DESIGN OF IIR DIGITAL FILTERS - PART 2

1. Lesson 15 - 41 minutes

Differentials → Differences for $z = e^{j\omega}$ $e^{-j\frac{\omega}{2}}\left[e^{+j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right]$

$s \to \dfrac{z+1}{T}$

$s = \dfrac{2}{T}\left[\dfrac{1-e^{-j\omega}}{1+e^{-j\omega}}\right]$

Impulse Invariance

$\displaystyle\sum_{k=1}^{N}\dfrac{A_k}{s-s_k} \to \sum_{k=1}^{N}\dfrac{A_k}{1-e^{s_k T}z^{-1}}$

$= \dfrac{2}{T}j\tan\dfrac{\omega}{2} = j\Omega$

Bilinear Transformation

$\Omega = \dfrac{2}{T}\tan\dfrac{\omega}{2}$

$H_a(s) \Rightarrow H(z)$

$j\Omega$ axis $\Longleftrightarrow$ unit circle

$s = \dfrac{2}{T}\left[\dfrac{1-z^{-1}}{1+z^{-1}}\right]$

$z = \dfrac{1+\frac{sT}{2}}{1-\frac{sT}{2}}$

s-plane

z-plane

a.

Algorithmic Design (IIR Filters)

$H(z) = A\displaystyle\prod_{k=1}^{K}\dfrac{1+a_k z^{-1}+b_k z^{-2}}{1+c_k z^{-1}+d_k z^{-2}}$

Least Squares Inverse Design

① Minization of mean square error

choose parameters of $H(z)$ to minimize E

Specify $h_d(n)$

$H_d(e^{j\omega})$ = Desired Frequency Responce

$H(z) = \dfrac{b_o}{1-\displaystyle\sum_{k=1}^{N}a_k z^{-k}} \longleftrightarrow h(n)$

$H_d(e^{j\omega})$ $L = 1,2,\dots,M$

$h_d(n) \to \boxed{\dfrac{1}{H(z)}} \to g(n)$

Error =

$\displaystyle\sum_{L=1}^{M}\left[|H(e^{j\omega_L})| - |H_d(e^{j\omega_L})|\right]^2$

$E = \displaystyle\sum_{n=0}^{\infty}\left[g(n) - \delta(n)\right]^2$

$\Rightarrow$ Linear Equations

b.

Example of impulse invariance.

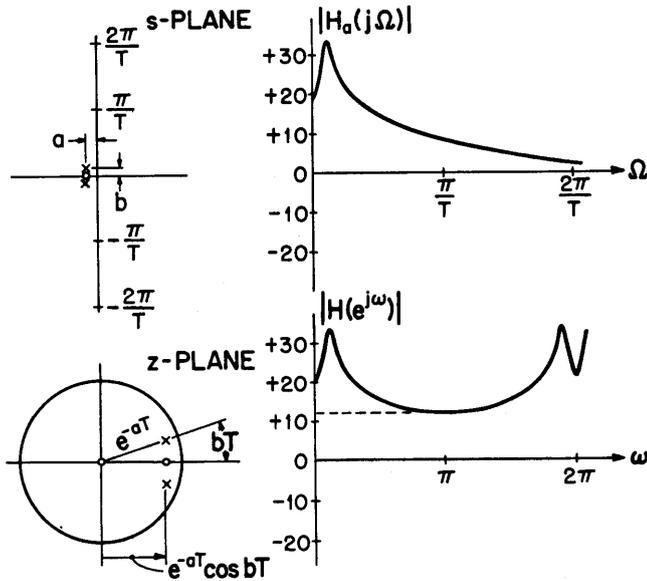$H_a(S) = \dfrac{(S+a)}{(S+a)^2+b^2} = \dfrac{1/2}{S+a+jb} + \dfrac{1/2}{S+a-jb}$

$H(z) = \dfrac{1/2}{1-e^{-aT}e^{-jbT}z^{-1}} + \dfrac{1/2}{1-e^{-aT}e^{jbT}z^{-1}}$

$= \dfrac{1-(e^{-aT}\cos bT)z^{-1}}{(1-e^{-aT}e^{-jbT}z^{-1})(1-e^{-aT}e^{jbT}z^{-1})}$

c.

s-PLANE

$\frac{2\pi}{T}$

$\frac{\pi}{T}$

a →

b

$-\frac{\pi}{T}$

$-\frac{2\pi}{T}$

$|H_a(j\Omega)|$

+30
+20
+10
0
-10
-20

$\frac{\pi}{T}$ $\frac{2\pi}{T}$ Ω

z-PLANE

e^{-aT} bT

$e^{-aT}\cos bT$

$|H(e^{j\omega})|$

+30
+20
+10
0
-10
-20

π 2π ω

d.

Pole-zero patterns
and frequency re-
sponse corresponding
to the example of
viewgraph a.

ω ω

π π

ω_S

ω_P

$\omega = 2\arctan\left(\frac{\Omega T}{2}\right)$

Ω

$|H(e^{j\omega})|$

$|H_a(j\Omega)|$ $\Omega_P = \frac{2}{T}\tan\left(\frac{\omega_P}{2}\right)$

$\Omega_S = \frac{2}{T}\tan\left(\frac{\omega_S}{2}\right)$

$\Omega_P \Omega_S$ Ω

e.
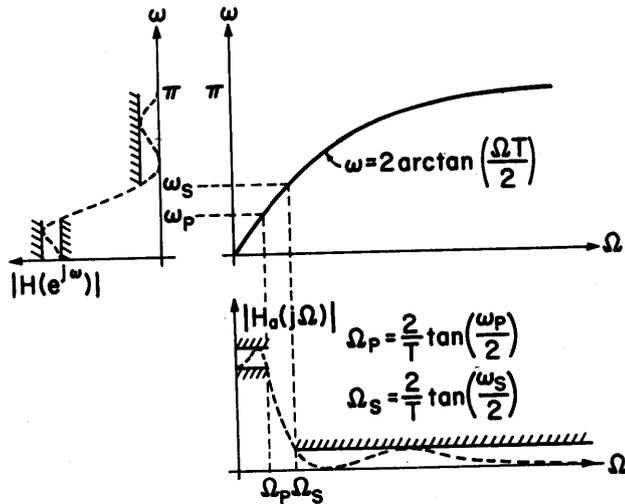
Illustration of effect
of frequency warping
inherent in the bi-
linear transformation.

ω ω

π π

$\omega = 2\arctan\left(\frac{\Omega T}{2}\right)$

Ω

$|H(e^{j\omega})|$

$|H_a(j\Omega)|$

Ω

f.
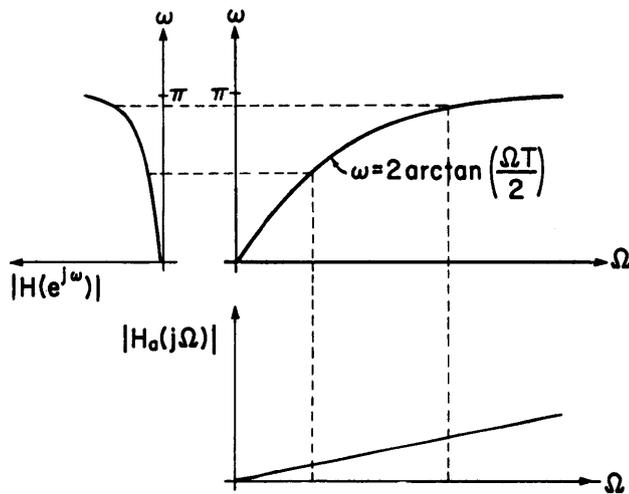
Illustration of effect
of bilinear trans-
formation on a piece-
wise constant
frequency response
characteristic.

$$\omega = 2\arctan\left(\frac{\Omega T}{2}\right)$$

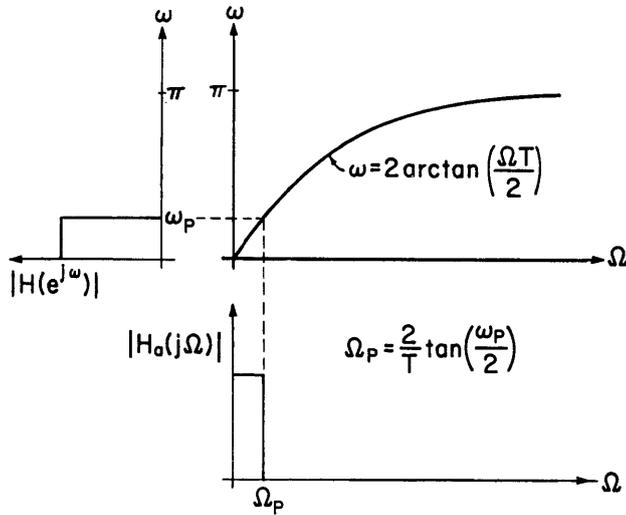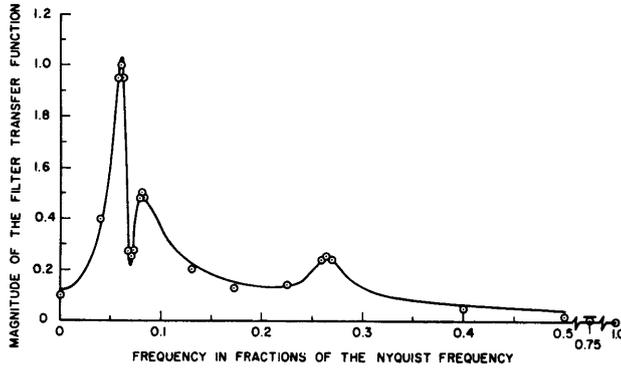$$\Omega_P = \frac{2}{T}\tan\left(\frac{\omega_P}{2}\right)$$

Illustration of effect of bilinear transformation on an equiripple frequency response characteristic.

g.

Example of frequency response obtained for an IIR filter designed by minimization of mean-square error.

h.

2. Correction

The first equation on board number 1 indicates that the mapping corresponding to replacing differentials by differences is $s \to \frac{z+1}{T}$. This should be corrected to $s \to \frac{z-1}{T}$.

3. Comments

In the previous lecture we introduced two techniques for transforming analog filter designs to digital filter designs. The first of these, corresponding to transforming differentials to differences is not a very useful technique for the reasons discussed. The second, that of

impulse invariance is a useful technique, although it introduces aliasing which must be accounted for.

In this lecture we begin with an illustration of impulse invariance. An important observation in this example is that the zeros of the analog transfer function don't map to the z-plane in the same way that the poles do. As we stressed in lecture 14, the impulse invariant design procedure corresponds to a mapping of the poles and residues rather than a mapping of the poles and zeros. We also observe in this example the effect of aliasing.

The second method for IIR filter design discussed in this lecture is the use of the bilinear transformation. This transformation has the advantage that it maps the entire jΩ axis in the s-plane into one revolution around the unit circle in the z-plane. Aliasing is therefore eliminated but in contrast to impulse invariance a nonlinear distortion in the frequency axis is introduced. This nonlinear distortion can be tolerated when the filter to be designed is piece-wise constant but usually cannot be for more general filter characteristics. In the next lecture a number of these points will be illustrated through a design example.

Impulse invariance and the bilinear transformation represent the two major analytical techniques for designing digital filters through the transformation of analog designs. There are also a number of algorithmic design procedures for digital filters. We conclude this lecture with a brief introduction to two, one of which requires the solution of a set of nonlinear equations and the second of which leads to a set of linear equations.

4. Reading

Text: Review section 7.1.1 (page 407) to example 7.3 and read section 7.1.2. (page 415) to example 7.4. Also read section 7.3 (page 438).

5. Problems

Problem 15.1

In Figure P15.1-1 is shown the frequency response of a digital filter. (a) Determine and sketch the analog frequency response characteristics which, without aliasing will map to this digital frequency response when the impulse invariant transformation is applied.

(b) Sketch the analog frequency response that will map to this
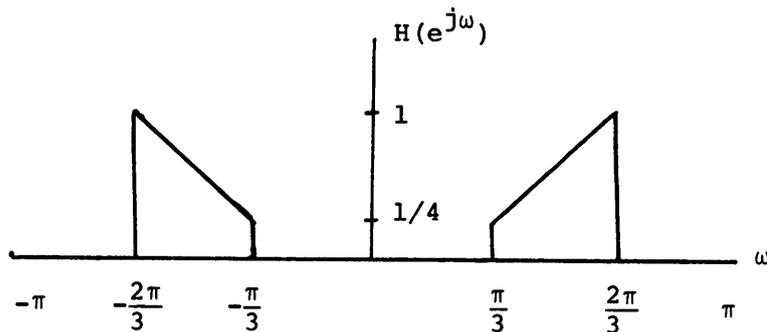digital frequency response when the bilinear transformation is applied.

Figure P15.1-1

Problem 15.2

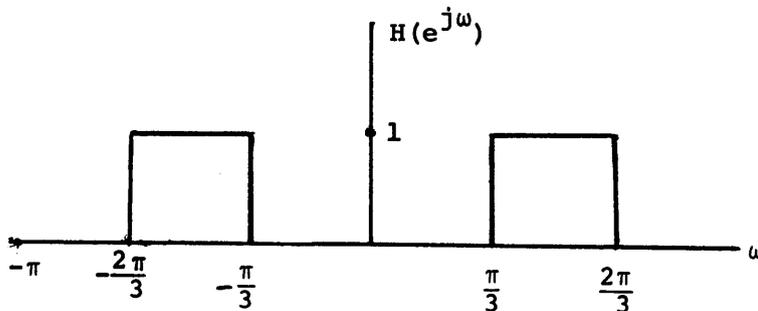Repeat problem 15.1 for the digital frequency response characteristic
in Figure P15.2-1.

Figure P15.2-1

Problem 15.3

A digital lowpass filter is to be designed using the bilinear trans-
formation. The specifications are that:

(i) $(.99) < |H(e^{j\omega})| \leq 1$ for $0 \leq \omega < \frac{\pi}{8}$

(ii) $|H(e^{j\omega})| < .001$ for $\frac{\pi}{6} \leq \omega < \pi$.

Determine the specifications on an analog lowpass filter which will
yield the desired digital filter when the bilinear transformation is
applied with T = 1.

Problem 15.4[*]

An analog highpass filter can be obtained from an analog lowpass
filter by replacing s by $1/s$ in the transfer function: i.e., if
$G_a(s)$ is the transfer function for a lowpass filter, then $H_a(s)$ is
the transfer function for a highpass filter if

$$H_a(s) = G_a\left(\frac{1}{s}\right) .$$

Also, a digital filter can be obtained by mapping an analog filter by
means of the bilinear transformation

$s = \frac{z-1}{z+1}$. [Assume for convenience that $T = 2$ in Eq. (5.22) of the text.]
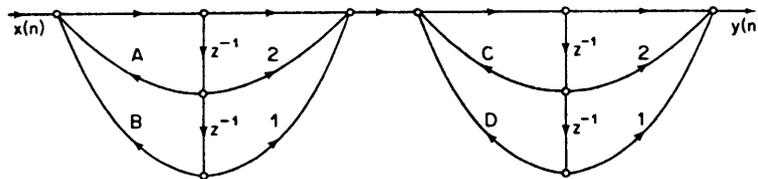

Figure P15.4-1

This mapping preserves the character of the magnitude characteristic,
although the frequency scale is distorted. The network in Figure
P 15.4-1 depicts a lowpass filter with cutoff frequency $\omega_L = \pi/2$.

The constants A,B,C, and D are real. Determine how to modify the
coefficients in the network to obtain a highpass filter with cutoff
frequency $\omega_H = \pi/2$.

<u>Problem 15.5</u>[*]

We wish to approximate a desired unit-sample response using the least-
squares inverse design procedure as outlined in section 5.3.3. of
text. The desired unit sample response $h_d(n)$ is given by

$$h_d(n) = \begin{cases} 1 & 0 \le n < 10 \\ \\ 0 & \text{otherwise} \end{cases}$$

15.6

corresponding to a 9th order FIR filter. This is to be approximated by a second order IIR filter of the form

$$H(z) = \frac{b_0}{1-a_1 z^{-1}-a_2 z^{-2}} \; .$$

Determine the coefficients b_0, a_1 and a_2.

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Resource: Digital Signal Processing
Prof. Alan V. Oppenheim

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