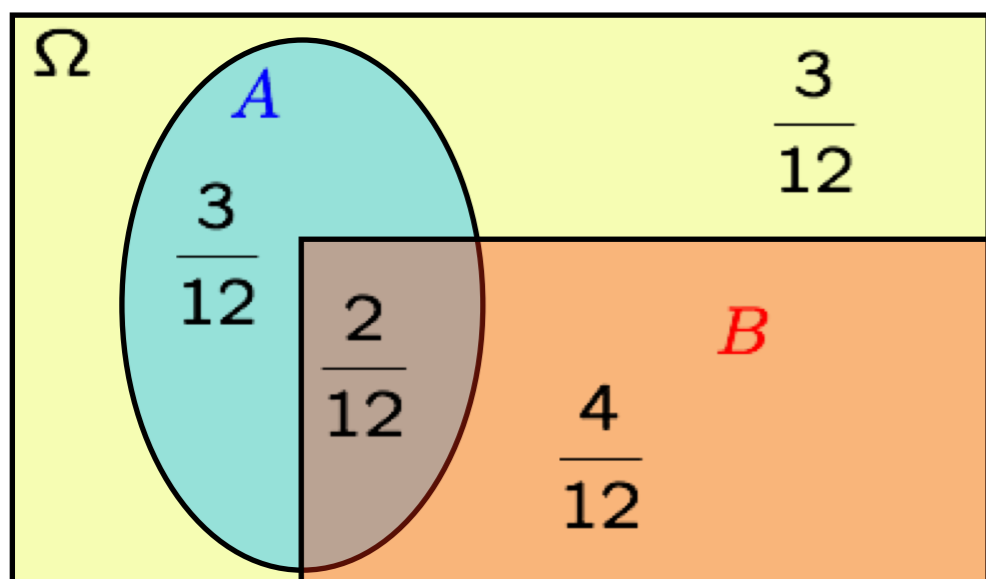
Assume 12 equally likely outcomes

If told  $B$  occurred:

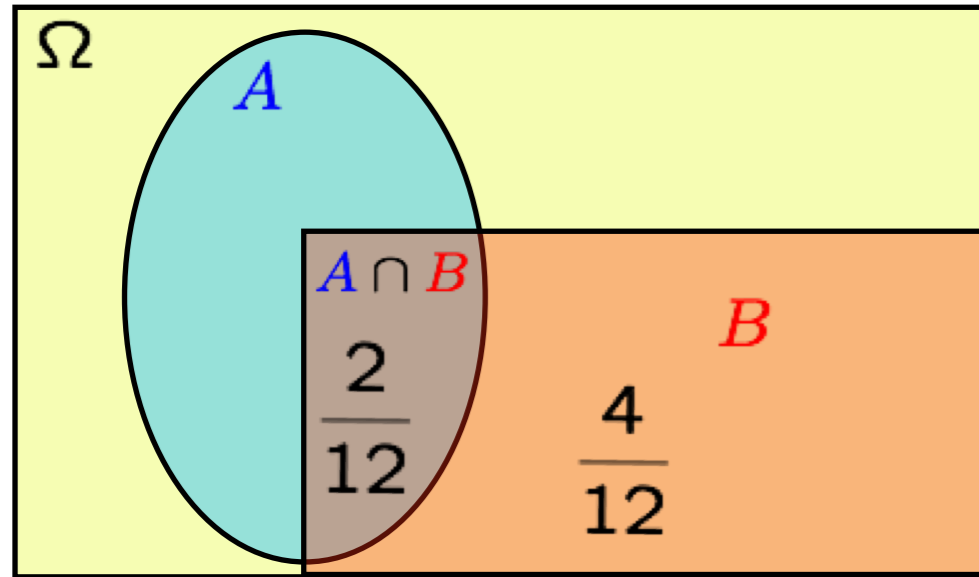


$$P(A) = \frac{5}{12} \quad P(B) = \frac{6}{12}$$

$$P(A | B) = \quad P(B | B) =$$



## Definition of conditional probability

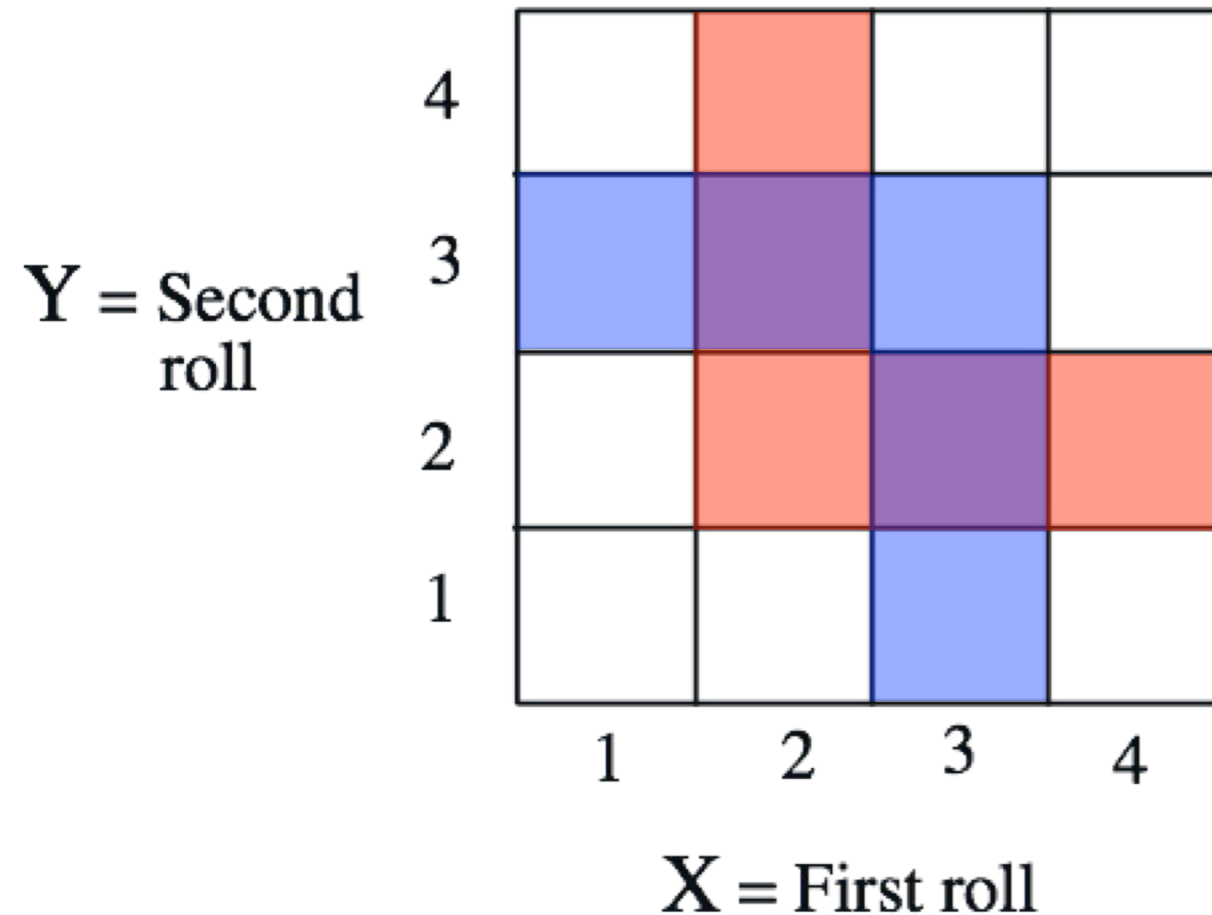


- $P(A|B)$  = “probability of  $A$ , given that  $B$  occurred”

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

defined only when  $P(B) > 0$

## Example: two rolls of a 4-sided die



- Let  $B$  be the event:  $\min(X, Y) = 2$

Let  $M = \max(X, Y)$

$$P(M = 1 \mid B) =$$

$$P(M = 3 \mid B) =$$

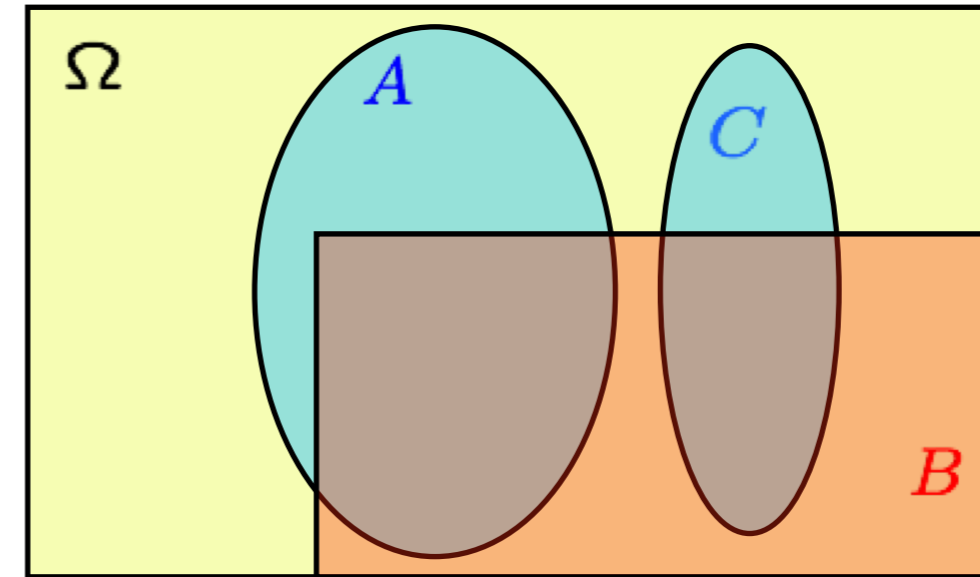
## Conditional probabilities share properties of ordinary probabilities

$$\mathbf{P}(A \mid B) \geq 0$$

assuming  $\mathbf{P}(B) > 0$

$$\mathbf{P}(\Omega \mid B) =$$

$$\mathbf{P}(B \mid B) =$$



If  $A \cap C = \emptyset$ , then  $\mathbf{P}(A \cup C \mid B) = \mathbf{P}(A \mid B) + \mathbf{P}(C \mid B)$

## Models based on conditional probabilities

$$P(A | B) = \frac{P(A \cap B)}{P(B)} \quad P(B | A) = \frac{P(A \cap B)}{P(A)}$$

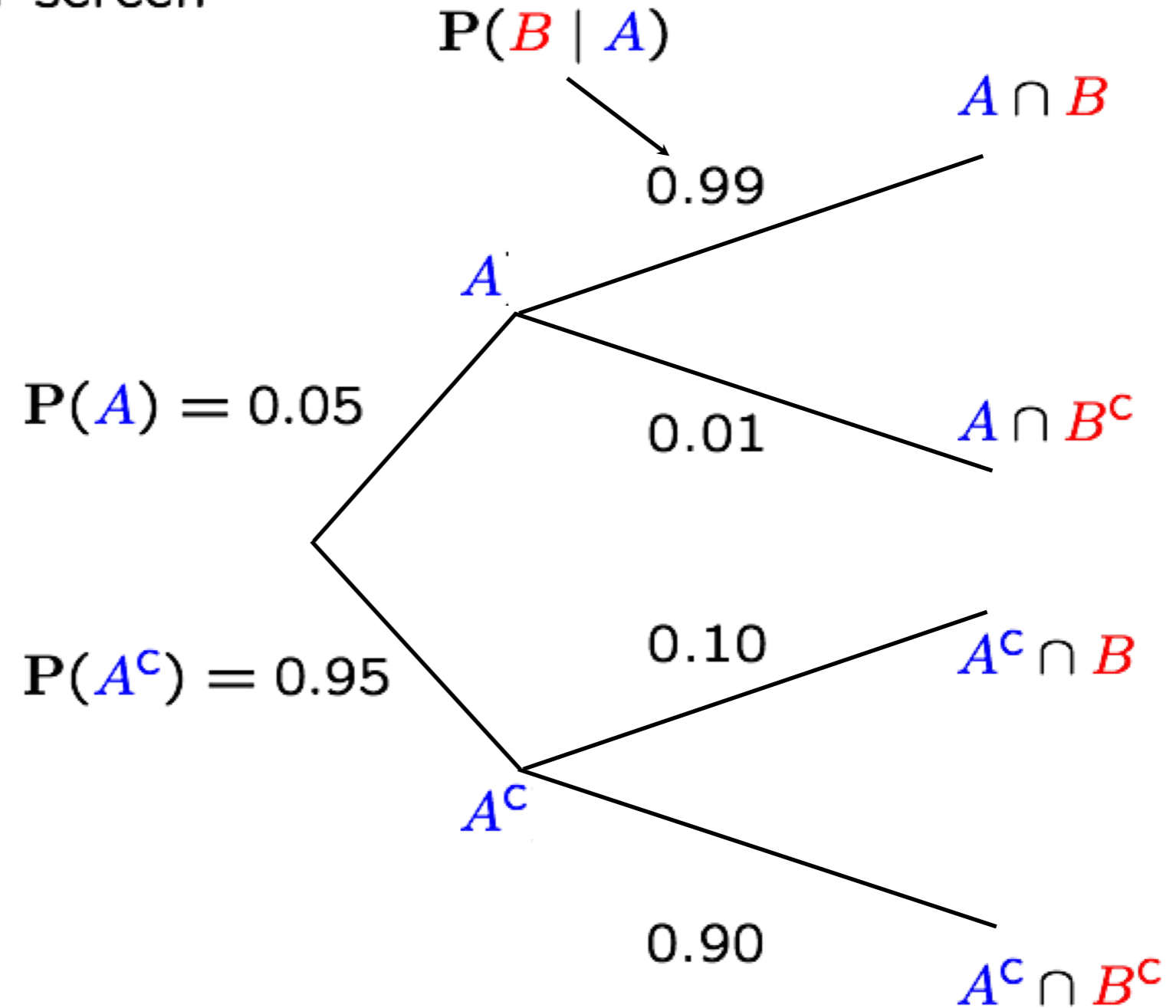
Event  $A$ : Airplane is flying above

Event  $B$ : Something registers on radar screen

- $P(A \cap B) =$

- $P(B) =$

- $P(A | B) =$

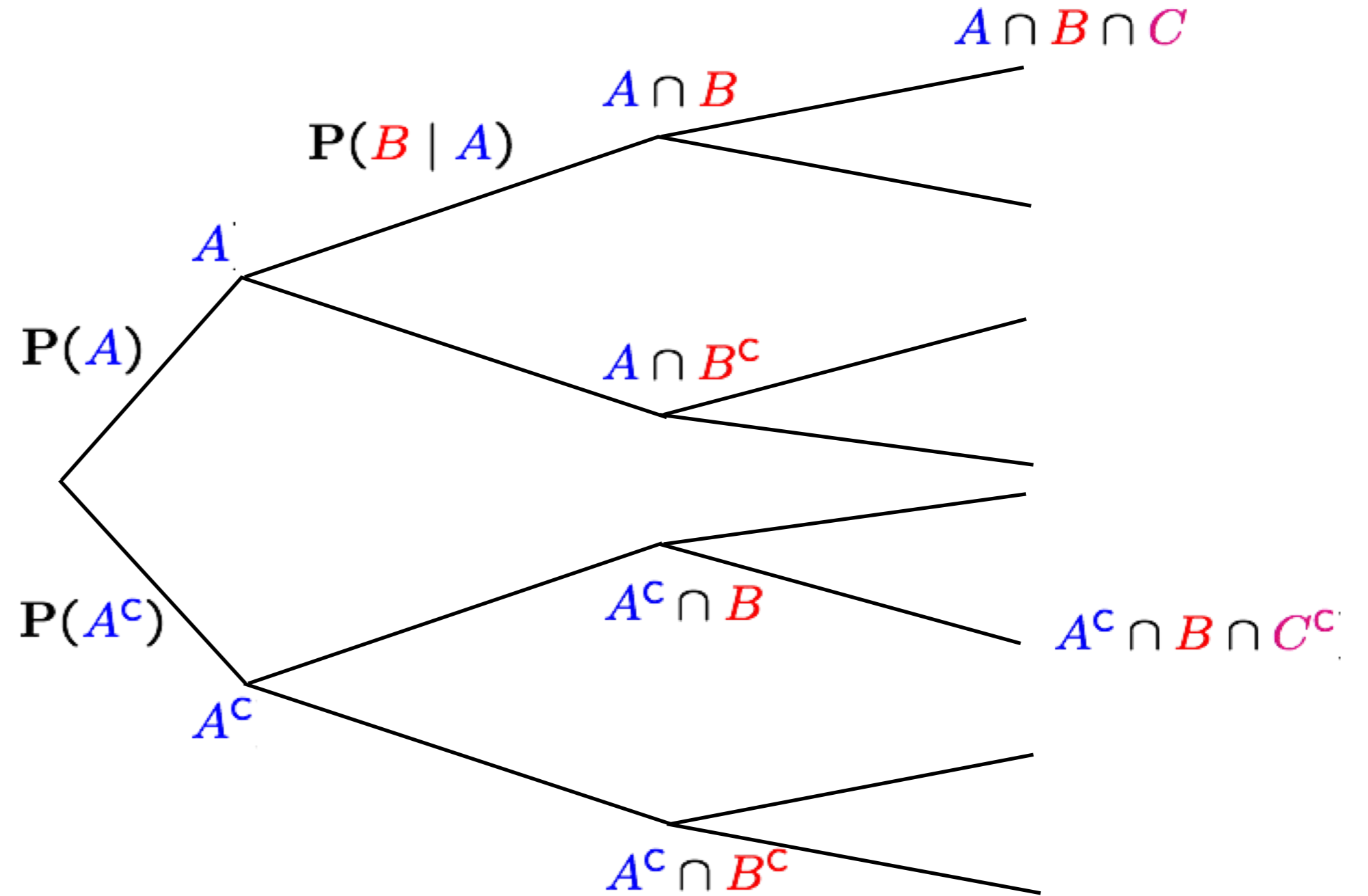


## The multiplication rule

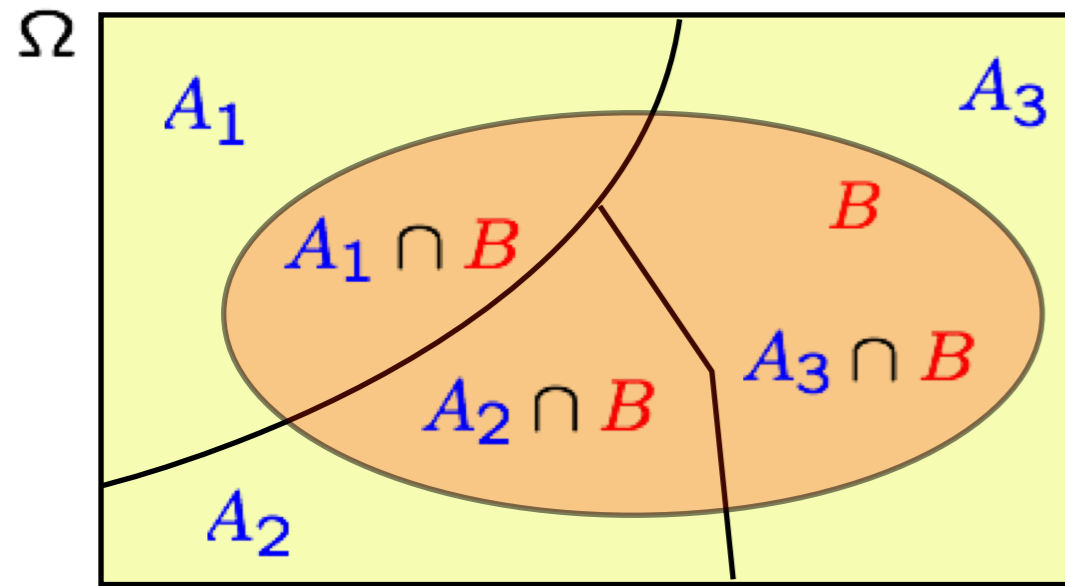
$$P(A | B) = \frac{P(A \cap B)}{P(B)}$$

$$\begin{aligned} P(A \cap B) &= P(B) P(A | B) \\ &= P(A) P(B | A) \end{aligned}$$

$$P(A^c \cap B \cap C^c) =$$

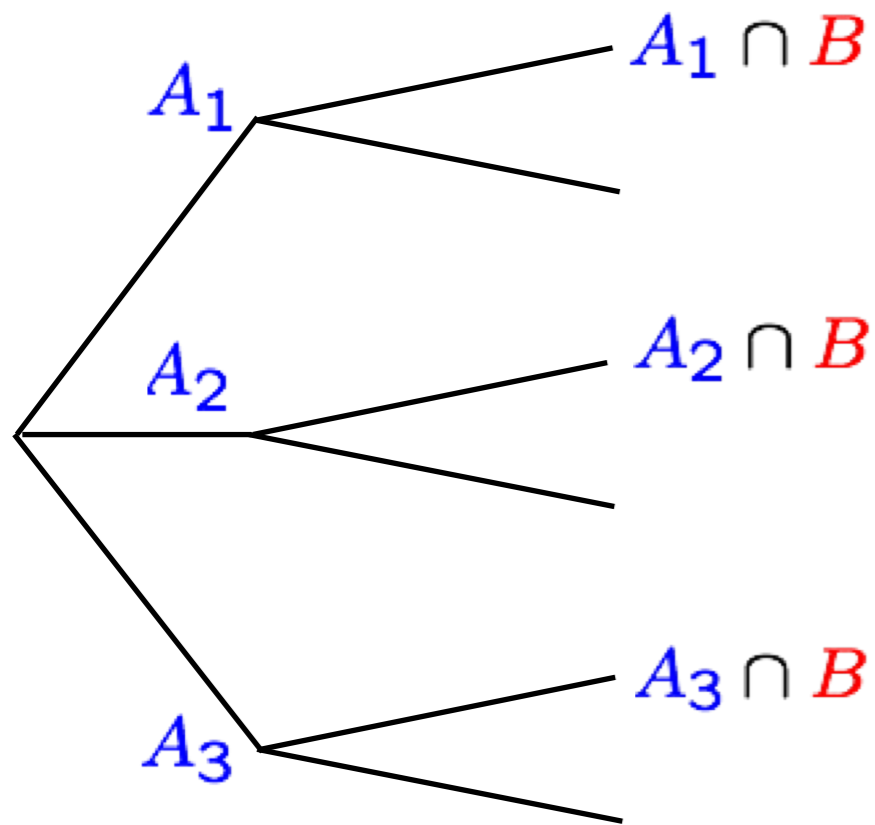


# Total probability theorem



- Partition of sample space into  $A_1, A_2, A_3$
- Have  $P(A_i)$ , for every  $i$
- Have  $P(B | A_i)$ , for every  $i$

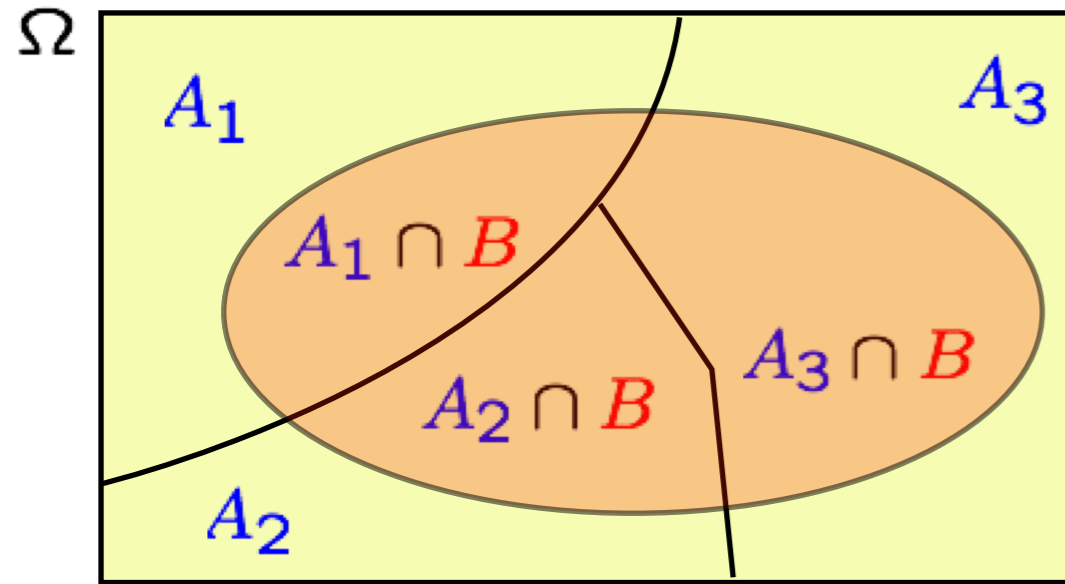
$$P(B) =$$



$$P(B) = \sum_i P(A_i) P(B | A_i)$$



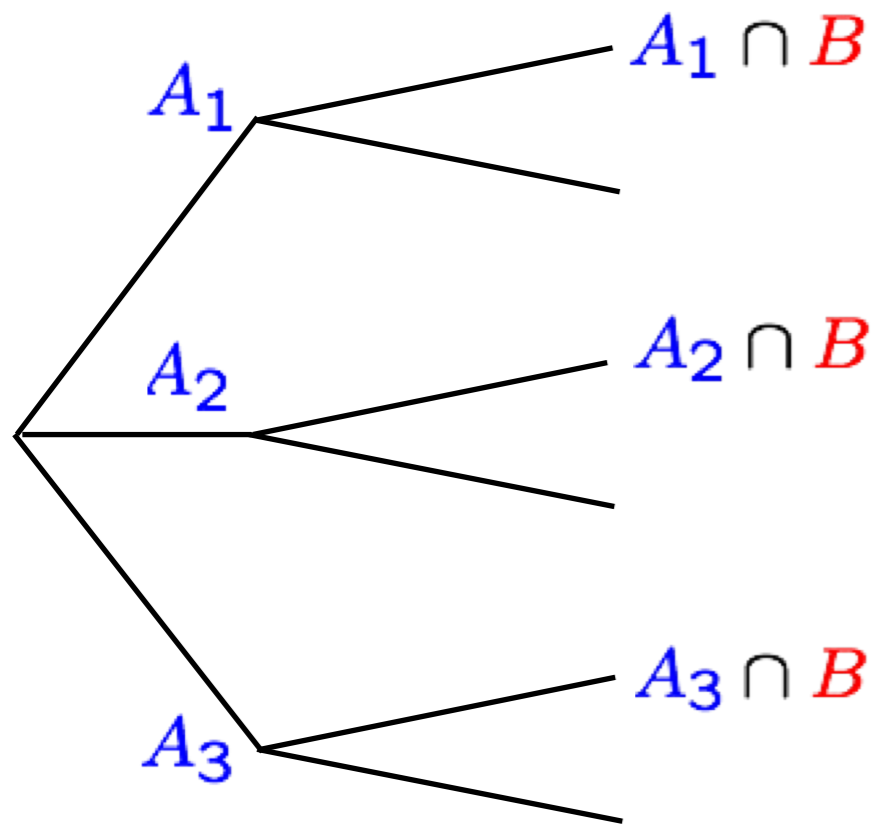
## Bayes' rule



- Partition of sample space into  $A_1, A_2, A_3$
- Have  $P(A_i)$ , for every  $i$       initial "beliefs"
- Have  $P(B | A_i)$ , for every  $i$

revised "beliefs," given that  $B$  occurred:

$$P(A_i | B) =$$



$$P(A_i | B) = \frac{P(A_i)P(B | A_i)}{\sum_j P(A_j)P(B | A_j)}$$

## Bayes' rule and inference

- Thomas Bayes, presbyterian minister (c. 1701-1761)
- “Bayes' theorem,” published posthumously
- systematic approach for incorporating new evidence
- Bayesian inference
  - initial beliefs  $P(A_i)$  on possible causes of an observed event  $B$
  - model of the world under each  $A_i$ :  $P(B | A_i)$

$$A_i \xrightarrow[\text{P}(B | A_i)]{\text{model}} B$$

- draw conclusions about causes

$$B \xrightarrow[\text{P}(A_i | B)]{\text{inference}} A_i$$

MIT OpenCourseWare  
<https://ocw.mit.edu>

Resource: Introduction to Probability  
John Tsitsiklis and Patrick Jaillet

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