

---

THE DISCRETE FOURIER TRANSFORM

Solution 9.1

---

(a)

$$X(k) = \left[ \sum_{n=0}^{N-1} x(n) W_N^{kn} \right] R_N(k)$$

$$= R_N(k)$$

(b)

$$X(k) = W_N^{kn_0} R_N(k)$$

(c)

$$X(k) = \left[ \sum_{n=0}^{N-1} a^n W_N^{kn} \right] R_N(k)$$

$$= \left[ \frac{1-a^N W_N^{kN}}{1-a W_N^k} \right] R_N(k)$$

$$= \left[ \frac{1-a^N}{1-a W_N^k} \right] R_N(k)$$

Solution 9.2

---

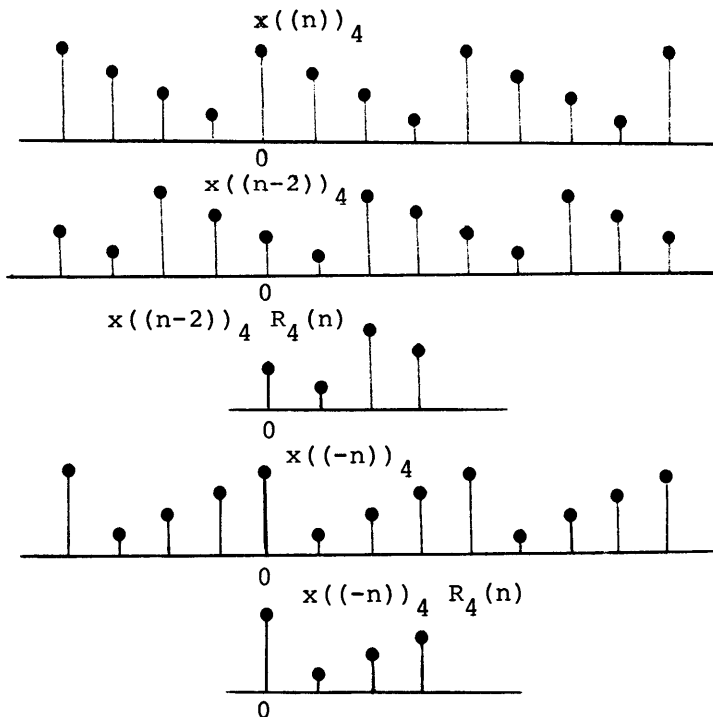


Figure S9.2-1

---

---

Solution 9.3

---

Since  $X_I(k) = -X_I((N-k)_N) R_N(k)$

$X_I(0) = -X_I((N)_N) R_N(k) = -X_I(0)$

Therefore  $X_I(0) = 0$

Also, for  $N$  even,

$$\begin{aligned} X_I\left(\frac{N}{2}\right) &= -X_I\left(\left(N - \frac{N}{2}\right)_N\right) R_N(k) \\ &= -X_I\left(\frac{N}{2}\right) \end{aligned}$$

Therefore  $X_I\left(\frac{N}{2}\right) = 0$ .

---

Solution 9.4

---

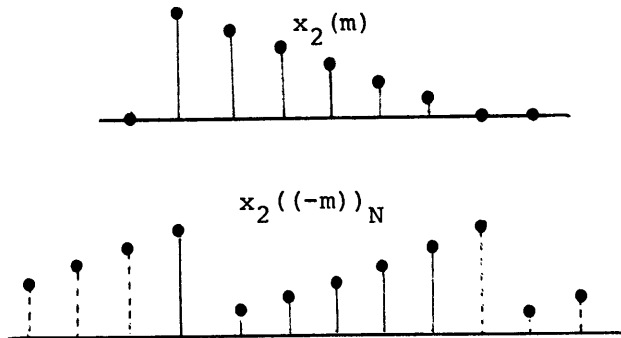


Figure S9.4-1

---

Solution 9.5

---

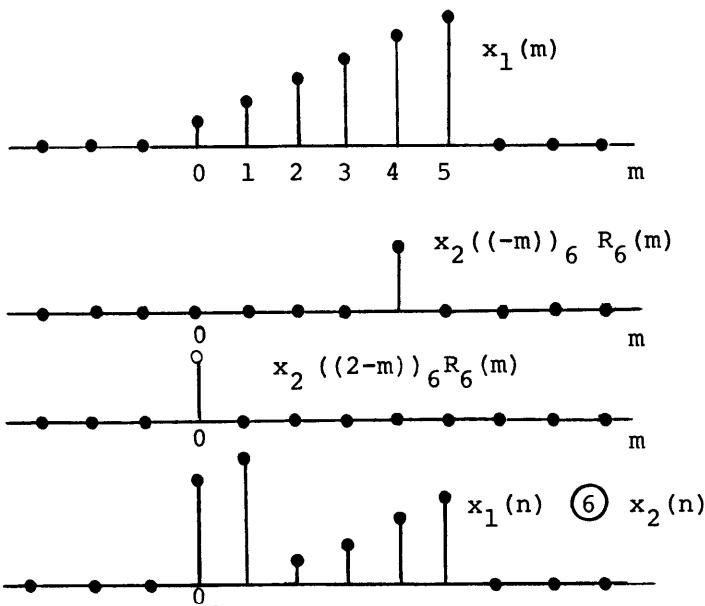


Figure S9.5-1

---

---

Note that this corresponds to  $x_1(n)$  circularly shifted to the right by two points.

Solution 9.6

---

We wish to compute  $X_1(k)$  given by

$$X_1(k) = \left\{ \sum_{n=0}^9 x(n) z_k^{-n} \right\} R_{10}(k)$$

$$\text{where } z_k = 0.5 e^{\frac{j2\pi k}{10}} e^{j\frac{\pi}{10}}$$

so that

$$\begin{aligned} X_1(k) &= \left\{ \sum_{n=0}^9 x(n) \left[ \frac{1}{2} e^{\frac{j2\pi k}{10}} e^{\frac{j\pi}{10}} \right]^{-n} \right\} R_{10}(k) \\ &= \left\{ \sum_{n=0}^9 x(n) \left[ \frac{1}{2} e^{\frac{j\pi}{10}} \right]^{-n} e^{-\frac{j2\pi kn}{10}} \right\} R_{10}(k) \end{aligned}$$

Thus  $X_1(k)$  is the 10-point DFT of the sequence

$$x_1(n) = x(n) \left[ \frac{1}{2} e^{\frac{j\pi}{10}} \right]^{-n}$$

Solution 9.7

---

In all of the following equations the DFT computed is valid only in the range  $0 \leq k \leq N-1$  and is zero outside that range. This permits us to keep the equations somewhat cleaner by suppressing the use of the function  $R_N(k)$ .

$$\begin{aligned} G_1(k) &= \sum_{n=0}^{N-1} x(N-1-n) W_N^{kn} \\ &= \sum_{m=0}^{N-1} x(m) W_N^{k(N-1-m)} \\ &= \sum_{m=0}^{N-1} x(m) e^{\frac{j2\pi k}{N}} e^{\frac{j2\pi}{N} mk} \\ &= e^{j\frac{2\pi k}{N}} X(e^{-j\frac{2\pi}{N} k}) = H_7(k) \end{aligned}$$

$$G_2(k) = \sum_{n=0}^{N-1} (-1)^n x(n) W_N^{kn} = \sum_{n=0}^{N-1} x(n) W_N^{\frac{Nn}{2}} W_N^{kn}$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}(k+\frac{N}{2})n} = X(e^{j\frac{2\pi}{N}(k+\frac{N}{2})}),$$

$$= H_8(k)$$

$$\begin{aligned} G_3(k) &= \sum_{n=0}^{N-1} x(n) W_{2N}^{nk} + \sum_{n=N}^{2N-1} x(n-N) W_{2N}^{nk} \\ &= \sum_{n=0}^{N-1} x(n) \left[ W_{2N}^{nk} + W_{2N}^{(n+N)k} \right] \\ &= \sum_{n=0}^{N-1} x(n) W_{2N}^{nk} \left[ 1 + W_{2N}^{Nk} \right] \\ &= \left[ 1 + (-1)^k \right] X(e^{j\frac{2\pi}{2N}k}) = H_3(k) \end{aligned}$$

$$\begin{aligned} G_4(k) &= \sum_{n=0}^{\frac{N}{2}-1} \left( x(n) + x(n + \frac{N}{2}) \right) W_{\frac{N}{2}}^{nk} \\ &= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_{\frac{N}{2}}^{nk} + \sum_{n=\frac{N}{2}}^{N-1} x(n) W_{\frac{N}{2}}^{(n-\frac{N}{2})k} \\ &= \sum_{n=0}^{N-1} x(n) W_{\frac{N}{2}}^{nk} = X(e^{j\frac{4\pi}{N}k}) = H_6(k) \end{aligned}$$

$$G_5(k) = \sum_{n=0}^{2N-1} x(n) W_{2N}^{nk} = X(e^{j\frac{\pi k}{N}}) = H_2(k)$$

$$G_6(k) = \sum_{n=0}^{N-1} x(n) W_{2N}^{2nk} = X(e^{j\frac{2\pi k}{N}}) = H_1(k)$$

$$\begin{aligned} G_7(k) &= \sum_{n=0}^{\frac{N}{2}-1} x(2n) W_{\frac{N}{2}}^{nk} \\ &= \sum_{n=0}^{N-1} x(n) \left[ \frac{1+(-1)^n}{2} \right] W_{\frac{N}{2}}^{nk/2} \\ &= \frac{1}{2} \sum_{n=0}^{N-1} x(n) \left[ W_N^{nk} + W_N^{n(k+N/2)} \right] \end{aligned}$$

---

$$= \frac{1}{2} \left[ X \left( e^{j \frac{2\pi k}{N}} \right) + X \left( e^{j \frac{2\pi}{N} (k+N/2)} \right) \right] = H_5(k)$$

All of the above properties can alternatively be obtained from the basic DFT properties of sections 8.7 and 8.8, or the z-transform properties of section 4.4. Many of the properties used in this problem have important practical applications.  $g_5(n)$ , for example, corresponds to augmenting a finite length sequence with zeros so that a computation of the DFT for this augmented sequence provides finer spectral sampling of the Fourier transform.

MIT OpenCourseWare  
<http://ocw.mit.edu>

Resource: Digital Signal Processing  
Prof. Alan V. Oppenheim

The following may not correspond to a particular course on MIT OpenCourseWare, but has been provided by the author as an individual learning resource.

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.