DISCRETE-TIME SIGNALS AND SYSTEMS, PART 2
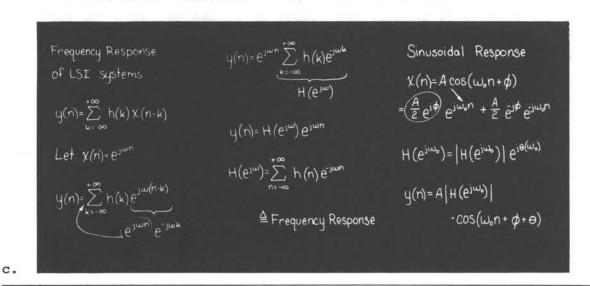
1. Lecture 3 - 50 minutes
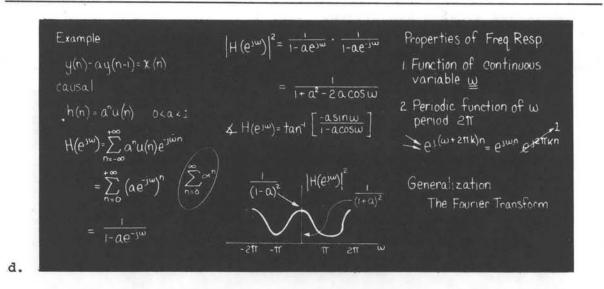
a.

$x(n) \rightarrow T[\ ] \rightarrow y(n)$

general

$$y(n) = T[x(n)]$$

LSI

$$y(n) = \sum_{k=-\infty}^{+\infty} x(k)h(n-k)$$

$$= \sum_{k=-\infty}^{+\infty} h(k)x(n-k)$$

Convolution Sum

Stability

general: If $x(n)$ bounded
ie $|x(n)| < \infty$ all n

then $y(n)$ bounded
ie $|y(n)| < \infty$ all n

LSI

$$\sum_{k=-\infty}^{+\infty} |h(k)| < \infty$$

$h(n) = 2^n\, u(n)$ — unstable
$h(n) = (\tfrac{1}{2})^n\, u(n)$ — stable

Causality

$y(n)$ for $n=n_1$ depends on $x(n)$ only for $n \le n_1$

LSI
$$h(n) = 0 \quad n < 0$$

$h(n) = (2)^n\, u(-n)$
noncausal
stable

b.

Linear Constant Coefficient Difference Equation

N^{th} order

$$\sum_{k=0}^{N} a_k\, y(n-k) = \sum_{r=0}^{M} b_r\, x(n-r)$$

$N=0 \quad a_0 = 1$

$$y(n) = \sum_{r=0}^{M} b_r\, x(n-r)$$

$h(n) = b_n \quad n = 0,1,\dots,M$
$= 0$ otherwise

$N \neq 0 \quad a_0 = 1$

$$y(n) = \sum_{r=0}^{M} b_r\, x(n-r) - \sum_{k=1}^{N} a_k\, y(n-k)$$

First-order

$$y(n) + a y(n-1) = x(n)$$
$$x(n) = \delta(n)$$

assume $y(n) = 0 \quad n < 0$

$$y(n) = \delta(n) + a y(n-1)$$

$\left.\begin{array}{l} y(-1) = 0 \\ y(0) = 1 \\ y(1) = a \\ y(2) = a^2 \end{array}\right\}$ $a^n u(n)$
$|a| < 1$
Stable

$x(n) = \delta(n)$

assume $y(n) = 0 \quad n > 0$

$$y(n-1) = a^{-1}\left[y(n) - \delta(n) \right]$$

$\left.\begin{array}{l} y(1) = 0 \\ y(0) = 0 \\ y(-1) = -a^{-1} \\ y(-2) = -a^{-2} \end{array}\right\}$ $-a^n u(-n-1)$
$|a| < 1$
unstable

c.

Frequency Response of LSI systems

$$y(n) = \sum_{k=-\infty}^{+\infty} h(k)x(n-k)$$

Let $x(n) = e^{jwn}$

$$y(n) = \sum_{k=-\infty}^{+\infty} h(k)\, e^{jw(n-k)}$$
$$e^{jwn} e^{-jwk}$$

$$y(n) = e^{jwn} \underbrace{\sum_{k=-\infty}^{+\infty} h(k)\, e^{-jwk}}_{H(e^{jw})}$$

$$y(n) = H(e^{jw})\, e^{jwn}$$

$$H(e^{jw}) = \sum_{n=-\infty}^{+\infty} h(n)\, e^{-jwn}$$

$\triangleq$ Frequency Response

Sinusoidal Response

$$x(n) = A\cos(w_0 n + \phi)$$

$$= \left(\frac{A}{2} e^{j\phi}\right) e^{jw_0 n} + \frac{A}{2} e^{-j\phi} e^{-jw_0 n}$$

$$H(e^{jw_0}) = |H(e^{jw_0})|\, e^{j\theta(w_0)}$$

$$y(n) = A|H(e^{jw_0})| \cdot \cos(w_0 n + \phi + \theta)$$

Example

$y(n) - ay(n-1) = x(n)$

causal

$h(n) = a^n u(n) \quad 0 < a < 1$

$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^n u(n) e^{-j\omega n}$

$= \sum_{n=0}^{+\infty} (ae^{-j\omega})^n \quad \left(\sum_{n=0}^{\infty} \alpha^n \right)$

$= \dfrac{1}{1-ae^{-j\omega}}$

$|H(e^{j\omega})|^2 = \dfrac{1}{1-ae^{j\omega}} \cdot \dfrac{1}{1-ae^{-j\omega}}$

$= \dfrac{1}{1+a^2-2a\cos\omega}$

$\angle H(e^{j\omega}) = \tan^{-1}\left[\dfrac{-a\sin\omega}{1-a\cos\omega} \right]$

Properties of Freq Resp

1. Function of continuous variable $\underline{\omega}$

2. Periodic function of ω period 2π

$\sum e^{j(\omega+2\pi k)n} = e^{j\omega n} e^{j2\pi k n}$

Generalization
The Fourier Transform

d.

2. Comments

This lecture continues the discussion of discrete-time systems. We consider, in particular, the definitions for stability and causality, first for discrete-time systems in general, and then for the particular class of linear shift-invariant systems. We also introduce in this lecture the class of linear shift-invariant systems characterized by linear constant-coefficient difference equations.

As the final topic in this lecture we discuss the notion of the frequency response of LSI systems. The description of LSI systems in terms of the frequency response is based on the fact that complex exponentials are eigenfunctions of this class of systems, i.e. the output due to a complex exponential input is of the same form as the input with only a change in the complex amplitude. This implies that for sinusoidal inputs, the output is sinusoidal with the same frequency but with a change in amplitude and phase. Just as with continuous-time systems, the frequency domain representation of discrete-time LSI systems is one of the cornerstones of digital signal processing.

3. Reading

Text: Section 2.4 starting after eq.(2.51) (page 28) sections 2.5 and 2.6.

4. Problems

Problem 3.1

Stability of a system was defined in the lecture and in section 2.2.5 of the text, and it was shown that for an LSI system a necessary and sufficient condition for stability is that $h(n)$ be absolutely summable. Show that each of the unit sample responses listed below corresponds to a stable system

(i) $h(n) = \delta(n + 2)$

(ii) $h(n) = (\frac{1}{2})^n u(n)$

(iii) $h(n) = 2^n u(-n)$

Problem 3.2

Causality of a system was defined in the lecture and in section 1.3 of the text. In this problem we wish to show that for an LSI system a necessary and sufficient condition for causality is that the unit sample response be zero for $n < 0$.

(a) Show from the convolution sum that if $h(n) = 0$ for $n < 0$ then the system must be causal, i.e. that $y(n_0)$ depends only on $x(n)$ for $n \le n_0$ where n_0 is arbitrary. This then shows that a sufficient condition for causality of an LSI system is that $h(n) = 0$, $n < 0$.

(b) Now, suppose that $h(n)$ is <u>not</u> zero for $n < 0$. Argue that the system <u>cannot</u> be causal. This can be done simply by showing that there is at least one input for which the output "anticipates" the input i.e. for which $y(n_0)$ depends on values of $x(n)$ for $n > n_0$. This then establishes that a necessary condition for causality of an LSI system is that $h(n) = 0$, $n < 0$.

Problem 3.3

Determine whether or not each of the following systems is stable and/or causal:

(a) $y(n) = g(n)x(n)$ where $g(n)$ is bounded

(b) $y(n) = \displaystyle\sum_{k=n_0}^{n} x(k)$

(c) $y(n) = x(n - n_0)$

Problem 3.4

Consider a causal system for which the input $x(n)$ and output $y(n)$ are related by the linear constant coefficient difference equation

$$y(n) - \frac{1}{2} y(n - 1) = x(n) + \frac{1}{2} x(n - 1)$$

(a) Determine the unit sample response of the system.

(b) Using your result in (a) and the convolution sum determine the response to the input

$$x(n) = e^{j\omega n}$$

(c) Determine the frequency response of the system;

(d) Determine the response of the system to the input

$$x(n) = \cos(\frac{\pi}{2} n + \frac{\pi}{4})$$

Problem 3.5*

Consider a system with input $x(n)$ and output $y(n)$. The input-output relation for the system is defined by the following two properties:

(1) $y(n) - ay(n - 1) = x(n)$.

(2) $y(0) = 1$.

(a) Determine whether the system is shift-invariant.

(b) Determine whether the system is linear.

Problem 3.6*

Consider a discrete-time system with input $x(n)$ and output $y(n)$. The system transformation $y(n) = F[x(n)]$ is arbitrary and may be nonlinear and time-varying. The only known property of the system is that it is well defined, i.e. the output for any given input is unique. Suppose that the input $x(n)$ is chosen as $x(n) = e^{j\omega n}$ and some parameter P of the output is measured (e.g. the maximum amplitude). P will in general be a function of ω. Let us consider P for different excitation frequencies. Show that P is _periodic in ω_ and determine the period.

Resource: Digital Signal Processing
Prof. Alan V. Oppenheim