

Note: All references to Figures and Equations whose numbers are *not* preceded by an “S” refer to the textbook.

From Figure 3.6 on page 79 of the textbook, a first-order system has a step response as shown in Figure S3.1:

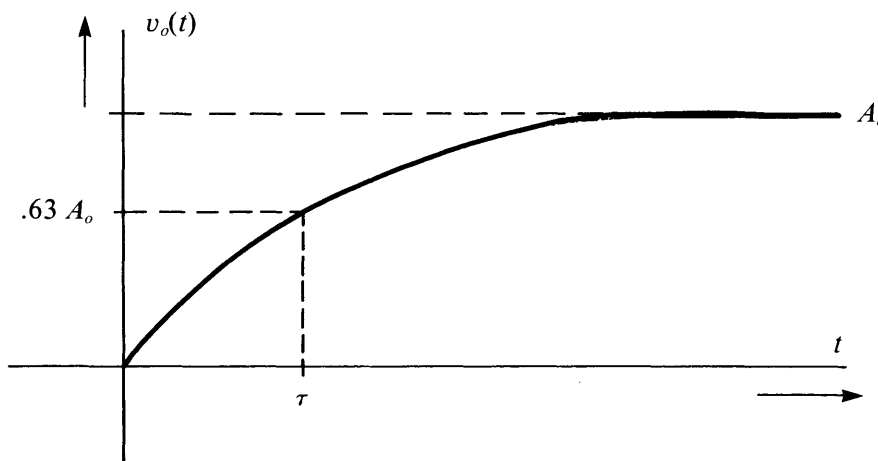
Solution 3.1 (P3.1)

Figure S3.1 First-order step response.

where A_o is the d-c gain of the system. This time response is described by

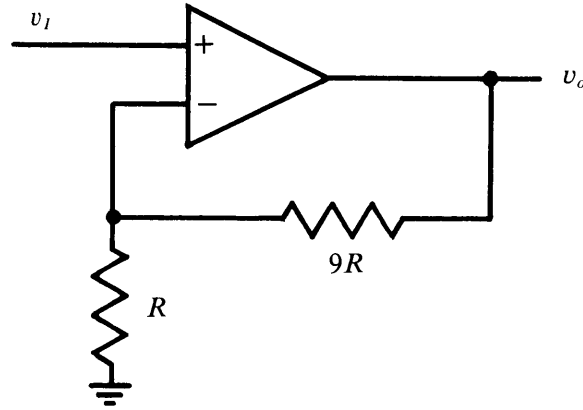
$$v_o(t) = A_o(1 - e^{-t/\tau}) \quad (\text{S3.1})$$

and the transfer function for this system is

$$\frac{V_o(s)}{V_i(s)} = \frac{A_o}{\tau s + 1} \quad (\text{S3.2})$$

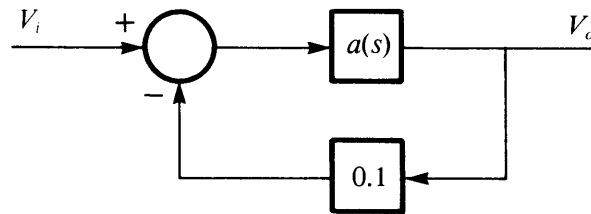
Here, we are given an operational amplifier connected for a non-inverting gain of 10, as shown in Figure S3.2:

Figure S3.2 Gain-of-ten connection.



This corresponds to the block diagram of Figure S3.3:

Figure S3.3 Block diagram for the gain-of-ten connection.



This connection has a transfer function of

$$\frac{V_o}{V_i}(s) = \frac{a(s)}{1 + 0.1 \times a(s)} = \frac{10 a(s)}{10 + a(s)} \quad (\text{S3.3})$$

We are given that this system is first order with $A_o = 10$, and $\tau = 10^{-6}$ sec. Thus, using Equation S3.2, we have

$$\frac{10 a(s)}{10 + a(s)} = \frac{10}{10^{-6}s + 1} \quad (\text{S3.4})$$

Solve this for $a(s)$:

$$a(s) = \frac{10 + a(s)}{10^{-6}s + 1} \quad (\text{S3.5})$$

Collect terms:

$$a(s)\left(1 - \frac{1}{10^{-6}s + 1}\right) = \frac{10}{10^{-6}s + 1} \quad (\text{S3.6})$$

or

$$a(s)\left(\frac{10^{-6}s}{10^{-6}s + 1}\right) = \frac{10}{10^{-6}s + 1} \quad (\text{S3.7})$$

which yields:

$$a(s) = \frac{10^7}{s} \quad (\text{S3.8})$$

That is, the op amp is modeled as a pole at the origin, which represents an integrator.

First, let's examine the pole locations for this system. There is a complex pair at $s_1 = -0.25 + j0.97$ and $s_2 = -0.25 - j0.97$. There is a real axis pole at $s_3 = -10$. These poles are shown on the s plane in Figure S3.4.

Solution 3.2 (P3.2)

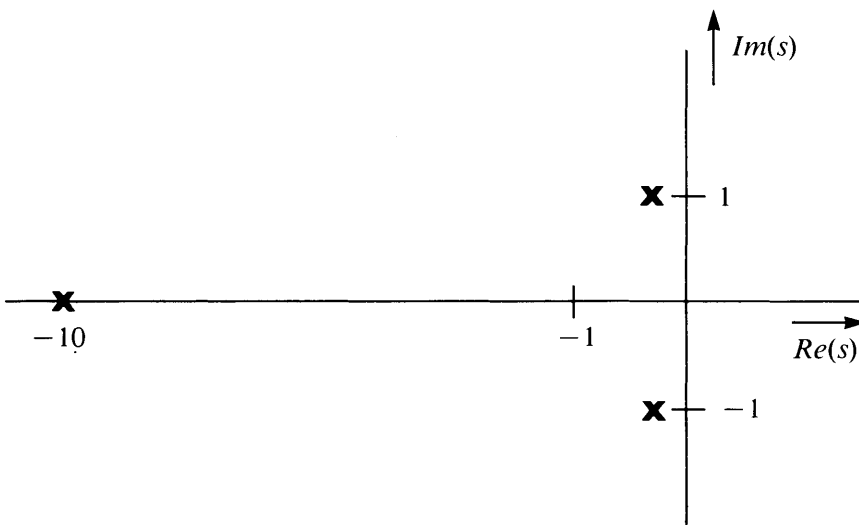


Figure S3.4 Pole locations for Problem 3.2 (P3.2).

Following the discussion of Section 3.3.2 of the textbook, because the real axis pole is a factor of 10 farther from the origin than the complex pair, the system is well approximated by the complex pair alone.

The complex pair has $\omega_n = 1$, and $\zeta = 0.25$. We consider the system to be approximated by the transfer function:

$$A(s) \simeq \frac{1}{s^2 + 0.5s + 1} \quad (\text{S3.9})$$

This has a step response given by Equation 3.41 of the textbook. That is:

$$v_o(t) = a_o \left[1 - \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \sin(\sqrt{1 - \zeta^2} \omega_n t + \Phi) \right]$$

where

$$\Phi = \tan^{-1} \left(\frac{\sqrt{1 - \zeta^2}}{\zeta} \right) \quad (\text{S3.10})$$

Here $a_o = 1$, and $\Phi = 1.32$ radians. Thus,

$$v_o(t) = [1 - 1.03e^{-t/4} \sin(0.97t + 1.32)] \quad (\text{S3.11})$$

For this second-order system, we can estimate the peak overshoot by using Equation 3.58 of the textbook. That is,

$$P_o = 1 + e^{-\pi/\tan \Phi} = 1.45 \quad (\text{S3.12})$$

Thus, there is a 45% overshoot.

Solution 3.3 (P3.5)

A system that is second order, with a d-c gain of 1, has a transfer function of the form

$$\frac{V_o}{V_i}(s) = \frac{1}{\frac{s^2}{\omega_n^2} + \frac{2\zeta s}{\omega_n} + 1} \quad (\text{S3.13})$$

Given that $P_o = 1.38$, we can use Equation 3.58 of the textbook to solve for ζ . We have:

$$P_o = 1 + e^{-\pi/\tan \theta} = 1.38 \quad (\text{S3.14})$$

Take the log of both sides

$$\ln(0.38) = \frac{-\pi}{\tan \theta} \quad (\text{S3.15})$$

or

$$\tan \theta = \frac{-\pi}{\ln(0.38)} = 3.25 \quad (\text{S3.16})$$

Thus, $\theta = \tan^{-1}(3.25) = 1.27$ radians.

Now, recall that $\zeta = \cos \theta$. Thus, $\zeta = \cos(1.27) = 0.29$. Then we can use the graph of Figure 3.8 in the textbook to find ω_n . For $\zeta = 0.3$, the time required to pass through unity is defined by:

$$\omega_n t = 2 \quad (\text{S3.17})$$

We are given that $t = 0.5 \times 10^{-6}$ sec. Thus

$$\omega_n = \frac{2}{0.5 \times 10^{-6}} = 4 \times 10^6 \text{ rad/sec} \quad (\text{S3.18})$$

The impulse response of this system may be found in Table 3.1 in the textbook. (We recall that the impulse response is the inverse transform of the transfer function for a system.) Thus, the impulse response, $h(t)$, for the system described by Equation S3.13 is

$$h(t) = \frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} (\sin \omega_n \sqrt{1 - \zeta^2} t) \quad (\text{S3.19})$$

Substituting in the values we have found for ω_n and ζ yields

$$h(t) = 4.2 \times 10^6 e^{-1.16 \times 10^6 t} (\sin 3.8 \times 10^6 t) \quad (\text{S3.20})$$

This impulse response is sketched in Figure S3.5:

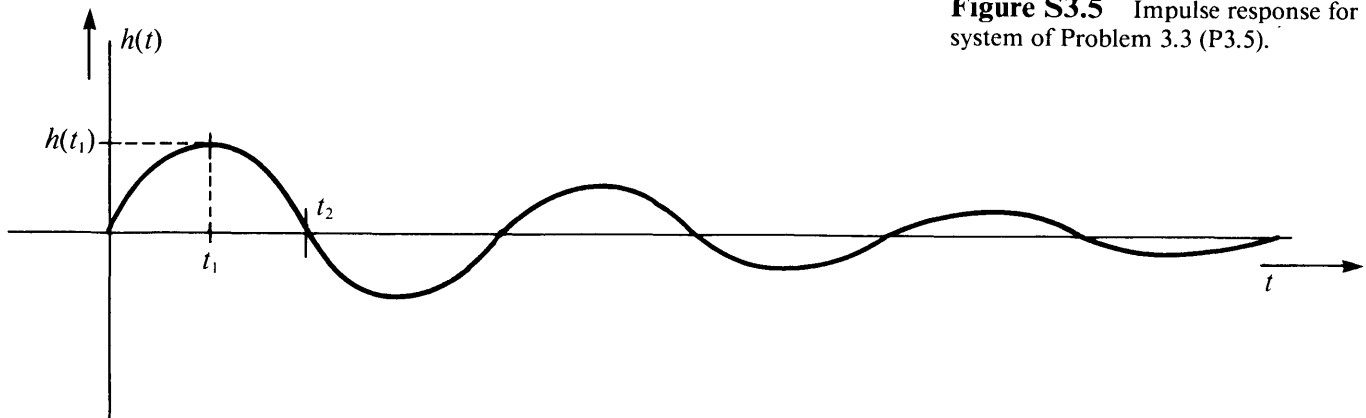


Figure S3.5 Impulse response for system of Problem 3.3 (P3.5).

The waveform is zero at $t = 0$, peaks at $t = t_1$, and is zero again at $t = t_2$. First we solve for t_2 . The waveform crosses zero at $t = t_2$ when the argument of the sine term in Equation S3.20 is equal to π . That is, $3.8 \times 10^6 t_2 = \pi$, which gives

$$t_2 = 0.82 \text{ } \mu\text{sec} \quad (\text{S3.21})$$

Because the system is lightly damped, the impulse response rings at a frequency of about ω_n , and thus the impulse response will peak approximately at the point where the argument of the sine term in Equation S3.20 is equal to $\frac{\pi}{2}$. Thus, $t_1 \simeq \frac{t_2}{2} = 0.41 \text{ } \mu\text{sec}$. Substituting into Equation S3.20 yields the peak value $h(t_1)$. That is,

$$\begin{aligned} h(t_1) &= 4.2 \times 10^6 e^{-(1.16 \times 10^6)(4.1 \times 10^{-7})} \sin \frac{\pi}{2} \\ &= 2.6 \times 10^6 \end{aligned} \quad (\text{S3.22})$$

We see that for a system with large ω_n , the impulse response has a large peak amplitude.

Using Equation 3.62 from the textbook, and the value of θ derived earlier, we find that

$$M_p = \frac{1}{\sin 2\theta} = \frac{1}{\sin(2 \times 1.27)} = 1.77 \quad (\text{S3.23})$$

Using Equation 3.64, we find:

$$\omega_h = \omega_n(1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4})^{1/2} \quad (\text{S3.24})$$

Substituting in $\zeta = 0.29$ yields

$$\begin{aligned} \omega_h &= 4 \times 10^6(1 - 0.17 + \sqrt{2 - 0.34 + 0.03})^{1/2} \\ &= 5.8 \times 10^6 \text{ rad/sec} \end{aligned} \quad (\text{S3.25})$$

Then, because $\omega = 2\pi f$, we have

$$f_h = \frac{\omega_h}{2\pi} = 9.2 \times 10^5 \text{ Hz} \quad (\text{S3.26})$$

First, we will find the frequency, ω_h , at which the transfer function magnitude is 0.707 of its midband value. That is, let $s = j\omega_h$. Then define ω_h by

$$\left| \frac{A_o}{(10^{-9}j\omega_h + 1)^5} \right| = 0.707 \times A_o \quad (\text{S3.27})$$

Taking the magnitude of the left-hand side gives

$$\left[\frac{1}{((10^{-9}\omega_h)^2 + 1)^5} \right]^{1/2} = 0.707 \quad (\text{S3.28})$$

Now, square both sides to find

$$\frac{1}{((10^{-9}\omega_h)^2 + 1)^5} = \frac{1}{2} \quad (\text{S3.29})$$

Then solve for ω_h by writing

$$(10^{-9}\omega_h)^2 + 1 = 2^{1/5} \quad (\text{S3.30})$$

or equivalently

$$\omega_h = \frac{\sqrt{2^{1/5} - 1}}{10^{-9}} = 3.86 \times 10^8 \text{ rad/sec} \quad (\text{S3.31})$$

Then, using the approximations of Equation 3.51 of the textbook, we can solve for the rise time as

$$t_r \simeq \frac{2.2}{\omega_h} = \frac{2.2}{3.86 \times 10^8} = 5.7 \text{ ns} \quad (\text{S3.32})$$

This rise time is typical for moderate bandwidth (50 MHz) oscilloscopes.

Solution 3.4 (P3.7)

To solve this problem we need an expression for the closed-loop transfer function from V_i to V_o . Because the open-loop amplifier response is second order (single poles at $\omega = 10$ and $\omega = 10^6$), the closed-loop system will be second order, and we can use the results of Section 3.3 of the textbook to determine the closed-loop damping, peak overshoot, and rise time.

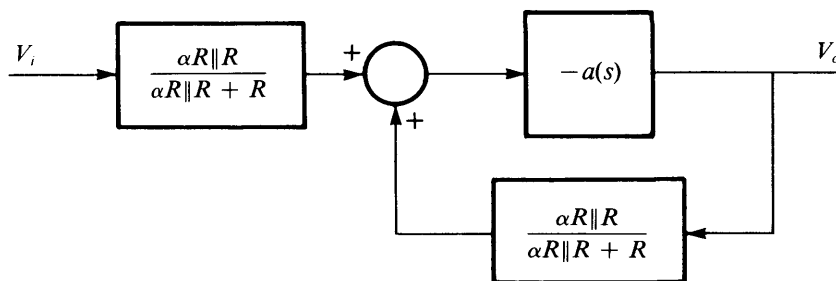
Solution 3.5 (P3.8)

The output V_o is given by

$$V_o = -a(s) \left(V_i \frac{\alpha R \parallel R}{\alpha R \parallel R + R} + V_o \frac{\alpha R \parallel R}{\alpha R \parallel R + R} \right) \quad (\text{S3.33})$$

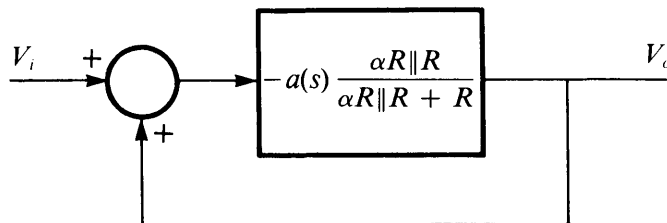
The equivalent block diagram for this expression is as shown in Figure S3.6.

Figure S3.6 Block diagram representation of Equation S3.33.



At this point, it is easier to manipulate the block diagram than to work through the algebra. An equivalent block diagram is given in Figure S3.7.

Figure S3.7 Manipulated block diagram for Problem 3.5 (P3.8)



A few steps of algebra reduce the expression $\frac{\alpha R \parallel R}{\alpha R \parallel R + R}$ to a simpler form:

$$\begin{aligned} \frac{\alpha R \parallel R}{\alpha R \parallel R + R} &= \frac{\frac{\alpha R^2}{\alpha R + R}}{\frac{\alpha R^2}{\alpha R + R} + R} = \frac{\alpha R}{\alpha R + R(\alpha + 1)} \\ &= \frac{\alpha}{2\alpha + 1} \end{aligned} \quad (\text{S3.34})$$

Substituting this result back into the block diagram, and pushing the minus sign back through the summing junction gives the familiar unity-feedback block diagram of Figure S3.8.

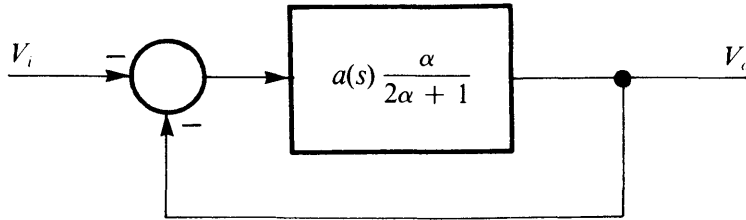


Figure S3.8 Fully reduced block diagram for Problem 3.5 (P3.8).

This has a closed-loop response given by

$$\frac{V_o}{V_i}(s) = - \frac{a(s) \frac{\alpha}{2\alpha + 1}}{1 + a(s) \frac{\alpha}{2\alpha + 1}} \quad (\text{S3.35})$$

By inspection of the Bode plot of Figure 3.24a, we conclude that $a(s)$ is given by

$$a(s) = \frac{10^6}{(0.1s + 1)(10^{-6}s + 1)} \quad (\text{S3.36})$$

Substituting this back into Equation S3.35 gives

$$\begin{aligned} \frac{V_o}{V_i}(s) &= - \frac{\frac{10^6}{(0.1s + 1)(10^{-6}s + 1)} \times \frac{\alpha}{2\alpha + 1}}{1 + \frac{10^6}{(0.1s + 1)(10^{-6}s + 1)} \times \frac{\alpha}{2\alpha + 1}} \quad (\text{S3.37}) \\ &= \frac{-1}{10^{-6} \left(2 + \frac{1}{\alpha} \right) (0.1s + 1)(10^{-6}s + 1) + 1} \end{aligned}$$

Collecting terms gives:

$$\frac{V_o}{V_i}(s) \simeq \frac{-1}{\left(2 + \frac{1}{\alpha} \right) 10^{-13} s^2 + \left(2 + \frac{1}{\alpha} \right) 10^{-7} s + 1} \quad (\text{S3.38})$$

The approximation sign is used because we have dropped a negligibly small term in the coefficient of s . Following the standard form for a second-order system from Equation 3.40 of the textbook we see that

$$\frac{1}{\omega_n^2} = \left(2 + \frac{1}{\alpha}\right)10^{-13} \quad (\text{S3.40})$$

and

$$\frac{2\zeta}{\omega_n} = \left(2 + \frac{1}{\alpha}\right)10^{-7} \quad (\text{S3.41})$$

Equation S3.40 gives

$$\frac{1}{\omega_n} = \sqrt{2 + 1/\alpha} \times 10^{-6.5} \quad (\text{S3.42})$$

Substituting this into Equation S3.41 yields

$$2\zeta = \frac{(2 + 1/\alpha)10^{-7}}{\sqrt{2 + 1/\alpha} 10^{-6.5}} = \sqrt{2 + 1/\alpha} \times 10^{-0.5} \quad (\text{S3.43})$$

Thus,

$$\zeta = \sqrt{2 + 1/\alpha} \times 0.158 \quad (\text{S3.44})$$

We can see then that lower values of α result in larger values of ζ , and consequently more heavily damped responses.

Using Equation 3.58 from the textbook, we can solve for the value of ζ required for 20% overshoot in the step response. That is, we set $P_o = 1.20$, and solve for ζ .

$$P_o = 1.20 = 1 + \exp \frac{-\pi\zeta}{\sqrt{1 - \zeta^2}} \quad (\text{S3.45})$$

Thus,

$$\exp \frac{-\pi\zeta}{\sqrt{1 - \zeta^2}} = 0.2 \quad (\text{S3.46})$$

Take the log of both sides.

$$\frac{-\pi\zeta}{\sqrt{1 - \zeta^2}} = \ln 0.2 \quad (\text{S3.47})$$

Square both sides.

$$\frac{\pi^2\zeta^2}{1 - \zeta^2} = (\ln 0.2)^2 \quad (\text{S3.48})$$

Then

$$\pi^2\zeta^2 = (\ln 0.2)^2(1 - \zeta^2) \quad (\text{S3.49})$$

or

$$\zeta^2[\pi^2 + (\ln 0.2)^2] = (\ln 0.2)^2 \quad (\text{S3.50})$$

Finally, choosing the solution for ζ with $\zeta > 0$, we have:

$$\zeta = \frac{-\ln 0.2}{\sqrt{\pi^2 + (\ln 0.2)^2}} = 0.46 \quad (\text{S3.51})$$

Then, we use Equation S3.44 to find the value of α that will give $\zeta = 0.46$.

$$0.46 = \sqrt{2 + 1/\alpha} \times 0.158 \quad (\text{S3.52})$$

or

$$2 + 1/\alpha = 8.33 \quad (\text{S3.53})$$

which requires that

$$\alpha = 0.16 \quad (\text{S3.54})$$

To find the rise time, we need ω_n . From Equation S3.42,

$$\omega_n = (2 + 1/\alpha)^{-1/2} 10^{6.5} \quad (\text{S3.55})$$

Substituting in $\alpha = 0.16$ gives

$$\omega_n = 1.1 \times 10^6 \text{ rad/sec} \quad (\text{S3.56})$$

Then, we use Equation 3.64 from the textbook to find ω_h .

$$\begin{aligned} \omega_h &= \omega_n (1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4})^{1/2} \\ &= 1.1 \times 10^6 (1 - 0.42 + \sqrt{2 - 0.85 + 0.18})^{1/2} \quad (\text{S3.57}) \\ &= 1.45 \times 10^6 \text{ rad/sec} \end{aligned}$$

Then using the by-now familiar approximation from Equation 3.57 of the textbook we find the rise time.

$$t_r \simeq \frac{2.2}{\omega_h} = \frac{2.2}{1.45 \times 10^6} = 1.52 \mu\text{sec} \quad (\text{S3.58})$$

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