Solutions Manual for Electromechanical Dynamics

PROBLEM 8.1

The identity to be verified is

$$\nabla\cdot(\psi\bar{A}) = \psi\nabla\cdot\bar{A} + \bar{A}\cdot\nabla\psi \tag{a}$$

First express the identity in index notation.

$$\frac{\partial}{\partial x_m}[\psi A_m] = \psi\frac{\partial A_m}{\partial x_m} + A_m\frac{\partial\psi}{\partial x_m} \tag{b}$$

The repeated subscript indicates summation. Thus, expanding the first term on the left yields:

$$\psi\frac{\partial A_m}{\partial x_m} + A_m\frac{\partial\psi}{\partial x_m} \equiv \psi\nabla\cdot\bar{A} + \bar{A}\cdot\nabla\psi \tag{c}$$

PROBLEM 8.2

We wish to show that

$$\bar{B}\cdot\nabla(\psi\bar{A}) = \psi\bar{B}\cdot\nabla\bar{A} + \bar{A}\bar{B}\cdot\nabla\psi \tag{a}$$

First, the identity is expressed in index notation, considering the m^{th} component of this vector equation. Note that the equation relates two vectors.

$$(\bar{B}\cdot[\nabla(\psi\bar{A})])_m = (\psi\bar{B}\cdot[\nabla\bar{A}])_m + A_m\bar{B}\cdot\nabla\psi \tag{b}$$

Now, consider each term separately

$$(\bar{B}\cdot[\nabla(\psi\bar{A})])_m = B_k\frac{\partial}{\partial x_k}(\psi A_m) = A_m B_k\frac{\partial\psi}{\partial x_k} + \psi B_k\frac{\partial A_m}{\partial x_k} \tag{c}$$

$$(\psi\bar{B}\cdot[\nabla\bar{A}])_m = \psi B_k\frac{\partial A_m}{\partial x_k} \tag{d}$$

$$A_m\bar{B}\cdot\nabla\psi = A_m B_k[\nabla\psi]_k = A_m B_k\frac{\partial\psi}{\partial x_k} \tag{e}$$

The sum of (d) and (e) give (c) so that the identity is verified.

PROBLEM 8.3

Part a

a_{ik} is the cosine of the angle between the x_i' axis and the x_k axis (see page 435). Thus for our geometry

$$a_{ik} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{a}$$

PROBLEM 8.3 (Continued)

Now, we may apply the transformation law for vectors (Eq. 8.2.10)

$$A_i' = a_{ik}A_k \tag{b}$$

where the components of $\bar{A}$ in the (x_1,x_2,x_3) system are given as

$$A_1 = 1;\ A_2 = 2;\ A_3 = -1 \tag{c}$$

Thus:

$$A_1' = a_{1k}A_k = a_{11}A_1 + a_{12}A_2 + a_{13}A_3 \tag{d}$$

$$A_1' = 1/2 + \sqrt{3} \tag{e}$$

$$A_2' = a_{2k}A_k = -\frac{\sqrt{3}}{2} + 1 \tag{f}$$

$$A_3' = a_{3k}A_k = -1 \tag{g}$$

Using matrix alegbra, we can write a more concise solution. That is:

$$\begin{bmatrix} A_1' \\ A_2' \\ A_3' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} (\frac{1}{2} + \sqrt{3}) \\ (-\frac{\sqrt{3}}{2} + 1) \\ (-1) \end{bmatrix} \tag{h}$$

Part b

The tensor a_{ik} is associated with coordinate transforms involving the direction of force while the tensor $a_{j\ell}$ is associated with coordinate transforms involving the direction of the area normal vectors. The tensor transformation is (Eq. 8.2.17), page 437;

$$T_{ij}' = a_{ik}\, a_{j\ell}T_{k\ell} \tag{i}$$

For example,

$$\begin{aligned} T_{11}' = a_{1k}a_{1\ell}T_{k\ell} = {} & a_{11}a_{11}T_{11} + a_{12}a_{11}T_{21} + a_{13}a_{11}T_{31} \\ & + a_{11}a_{12}T_{12} + a_{12}a_{12}T_{22} + a_{13}a_{12}T_{32} \\ & + a_{11}a_{13}T_{13} + a_{12}a_{13}T_{23} + a_{13}a_{13}T_{33} \end{aligned} \tag{j}$$

Thus:

$$T_{11}' = \frac{7}{4} + \frac{6\sqrt{3}}{4} \tag{k}$$

PROBLEM 8.3 (Continued)

Similarly

$$T'_{12} = -\frac{3}{2} + \frac{\sqrt{3}}{4} \quad (\ell)$$

$$T'_{13} = 0 \quad (m)$$

$$T'_{21} = -\frac{3}{2} + \frac{\sqrt{3}}{4} \quad (n)$$

$$T'_{22} = \frac{5}{4} - \frac{3\sqrt{3}}{2} \quad (o)$$

$$T'_{23} = 0 \quad (p)$$

$$T'_{31} = 0 \quad (q)$$

$$T'_{32} = 0 \quad (r)$$

$$T'_{33} = 1 \quad (s)$$

Written in matrix algebra, the problem is solved below:

$$\begin{bmatrix} T'_{11} & T'_{12} & T'_{13} \\ T'_{21} & T'_{22} & T'_{23} \\ T'_{31} & T'_{32} & T'_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \quad (t)$$

Note that the third matrix on the right is the transpose of a_{ij}. Matrix multiplication of (t) gives

$$T'_{ij} = \begin{bmatrix} (\frac{7}{4} + \frac{6\sqrt{3}}{2}) & (-\frac{3}{2} + \frac{\sqrt{3}}{4}) & 0 \\ (-\frac{3}{2} + \frac{\sqrt{3}}{4}) & (\frac{5}{4} - \frac{3\sqrt{3}}{2}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (u)$$

PROBLEM 8.4

The m^{th} component of the force density at a point is (Eq. 8.1.10)

$$F_i = \frac{\partial T_{ij}}{\partial x_j} \quad (a)$$

Thus in the $\vec{i}_1$ direction,

PROBLEM 8.4 (Continued)

$$F_1 = \left(\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3}\right) = \left(\frac{P_o^2}{a^2}x_1 - \frac{P_o^2}{a^2}x_1 + 0\right) = 0 \qquad \text{(b)}$$

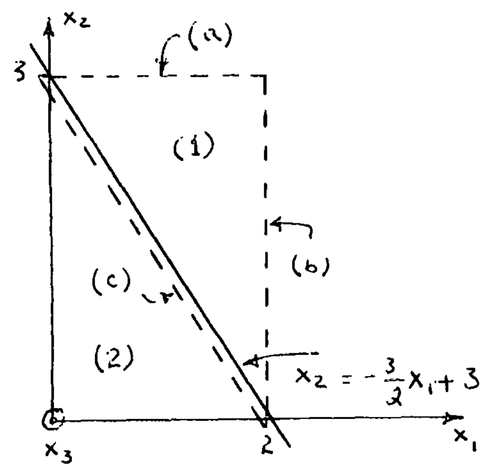
Similarly in the $\vec{i}_2$ and $\vec{i}_3$ directions we find

$$F_2 = \left(\frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3}\right) = 0 \qquad \text{(c)}$$

$$F_3 = \left(\frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3}\right) = 0 \qquad \text{(d)}$$

Hence, the total volume force density resulting from the given stress tensor is zero.

PROBLEM 8.5



in region (1) $\bar{E}=E_o(\frac{3}{2}\vec{i}_1 + \vec{i}_2)$

in region (2) $\bar{E} = 0$

$$T_{ij} = \varepsilon E_i E_j - \frac{\delta_{ij}}{2}\varepsilon_o E_k E_k \qquad \text{(a)}$$

Thus in region (2)

$$T_{ij} = [0] \qquad \text{(b)}$$

in region (1)

PROBLEM 8.5 (Continued)

$$T_{ij} = \begin{bmatrix} \frac{5}{8}\varepsilon_o E_o^2 & \frac{3}{2}\varepsilon_o E_o^2 & 0 \\ \frac{3}{2}\varepsilon_o E_o^2 & -\frac{5}{8}\varepsilon_o E_o^2 & 0 \\ 0 & 0 & -\frac{13}{8}\varepsilon_o E_o^2 \end{bmatrix} \tag{c}$$

The total contribution to the forces found by integrating the stress tensor over surface (c) is zero, because surface (c) lies in region (2) where the stress tensor is zero. By symmetry the sum of contributions to the force resulting from integrations over the two surfaces perpendicular to the x_3 axis is zero.

Now let us note the fact that:

$$\text{area (a)} = 2 \tag{d}$$

$$\text{area (b)} = 3 \tag{e}$$

Thus:

$$f_i = \oint T_{ij}\, n_j\, da \tag{f}$$

$$f_1 = \int_{(b)} T_{11} da + \int_{(a)} T_{12} da + \int T_{13} da$$

$$= \frac{5}{8}\varepsilon_o E_o^2 (3) + \frac{3}{2}\varepsilon_o E_o^2 (2) \tag{g}$$

$$f_1 = 4\,\frac{7}{8}\,\varepsilon_o E_o^2 \tag{h}$$

$$f_2 = \int_{(b)} T_{21} da + \int_{(a)} T_{22} da + \int T_{23} da$$

$$= \frac{3}{2}\varepsilon_o E_o^2 (3) - \frac{5}{8}\varepsilon_o E_o^2 (2) \tag{i}$$

$$f_2 = 3\,\frac{1}{4}\,\varepsilon_o E_o^2 \tag{j}$$

$$f_3 = \int T_{31} da + \int T_{32} da + \int T_{33} da$$

$$= 0 \tag{k}$$

Hence, the total force is:

$$\bar{f} = 4\,\frac{7}{8}\,\varepsilon_o E_o^2\, \vec{i}_1 + 3\,\frac{1}{4}\,\varepsilon_o E_o^2\, \vec{i}_2 \tag{ℓ}$$

PROBLEM 8.6

Part a

At point A, the electric field intensity is a superposition of the imposed field and the field due to the surface charges; $\bar{E} = (\sigma_f/\varepsilon_o)\bar{i}_y$. Thus at A,

$$\bar{E} = \bar{i}_x(E_o) + \bar{i}_y(E_o + \frac{\sigma_f}{\varepsilon_o}) \tag{a}$$

while at B,

$$\bar{E} = \bar{i}_x(E_o) + \bar{i}_y(E_o) \tag{b}$$

Thus, from Eq. 8.3.10, at A,

$$T_{ij} = \begin{bmatrix} \frac{1}{2}\varepsilon_o[E_o^2 - (E_o + \frac{\sigma_f}{\varepsilon_o})^2] & \varepsilon_o E_o(E_o + \frac{\sigma_f}{\varepsilon_o}) & 0 \\ \varepsilon_o E_o(E_o + \frac{\sigma_f}{\varepsilon_o}) & \frac{1}{2}\varepsilon_o[(E_o + \frac{\sigma_f}{\varepsilon_o})^2 - E_o^2] & 0 \\ 0 & 0 & -\frac{1}{2}\varepsilon_o[E_o^2 + (E_o + \frac{\sigma_f}{\varepsilon_o})^2] \end{bmatrix} \tag{c}$$

while at B the components are given by (c) with $\sigma_f \to 0$.

Part b

In the x direction, because the fields are independent of x and z,

$$f_x = (b-a)\left[(T_{xy})\Big|_A - (T_{xy})\Big|_B\right]D = (b-a)DE_o\sigma_f \tag{d}$$

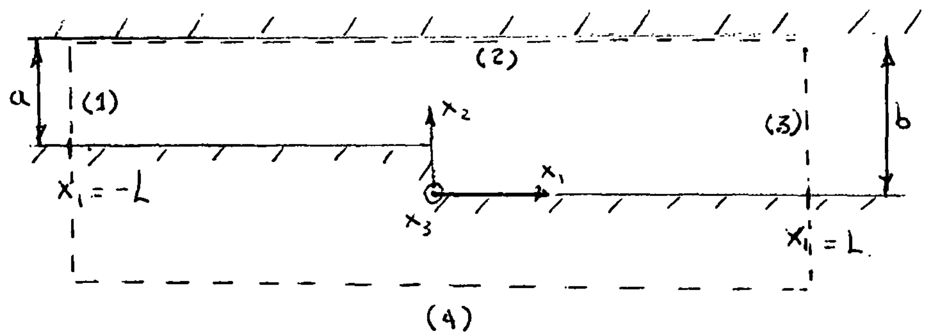
or simply the area multiplied by the surface charge density and x component of electric field intensity.

In the y direction

$$f_y = (b-a)(T_{yy}\Big|_A - T_{yy}\Big|_B)D = (b-a)D[E_o\sigma_f + \frac{\sigma_f^2}{2\varepsilon_o}] \tag{e}$$

Note that both (d) and (e) could be found by multiplying the surface charge density by the average electric field intensity and the area, as shown by Eq. 8.4.8.

PROBLEM 8.7



Before finding the force, we must calculate the $\bar{H}$ field at $x_1 = L$. To find this field let us use

$$\oint \bar{B}\cdot\bar{n}da = 0 \tag{a}$$

over the dotted surface. At $x_1 = + L$,

$$\bar{H}(x_1{=}L) = H_o\bar{i}_1 \tag{b}$$

over surface (4) $\bar{H} = 0$, and over surface (2), $\bar{H}$ is in the $\bar{i}_1$ direction, where $\bar{n} = \bar{i}_2$. Thus over surface (2) $\bar{B}\cdot\bar{n} = 0$.

Hence, the integral in (a) reduces to

$$-\int_{(1)} \mu_o H_o da + \int_{(3)} \mu_o H(x_1 = + L)da = 0$$

$$-\mu_o H_o a + \mu_o Hb = 0 \qquad \text{per unit depth} \tag{d}$$

Thus:

$$\bar{H}(x_1 = + L) = (a/b)H_o\bar{i}_1 \tag{e}$$

$$T_{ij} = \mu_o H_i H_j - \frac{\delta_{ij}}{2}\mu_o H_k H_k \tag{f}$$

Hence, the stress tensor over surfaces (1), (2) and (3) is:

$$T_{ij} = \begin{bmatrix} \frac{\mu_o}{2}H_1^2 & 0 & 0 \\ 0 & -\frac{\mu_o}{2}H_1^2 & 0 \\ 0 & 0 & -\frac{\mu_o}{2}H_1^2 \end{bmatrix} \tag{g}$$

PROBLEM 8.7 (Continued)

over surface (4)

$$T_{ij} = [0] \tag{h}$$

Thus the force in the 1 direction is

$$f_1 = \int T_{ij}\, n_j\, da \tag{i}$$

$$f_1 = -\int_{(1)} T_{11} da + \int_{(3)} T_{11} da + \int_{(2)} T_{12} da \tag{j}$$

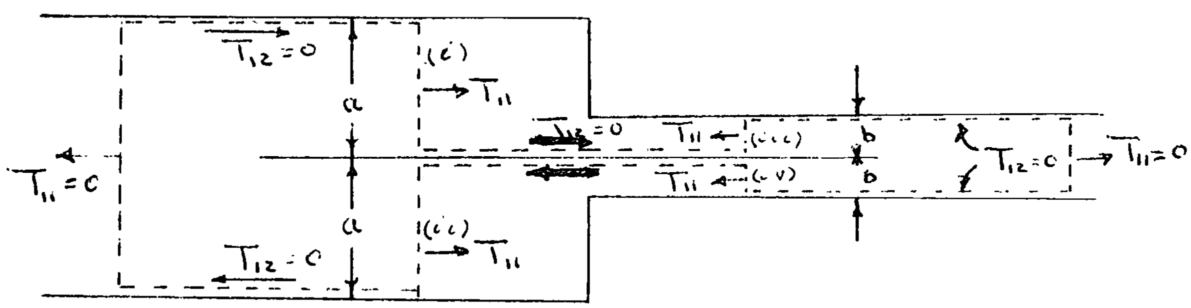
Thus, since the last integral makes no contribution,

$$f_1 = -\frac{\mu_o}{2} H_o^2 (a) + \frac{\mu_o}{2} H_o^2 \left(\frac{a}{b}\right)^2 \cdot b = \frac{\mu_o}{2} H_o^2\, a \left\{\frac{a}{b} - 1\right\} \tag{k}$$

Since $T_{ij} = 0$ over surface (4) there is no contribution to the force from this surface and by symmetry, there is no contribution to the force from the surfaces perpendicular to the x_3 axis. Thus, the force per unit depth in 1 direction is (k).

PROBLEM 8.8

The appropriate surface of integration is shown in the figure



The stresses acting in the x direction on the respective surfaces are as shown. Because the plates are perfectly conducting, all shear stresses required to complete the integration of Eq. 8.1.17 vanish. The only contributions are from surfaces (i), (ii), (iii) and (iv), where the fields

PROBLEM 8.8 (Continued)

are known to be

$$\bar{E} = -\frac{V_o}{a}\bar{i}_y \quad (i) \quad ; \quad \bar{E} = -\frac{V_o}{b}\bar{i}_y \quad (iii)$$

$$\bar{E} = \frac{V_o}{a}\bar{i}_y \quad (ii) \quad ; \quad \bar{E} = \frac{V_o}{b}\bar{i}_y \quad (iv) \qquad (a)$$

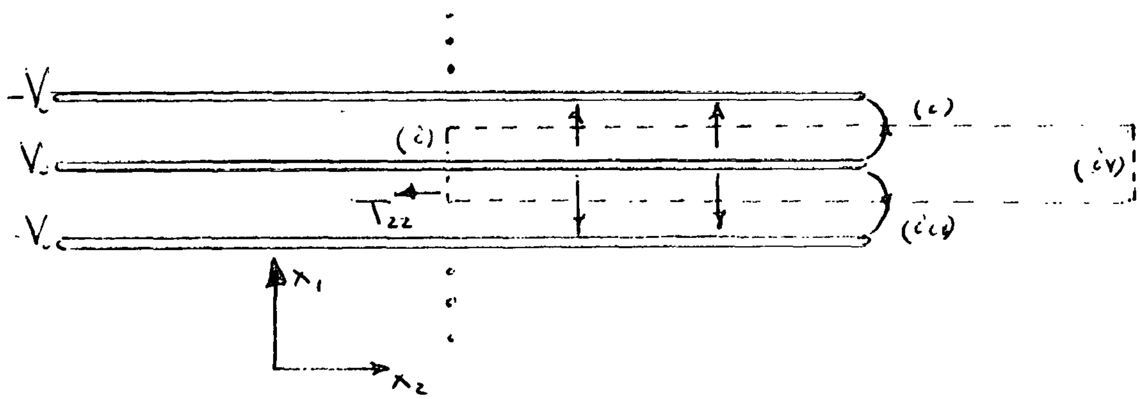
Thus,

$$f_1 = (T_{11})_i\, ad + (T_{11})_{ii}\, ad - (T_{11})_{iii}\, bd - (T_{11})_{iv}\, bd \qquad (b)$$

$$= dV_o^2\varepsilon_o[\frac{1}{b} - \frac{1}{a}] \qquad (c)$$

The plate tends to be drawn to the right, where the fields are greater.

PROBLEM 8.9



The volume enclosing the half of the plate is arbitrary so long as it is defined so that it does not include additional charge. Thus the volume shown in the figure encloses no more than the desired distribution of charge. Moreover, surfaces (i) and (iii) pass through the fringing fields half way between the plates where by symmetry there is no x_2 component of $\bar{E}$. Thus surfaces (i) and (iii) support no shear stress T_{21}. There is no field at surface (iv) and hence the only contribution is from surface (i), where the square of the field is known to be

$$E_1^2 = \frac{4V_o^2}{s^2} \qquad (a)$$

and it follows that because T_{22} on (i) is $-\frac{1}{2}\varepsilon_o E_1^2$ and the normal vector is negative

$$f_2 = \frac{4ws\varepsilon_o}{2}\frac{V_o^2}{s^2} \qquad (b)$$

PROBLEM 8.9 (Continued)

The fringing field tends to pull the end of the plate in the $+ x_2$ direction.

PROBLEM 8.10

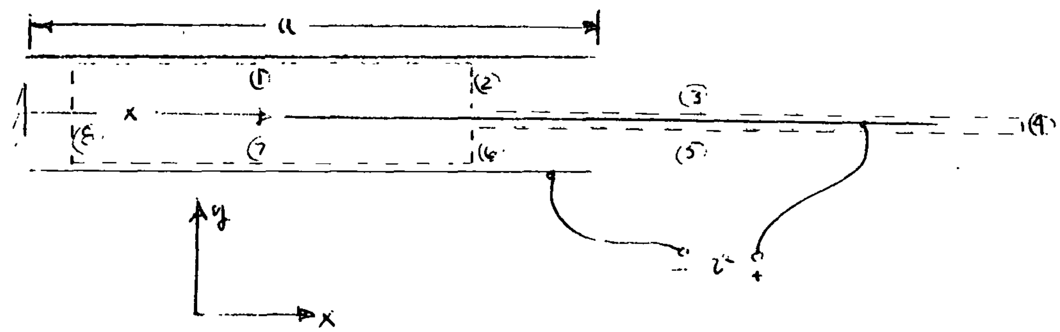


Figure 1

Part a

Consider the surface shown in Figure 1. The total force in the x direction is:

$$f_x = \int_{1,3} T_{xy}\, da - \int_{5,7} T_{xy}\, da + \int_{4} T_{xx}\, da - \int_{8} T_{xx}\, da + \int_{2,6} T_{xx}\, da \tag{a}$$

The first four integrals disappear because:

$T_{xy} = \varepsilon E_x E_y = 0$ on 1, 3, 5 and 7 because we are next to the conducting plates ($E_x = 0$)

$T_{xx} = 0$ an 4 and 8 because the $\bar{E}$ field = 0 there

Hence

$$f_x = \int_{2,6} T_{xx} da = \int_{2,6} - \frac{1}{2}\, E_y^2\, da \tag{b}$$

where T_{ij} is evaluated using Eq. 8.3.10.

$$E_y = \frac{v}{s} \tag{c}$$

and hence:

PROBLEM 8.10 (Continued)

$$f_x = \int_{2,6} -\frac{\varepsilon}{2}\left(\frac{v}{s}\right)^2 da = -\frac{\varepsilon d}{s} v^2 \tag{d}$$

Part b

The coenergy of the system is

$$W' = \frac{1}{2} C(x) v^2 \tag{e}$$

where $$C(x) = \frac{2(a-x)d\varepsilon}{s} \tag{f}$$

Thus, (see Sec. 3.1.2b)

$$f_x = \frac{\partial W'}{\partial x} = \frac{1}{2}\frac{\partial C(x)}{\partial x} v^2 = -\frac{d\varepsilon}{s} v^2 \tag{g}$$

which is the same value determined in part (a).

Part c

The equation of motion of the plate is:

$$M\frac{d^2x}{dt^2} + K(x-a) = f_x = -\frac{d\varepsilon}{s} V_o^2 \tag{h}$$

When the system reaches equilibrium with the switch closed,

$$K(X_o - a) = -\frac{d\varepsilon}{s} V_o^2 \tag{i}$$

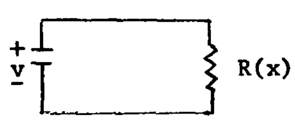
thus

$$X_o = a - \frac{d\varepsilon}{sK} V_o^2 \tag{j}$$

After the switch is opened,

$$M\frac{d^2x}{dt^2} + K(x-a) = -\frac{d\varepsilon}{s} v^2(t) \tag{k}$$

The electrical circuit is like an R-C circuit with time varying elements



$$v + R(x)i(t) = 0 \tag{ℓ}$$

$$v + R(x)\frac{d}{dt}[C(x)v] = 0 \tag{m}$$

$$v + R(x)C(x)\frac{dv}{dt} + R(x)\frac{dC(x)}{dx}\frac{dx}{dt} v = 0 \tag{n}$$

PROBLEM 8.10 (Continued)

where:

$$R(x) = \frac{s}{2\sigma d(a-x)} \qquad \text{and} \quad C(x) \; \frac{2d(a-x)\varepsilon}{s} \tag{o}$$

Hence

$$v + \frac{\varepsilon}{\sigma}\frac{dv}{dt} - \left[\frac{\varepsilon}{\sigma}\,\frac{1}{(a-x)}\,\frac{dx}{dt}\right] v = 0 \tag{p}$$

Part d

Dropping the inertial term from (h) leaves:

$$K(x-a) = -\frac{d\varepsilon}{s}\,v^2(t) \qquad \text{from (k)} \tag{q}$$

But we may write the identity

$$-\frac{1}{(a-x)}\,\frac{dx}{dt} = \frac{1}{K(x-a)}\,\frac{d}{dt}\;K(x-a) \tag{r}$$

and then, from (q)

$$-\frac{1}{(a-x)}\,\frac{dx}{dt} = -\frac{s}{d\varepsilon v^2(t)}\,\frac{d}{dt}\cdot\left[-\frac{d\varepsilon}{s}\,v^2(t)\right]$$

$$= \frac{1}{v^2(t)}\,\frac{d}{dt}\,v^2(t) = \frac{2}{v}\,\frac{dv}{dt} \tag{s}$$

Substituting back into (p) we have

$$v + \frac{\varepsilon}{\sigma}\frac{dv}{dt} + \frac{2\varepsilon}{\sigma}\frac{dv}{dt} = 0 \tag{t}$$

Solving we find

$$v = V_o\; e^{-(\sigma/3\varepsilon)\,t} \tag{u}$$

and substituting back into (q),

$$x = a - \frac{d\varepsilon}{sK}\,V_o^2\; e^{-\frac{2}{3}\frac{\sigma}{\varepsilon}\,t} \tag{v}$$

A long relaxation time is consistent with neglecting the inertial terms, as then x(t) varies slowly.

Part e

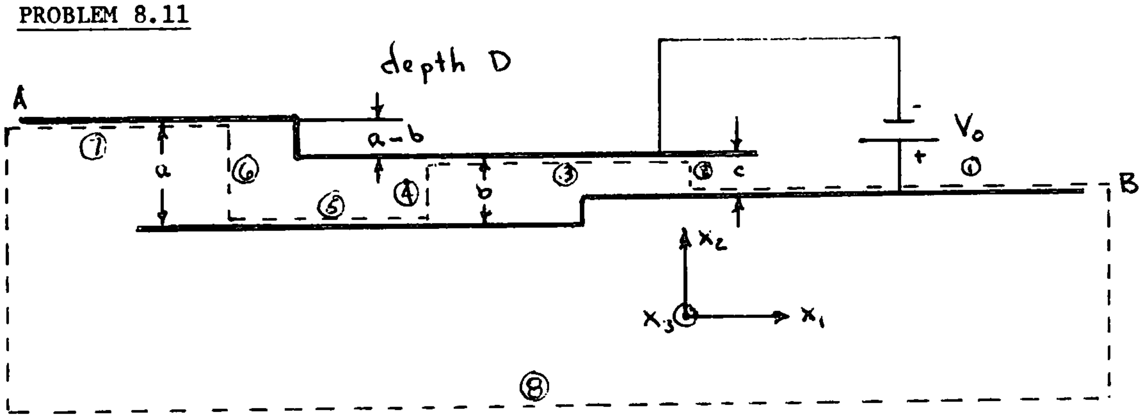
Proceed as in (c), and record the time constant τ of a-x(t) by measuring the mechanical displacement. Then,

$$\frac{\varepsilon}{\sigma} = \frac{2}{3}\,\tau \tag{w}$$

PROBLEM 8.10 (Continued)

This problem should raise questions as to the appropriate form of T_{ij} used in (b). Note that the surface of integration encloses liquid as well as the plate. We want only the force on the plate, so our calculation is correct only if there is no net force on the enclosed liquid. The electrical force density in the liquid is given by Eq. 8.5.45. There is no free charge or gradient of permittivity in the bulk of the liquid and hence the first two of the three contributions to this force density vanish in the liquid. However, there remains the electrostriction force density. Note that it is ignored in our calculation because the electrostriction term was not included in the stress tensor (we used Eq. 8.3.10 rather than 8.5.46). Our reason for ignoring the electrostriction is this: it gives rise to a force density that takes the form of the gradient of a pressure. Hence, it simply alters the distribution of liquid pressure around the plate. Because each element of the liquid is in static equilibrium and can give way to motions of the plate without changing its volume, the "hydrostatic pressure" of the liquid is altered by the electric field so as to exactly cancel the effect of the electrostriction force density. Hence, to correctly include the effect of electrostriction in integrating the stresses over the surface, we must also include the hydrostatic pressure of the liquid. If this is done, the effect of the electrostriction will cancel out, leaving the force on the plate we have derived by two alternative methods here.

PROBLEM 8.11



PROBLEM 8.11 (Continued)

First, let us note the $\bar{E}$ fields on each of the surfaces of the figure over surfaces (1), (3), (5) and (7), $E_1 = 0$ (a)

over surface

$$(6)\quad E_2 = \frac{V_o}{a} \quad E_1 = 0 \tag{b}$$

$$(4)\quad E_2 = \frac{V_o}{b} \quad E_1 = 0 \tag{c}$$

$$(2)\quad E_2 = \frac{V_o}{c} \quad E_1 = 0 \tag{d}$$

From Eq. 8.3.10,

$$T_{ij} = \varepsilon_o E_i E_j - \frac{\delta_{ij}}{2} \varepsilon_o E_k E_k \tag{e}$$

Hence, over surfaces (1), (3), (5) and (7)

$$T_{12} = 0 \tag{f}$$

and over surfaces

$$(6)\quad T_{11} = -\frac{\varepsilon_o}{2} \left(\frac{V_o}{a}\right)^2 \tag{g}$$

$$(4)\quad T_{11} = -\frac{\varepsilon_o}{2} \left(\frac{V_o}{b}\right)^2 \tag{h}$$

$$(2)\quad T_{11} = -\frac{\varepsilon_o}{2} \left(\frac{V_o}{c}\right)^2 \tag{i}$$

Now;

$$f_1 = \int T_{ij} n_j da = \int T_{11} n_1 da + \int T_{12} n_2 da + \int T_{13} n_3 da \tag{j}$$

$$\int T_{13} n_3 da = 0 \quad \text{because the problem is two dimensional.} \tag{k}$$

Let us consider each of the other integrals:

$$\int T_{12} n_2 da = 0 \tag{\ell}$$

because the surfaces which have normal n_2 are (1), (3), (5) and (7) and by (f) we have shown that $T_{12} = 0$ over these surfaces. Also, we get no contribution to the force over surface (8), because $\bar{E} \to 0$ faster than the area $\to \infty$.

Hence the calculation of the force reduces tc

PROBLEM 8.11 (Continued)

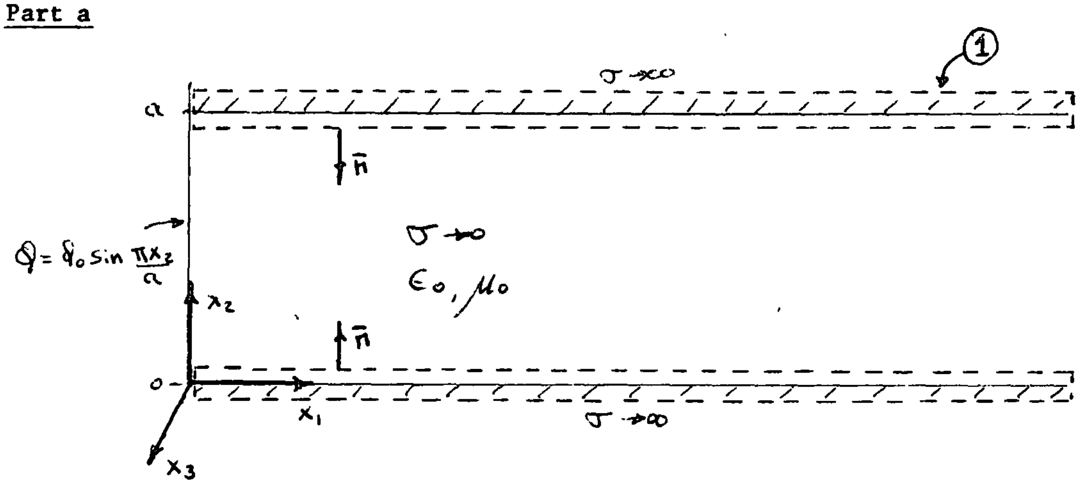
$$f_1 = \int_{(6)} T_{11}^{(6)}\, da_6 - \int_{(4)} T_{11}^{(4)}\, da_4 - \int_{(2)} T_{11}^{(2)}\, da_2 \tag{m}$$

$$f_1 = -\frac{\varepsilon_o D V_o^2}{2}\left\{\frac{1}{a} - \frac{1}{b} + \frac{1}{c}\right\} \tag{n}$$

Note: by symmetry, there is no contribution to the force from the surfaces perpendicular to the x_3 axis.

PROBLEM 8.12

Part a



From elementary field theory, we find that

$$\phi = \phi_o \sin\frac{\pi x_2}{a}\, e^{-\pi x_1/a} \tag{a}$$

satisfies $\nabla^2\phi = 0$ in the region between the plates and the required boundary conditions. The distribution of $\bar{E}$ follows from

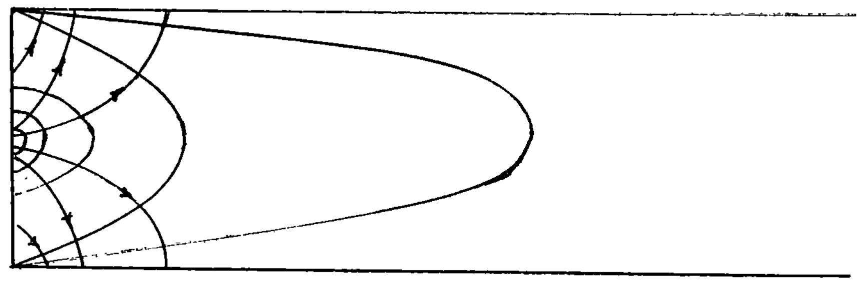
$$\bar{E} = -\nabla\phi \tag{b}$$

Hence,

$$\bar{E} = \frac{\pi\phi_o}{a} e^{-\pi x_1/a}\left[\sin\frac{\pi x_2}{a}\,\bar{i}_1 - \cos\frac{\pi x_2}{a}\,\bar{i}_2\right] \tag{c}$$

The sketch of the $\bar{E}$ field is obtained by recognizing that $\bar{E}$ is directed perpendicular to contours of constant ϕ.

PROBLEM 8.12 (Continued)



Part b

To find the force as the bottom plate, we use surface (2). $\bar{E} = 0$ everywhere except on the upper side where the normal $\bar{n} = \bar{i}_2$ (d)

and the field is

$$\bar{E} = -\frac{\pi\phi_o}{a} e^{-\pi x_1/a} \bar{i}_2 \tag{e}$$

Hence,

$$f_1 = \int T_{1j} n_j \, da = 0 \tag{f}$$

$$f_2 = \int T_{2j} n_j \, da = \int T_{22} n_2 \, da_2 \tag{g}$$

per unit x_3, this reduces to

$$f_2 = \int_o^\infty T_{22} dx_1 \tag{h}$$

but, $T_{22} = \frac{1}{2} \varepsilon_o E_2 E_2 = \frac{1}{2} \varepsilon_o \frac{\pi^2 \phi_o^2}{a^2} e^{-\frac{2\pi x_1}{a}}$ (i)

and thus

$$f_2 = \frac{\varepsilon_o \pi^2 \phi_o^2}{2a^2} \int_o^\infty e^{-\frac{2\pi x_1}{a}} dx_1 \tag{j}$$

$$f_2 = \frac{\varepsilon_o \pi \phi_o^2}{4a} \tag{k}$$

PROBLEM 8.12 (Continued)

Part c

On the top plate, use surface (1). Only the sign of the normal changes, and the result is

$$f_1 = 0 \tag{ℓ}$$

$$f_2 = -\frac{\varepsilon_o \pi \phi_o^2}{4a} \tag{m}$$

or the force is equal and opposite to that on the bottom plate.

PROBLEM 8.13

Part a

$$T_{ij} = \varepsilon E_i E_j - \frac{\delta_{ij}}{2} \varepsilon E_k E_k \tag{a}$$

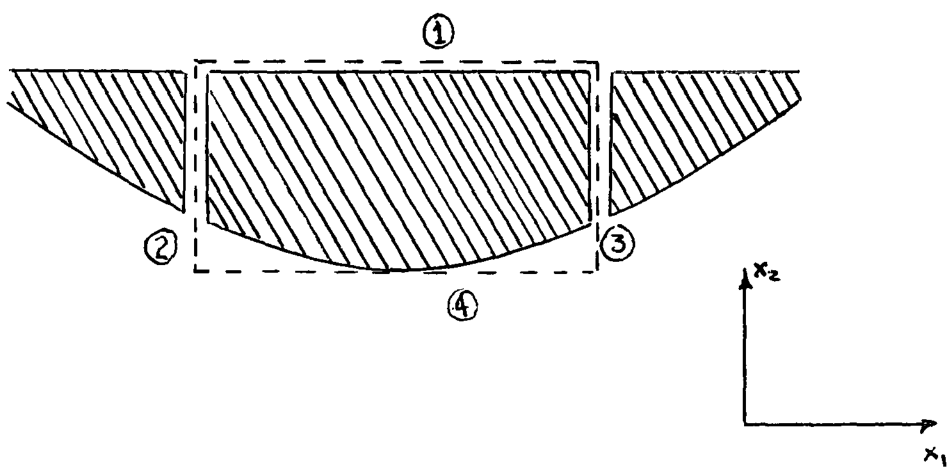
Hence:

$$T_{22} = \frac{1}{2} \varepsilon_o \left(\frac{2V_o}{3a^2}\right)^2 (x_2^2 - x_1^2) \tag{b}$$

$$T_{21} = \varepsilon_o E_2 E_1 = -\varepsilon_o \left(\frac{2V_o}{3a^2}\right)^2 x_1 x_2 \tag{c}$$

Part b

Consider the surface of integration shown in the figure.



$$f_2 = \int T_{2j} n_j da = \underset{(2)(3)}{\int} T_{21} n_1 da + \underset{(1)(4)}{\int} T_{22} n_2 da + \underset{\text{by symmetry}}{\int \cancel{T_{23}}^{\,0} n_3 da} \tag{d}$$

PROBLEM 8.13 (Continued)

Let us look at each of these integrals separately

$$\int_{(1)(4)} T_{22}n_2 da = \int_{(1)} T_{22} da_1 - \int_{(4)} T_{22} da_4 \qquad \text{(e)}$$

over surface (1), $\bar{E} \simeq 0$; $T_{22} \simeq 0$ and hence, the integral is merely:

$$-\int_{(4)} T_{22} da_4 = -\int_{x_1=-a}^{x_1=a} \frac{\varepsilon_o}{2}\left(\frac{2V_o}{3a^2}\right)^2 \left(x_2^2 - x_1^2\right) w dx_1 \Big|_{x_2 = 2a}$$

$$= -\frac{44}{27}\frac{\varepsilon_o V_o^2 w}{a} \qquad \text{(f)}$$

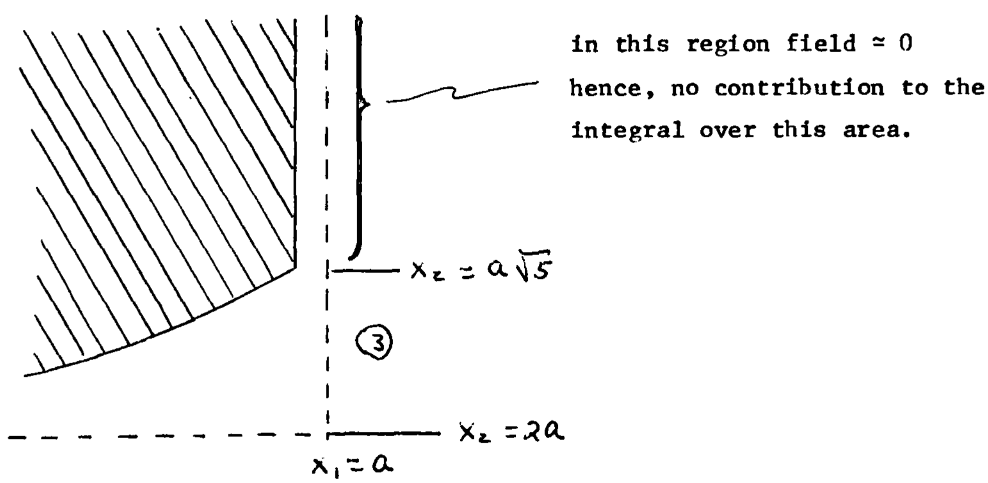
Thus,

$$\int_{(1)(4)} T_{22}n_2 da = -\frac{44}{27}\frac{\varepsilon_o V_o^2 w}{a} \qquad \text{(g)}$$

Let us now evaluate:

$$\int_{(2)(3)} T_{21}n_1 da$$

Consider the surface shown.



PROBLEM 8.13 (Continued)

Thus;

$$\int_{(3)} T_{21}da_3 = \int_{\substack{x_2=2a \\ x_1=a}}^{x_2=a\sqrt{5}} -\varepsilon_o \left(\frac{2V_o}{3a^2}\right)^2 awx_2dx_2$$

$$= -\frac{2}{9}\frac{\varepsilon_o V_o^2 w}{a} \tag{h}$$

Over surface (2), we have essentially the same thing, except $\bar{n} = -\bar{i}_1$ and $x_1 = -a$. Hence:

$$\int_{(2)} T_{21}da_2 = -\frac{2}{9}\frac{\varepsilon_o V_o^2 w}{a} \tag{i}$$

Therefore, the total force in the f_2 direction is

$$f_2 = -\frac{56}{27}\frac{\varepsilon_o V_o^2 w}{a} \tag{j}$$

Part c

$$f_1 = \int_{(2)(3)} T_{11}n_1da + \int_{(1)(4)} T_{12}n_2da + \underbrace{\int \cancel{T_{13}n_3}^{\,0}da}_{\text{by symmetry}} \tag{k}$$

$$\int_{(1)(4)} T_{12}n_2da = -\int_{(4)} T_{12}da_4$$

over (1) we get 0 as before

$$= \varepsilon_o \left(\frac{2V_o}{3a^2}\right)^2 \int_{\substack{-a \\ x_2=2a}}^{a} x_1x_2wdx_1 = 0 \tag{ℓ}$$

Now, over surfaces, 2 and 3

$$\int_{(2)(3)} T_{11}n_1da = -\int_{(2)} T_{11}da_2 + \int_{(3)} T_{11}da_3 = 0 \tag{m}$$

because,

$$T_{11}\Big|_2 = T_{11}\Big|_3$$

hence $f_1 = 0$. (n)

PROBLEM 8.13 (Continued)

Part d

$$\sigma_f = \bar{n} \cdot \varepsilon_o \bar{E} \tag{o}$$

at the lower surface of the movable conductor. The functional relation, $f(x_1 x_2)$, for the lower surface if the movable conductor is given as

$$f(x_1 x_2) = \sqrt{4a^2 + x_1^2} - x_2 = 0 \tag{p}$$

the outward unit normal to this surface is

$$\bar{n} = \frac{\nabla f(x_1 x_2)}{|\nabla f(x_1 x_2)|} = \left[\frac{x_1}{x_2}\bar{i}_1 - \bar{i}_2\right]\left[\frac{1}{\sqrt{\left(\frac{x_1}{x_2}\right)^2 + 1}}\right] \tag{q}$$

at $x_2 = \sqrt{4a^2 + x_1^2}$,

$$\sigma_f = \varepsilon_o[n_1 E_1 + n_2 E_2] = \frac{2V_o\varepsilon_o}{3a^2}\left[\frac{x_1^2}{x_2} + x_2\right]\left[\frac{4a^2 + x_1^2}{4a^2 + 2x_1^2}\right]^{1/2} \tag{r}$$

The surface force density (see Eq. 8.4.8) is equal to:

$$\bar{T} = \sigma_f \frac{\bar{E}^b + \bar{E}^a}{2} \tag{s}$$

where, $\bar{E}^b$ = field just below the charge sheet

$\bar{E}^a$ = field just above the charge sheet

Since

$$\bar{E}^a = 0, \quad \bar{T} = \frac{1}{2}\sigma_f \bar{E}^b \tag{t}$$

thus

$$\bar{T} = \frac{\varepsilon_o}{2}\left(\frac{2V_o}{3a^2}\right)^2\left[\frac{x_1^2}{x_2} + x_2\right]\left[x_1\bar{i}_1 - x_2\bar{i}_2\right]\left[\frac{4a^2 + x_1^2}{4a^2 + 2x_1^2}\right]^{1/2} \tag{u}$$

To find the total force, the surface force density must be integrated over the surface. Hence, we find

$$f_2 = -2\varepsilon_o\left(\frac{V_o}{3a^2}\right)^2\int_{-a}^{+a}\{2x_1^2 + 4a^2\}^{1/2}\{x_1^2 + 4a^2\}^{1/2}\,dx_1 \tag{v}$$

If the student wishes, he may **carry** out this integral, but the complexity of the integration shows the value of the stress tensor in calculating such a

PROBLEM 8.13 (Continued)

force. We realize that by using the stress tensor, we have essentially carried out this difficult integral by an integration by parts.

PROBLEM 8.14

Part a

$$\phi = \frac{V_o}{a^2} x_1 x_2 \tag{a}$$

$$\bar{E} = -\nabla\phi \tag{b}$$

hence,

$$\bar{E} = \bar{i}_1\left(-\frac{V_o}{a^2} x_2\right) + \bar{i}_2\left(-\frac{V_o}{a^2} x_1\right) \tag{c}$$

and, from Eq. 8.3.10

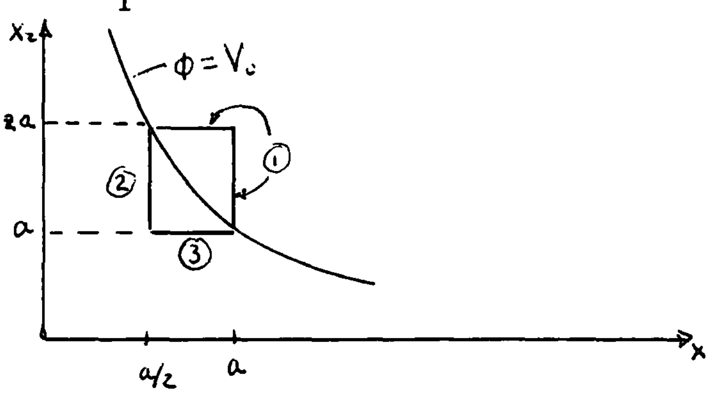
$$T_{ij} = \varepsilon E_i E_j - \delta_{ij}\frac{\varepsilon}{2} E_k E_k \tag{d}$$

Thus, the stress tensor becomes:

$$T_{ij} = \begin{bmatrix} \left(\frac{V_o}{a^2}\right)^2 \frac{\varepsilon_o}{2}(x_2^2 - x_1^2) & \left(\frac{V_o}{a^2}\right)^2 \varepsilon_o (x_1 x_2) & 0 \\ \left(\frac{V_o}{a^2}\right) \varepsilon_o (x_1 x_2) & \left(\frac{V_o}{a^2}\right)^2 \frac{\varepsilon_o}{2}(x_1^2 - x_2^2) & 0 \\ 0 & 0 & -\left(\frac{V_o}{a^2}\right)^2 \frac{\varepsilon_o}{2}(x_1^2 + x_2^2) \end{bmatrix} \tag{e}$$

Part b

Consider the surface shown, bounded by the line segment $x_2 = 2a$, $x_2 = a$, and $x_1 = a/2$ and $x_1 = a$.



PROBLEM 8.14 (Continued)

As before, because the geometry and fields are two-dimensional, the force in the $\bar{i}_3$ direction is zero. Also, since along surface (1) ϕ = constant, then the $\bar{E}$ field = 0, and hence $T_{ij} = 0$ along this surface. Thus the calculation of the force on AB reduces to:

$$f_1 = -\int_{(2)} T_{11}da - \int_{(3)} T_{12}da \tag{f}$$

$$f_2 = -\int_{(2)} T_{21}da - \int_{(3)} T_{22}da \tag{g}$$

$$f_1 = -\left(\frac{V_o}{a^2}\right)^2 \varepsilon_o D \left[\frac{1}{2}\int_a^{2a} [x_2^2 - (\tfrac{a}{2})^2]dx_2 + \int_{a/2}^{a} x_1 a dx_1\right] \tag{h}$$

and hence

$$f_1 = -\varepsilon_o \left(\frac{V_o}{a^2}\right)^2 Da^3 \left[\frac{17}{12}\right] \tag{i}$$

Similarly:

$$f_2 = -\left(\frac{V_o}{a^2}\right)^2 D\varepsilon_o \left[\int_a^{2a} \frac{a}{2} x_2 dx_2 + \frac{1}{2}\int_{a/2}^{a} (x_1^2 - a^2) dx_1\right] \tag{j}$$

and hence

$$f_2 = -\varepsilon_o D\, a^3 \left(\frac{V_o}{a^2}\right)^2 \left[\frac{31}{48}\right] \tag{k}$$

Thus,

$$\bar{f} = -\varepsilon_o \frac{V_o^2}{a} D \left[\bar{i}_1 \frac{17}{12} + \bar{i}_2 \frac{31}{48}\right] \tag{ℓ}$$

PROBLEM 8.15

Part a

The $\bar{E}$ field in the laboratory frame is zero since the two perfectly conducting plates are shorted. This can be seen by integrating $\bar{E}$ around a fixed contour through the block and short and recognizing that the enclosed flux is constant. Hence,

$$\bar{E}' = \bar{E} + \bar{v} \times \bar{B}\,, \quad \bar{E} = 0 \tag{a}$$

and thus

$$\bar{E}' = \bar{v} \times \bar{B} = -V\mu_o H_o \bar{i}_2 \tag{b}$$

Therefore we may now calculate $\bar{J}$ in the moving block.

PROBLEM 8.15 (Continued)

$$\bar{J}' = \sigma\bar{E}' = -\sigma\mu_o V H_o \bar{i}_2 \quad \text{(c)}$$

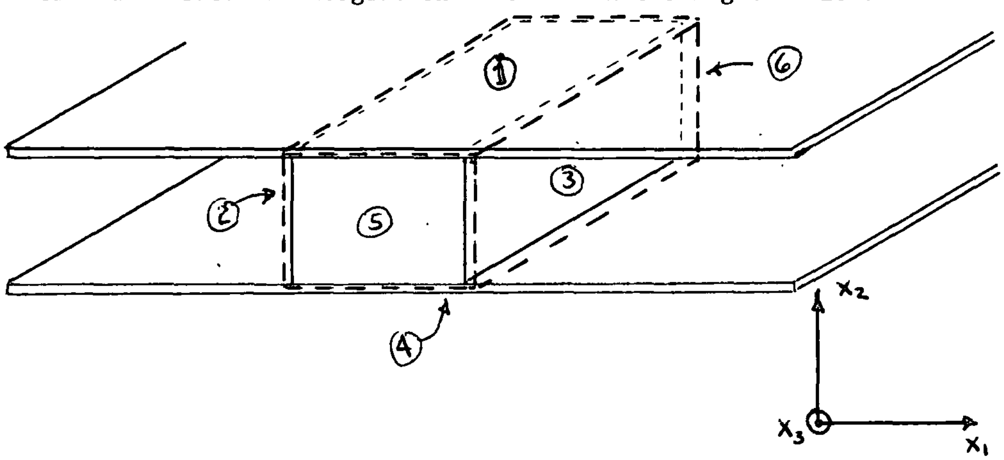
Thus:

$$\bar{F} = \bar{J} \times \bar{B} = -\sigma\mu_o^2 V H_o^2 \bar{i}_1 \quad \text{(d)}$$

$$\bar{f} = \int_{\text{volume}} (\bar{J} \times \bar{B})\, dV = -\mu_o^2 \sigma V H_o^2 (abD)\bar{i}_1 \quad \text{(e)}$$

Part b

The closed surface of integration is shown in the figure below.



Since the field is uniform everywhere, the only non-zero components of the stress tensor are the diagonal elements

$$T_{11} = T_{22} = -\frac{1}{2}\mu_o H_o^2 \qquad T_{33} = \frac{1}{2}\mu_o H_o^2 \quad \text{(f)}$$

Thus

$$f_1 = \int_{(3)} T_{11} da_3 - \int_{(2)} T_{11} da_2$$

$$= \frac{\mu_o}{2} H_o^2\, bD - \frac{\mu_o}{2} H_o^2 bD = 0 \quad \text{(g)}$$

Similarly

$$f_2 = \int_{(1)} T_{22} da_1 - \int_{(4)} T_{22} da_4 = 0 \quad \text{(h)}$$

$$f_3 = \int_{(5)} T_{33} da_5 - \int_{(6)} T_{33} da_6 = 0 \quad \text{(i)}$$

Hence:

$$\bar{f} = 0 \quad \text{(j)}$$

PROBLEM 8.15 (Continued)

Part c

The magnetic field strength and the current density are inconsistant. The quasi-static magnetic field cannot be uniform and irrotational in a region where a finite current density exists. The Maxwell stress tensor was developed with the aid of Ampere's Law (quasi-static) which relates current density and magnetic field rotation.

$$\bar{J} = \nabla \times \bar{H} \tag{k}$$

$$\bar{F} = \mu_o \bar{J} \times \bar{H} = \mu_o (\nabla \times \bar{H}) \times \bar{H} \tag{ℓ}$$

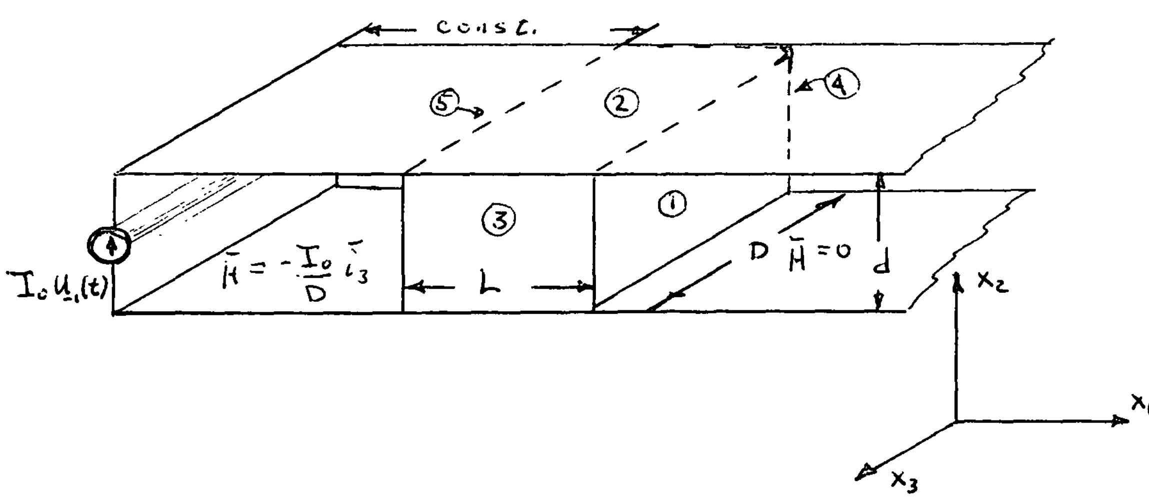
For this case, we have assumed that

$$\nabla \times \bar{H} = 0 \tag{m}$$

In the limit of small magnetic Reynold's number, ($R_m << 1$), the motion does not appreciably affect the field, and the answer found in part a is a good approximation. There are some problems more easily handled with the stress tensor. This problem illustrates that in other cases it is easiest to use the force density $\bar{J} \times \bar{B}$ directly. Note that we could compute the field induced by $\bar{J}$ and then use the Maxwell stress tensor and the self-consistent fields to find the same force as given by (e).

PROBLEM 8.16

To find the force on the block, we will use the stress tensor over the surface shown in the figure. Note that the surface is just outside the block.



PROBLEM 8.16 (Continued)

In the region to the left of the block

$$\bar{H} = -\frac{I_o}{D}\bar{i}_3 \text{ , and to the right } \bar{H}=0$$

Thus:

$$f_1 = \int T_{11}n_1 da + \int T_{12}n_2 da + \int T_{13}n_3 da \tag{a}$$

but, since

$$H_1 = H_2 = 0; \; T_{12} = T_{13} = 0 \tag{b}$$

hence,

$$f_1 = -\int_{(5)} T_{11} da_5 + \int_{(1)} T_{11} da_1 \tag{c}$$

on surface (5),

$$T_{11} = -\frac{\mu_o}{2}\frac{I_o^2}{D^2} \tag{d}$$

on surface (1)

$$T_{11} = 0 \tag{e}$$

therefore

$$f_1 = +\frac{\mu_o}{2}\frac{I_o^2}{D^2}\cdot Dd = +\frac{\mu_o I_o^2 d}{2D} \tag{f}$$

Similarly, f_2 reduces to

$$f_2 = \int_2 T_{22} da_2 - \int_6 T_{22} da_6 \tag{g}$$

But, since T_{22} is a function of x_1 alone ($\bar{H}$ is a function of x_1 alone) the two surface integrals are identical, and hence $f_2 = 0$. Similar reasoning shows that $f_3 = 0$ and thus the total force is

$$\bar{f} = \frac{\mu_o I_o^2 d}{2D}\bar{i}_1$$

PROBLEM 8.17

Part a

$$\nabla^2 \bar{H} = \mu_o \sigma \frac{\partial \bar{H}}{\partial t} \tag{a}$$

Assume a solution of the form:

$$\bar{H} = R_e[\underline{H}_z(x) e^{j\omega t}\bar{i}_z] \tag{b}$$

$$\frac{\partial^2 \underline{H}_z}{\partial x^2} = j\omega\sigma\mu_o \underline{H}_z \tag{c}$$

PROBLEM 8.17 (Continued)

Try

$$\underline{H}_z(x) = \underline{H}_o e^{Kx} \tag{d}$$

where

$$K^2 = j\omega\sigma\mu_o \tag{e}$$

and hence

$$K = \pm\sqrt{\frac{\omega\mu_o\sigma}{2}}(1+j) \tag{f}$$

Let us define the skin depth as:

$$\delta \equiv \sqrt{\frac{2}{\omega\mu_o\sigma}} \tag{g}$$

And thus

$$\bar{\underline{H}} = \left[\underline{H}_1 e^{-\frac{x}{\delta}(1+j)} + \underline{H}_2 e^{+\frac{x}{\delta}(1+j)}\right] e^{j\omega t}\,\bar{i}_z \tag{h}$$

Because the skin depth δ is assumed to be small, and the excitation is on the left,

$$H_z(\text{large } x) \to 0 \qquad \text{which implies } \underline{H}_2 = 0$$

Hence,

$$\bar{\underline{H}}(x_1t) = \underline{H}_1 e^{-\frac{x}{\delta}(1+j)} e^{j\omega t}\bar{i}_z \tag{i}$$

But, our boundary condition at $x = 0$ is

$$\underline{H}(x=0,t) = \text{Re}H_1 e^{-j\omega t}\,\bar{i}_z = -\,\text{Re}\,\frac{I}{D}\,e^{j\omega t}\,\bar{i}_z \tag{j}$$

and thus

$$\bar{\underline{H}}(x_1t) = -\frac{I}{D}\,e^{-\frac{x}{\delta}(1+j)} e^{j\omega t}\,\bar{i}_z \tag{k}$$

$$\underline{\bar{J}} = \nabla \times \underline{\bar{H}} = -\left(\frac{\partial H_z}{\partial x}\right)\bar{i}_y = -\frac{I}{D}\,\frac{(1+j)}{\delta}\,e^{-\frac{x}{\delta}(1+j)} e^{j\omega t}\,\bar{i}_y \tag{ℓ}$$

Part b

$$\bar{f} = \int \bar{J} \times \bar{B}\, dV = \int \bar{J} \times \mu_o\bar{H}dV \tag{m}$$

$$\bar{f} = \text{Re}\left[\frac{\int\underline{\bar{J}}\times\mu_o\underline{\bar{H}}^*}{2}\,dV\right] + \text{Re}\left[\frac{\int\underline{\bar{J}}\times\mu_o\underline{\bar{H}}}{2}\,e^{2j\omega t}\,dV\right] \tag{n}$$

Now, solving each of these integrals:

$$\frac{\int\underline{\bar{J}}\times\mu_o\underline{\bar{H}}^*}{2}\,dV = \frac{1}{2}\,\mu_o aD\left(\frac{I}{D}\right)^2\frac{1}{\delta}\,(1+j)\int_o^\infty e^{-\frac{2x}{\delta}}\,dx\,\bar{i}_x$$

$$= \frac{1}{4}\,\frac{\mu_o a}{D}\,I^2(1+j)\bar{i}_x \tag{o}$$

PROBLEM 8.17 (Continued)

$$\frac{\int \bar{J}\times\mu_o\bar{H}}{2} e^{2j\omega t}dV = \frac{1}{2}\mu_o aD\left(\frac{I}{D}\right)^2 \frac{1}{\delta}(1+j)e^{2j\omega t}\int_o^\infty e^{-\frac{2x}{\delta}(1+j)}dx\,\bar{i}_x$$

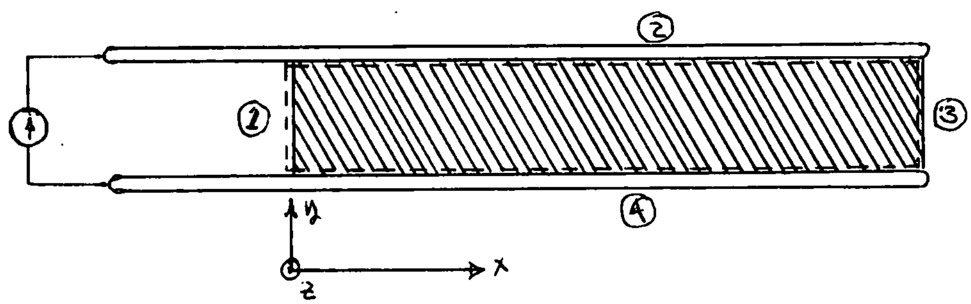
$$= \frac{1}{4}\frac{\mu_o a}{D} I^2 e^{2j\omega t}\,\bar{i}_x \qquad \text{(p)}$$

Hence, taking the real part, the force as in equation (n) is:

$$\bar{f} = \frac{1}{4}\frac{\mu_o a}{D} I^2(1 + \cos 2\omega t)\bar{i}_x \qquad \text{(q)}$$

Part c

Using the Maxwell stress tensor, we choose the surface shown in the figure,



$$f_x = \int T_{xj}n_j da = \int_{(1)(3)} T_{xx}n_x da + \int_{(2)(4)} T_{xy}n_y da \qquad \text{(r)}$$

Along surfaces (2) and (4), $H_x = 0$ along the interface between the perfect conductors and the finite conductivity block. Thus,

$$T_{xy} = \mu_o H_x H_y = 0 \qquad \text{(s)}$$

At surface (3), the field is zero since all current filaments complete a closed loop circuit with the source through the block. Hence

$$T_{xx} = 0 \text{ on surface (3)} \qquad \text{(t)}$$

Therefore the calculation of the force reduces to

$$f_x = -\int_1 T_{xx}da \qquad \text{(u)}$$

$$T_{xx} = -\frac{\mu_o}{2}H_z^2 \qquad \text{(v)}$$

And thus,

$$f_x = \frac{aD\mu_o}{2}H_z^2 \qquad \text{(w)}$$

PROBLEM 8.17 (Continued)

where the field H_z is evaluated on surface 1, i.e. $x = 0$, and is simply given by the boundary condition (j). Thus it follows

$$\bar{f} = \frac{a\mu_o}{4D} I^2 \{1 + \cos 2\omega t\}\bar{i}_x \qquad \text{(x)}$$

which checks with (q). Note that the distribution of $\bar{J}$ and $\bar{H}$, as found in part (a), are not required to find the total force in this problem. Even more, (x) is not limited to δ << x block dimension, while the detailed integration is. Note: We have made use of the rule for products, namely of:

$$a(t) = \mathrm{Re}[Ae^{j\omega t}] = \frac{Ae^{j\omega t} + A^*e^{-j\omega t}}{2}$$

$$b(t) = \mathrm{Re}[Be^{j\omega t}] = \frac{Be^{j\omega t} + B^*e^{-j\omega t}}{2}$$

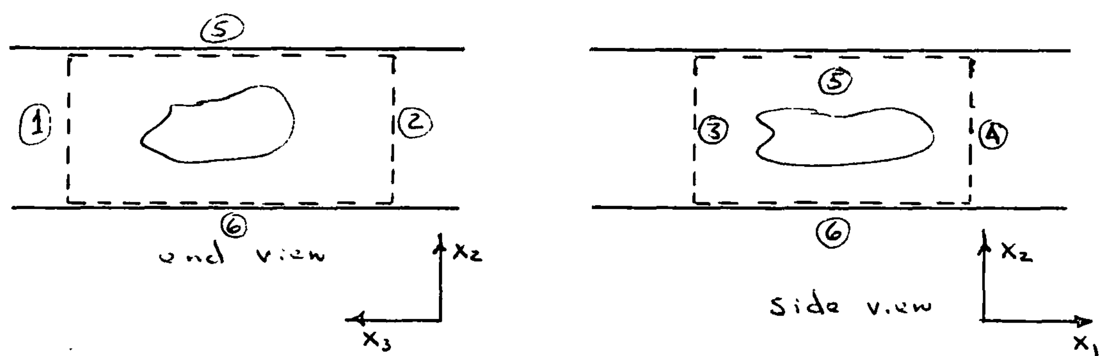
then

$$a(t)b(t) = \frac{AB^* + A^*B}{4} + \frac{ABe^{2j\omega t} + A^*B^*e^{-2j\omega t}}{4}$$

$$= \underbrace{\mathrm{Re}[\frac{AB^*}{2}]}_{\text{avg. value}} + \underbrace{\mathrm{Re}[\frac{AB}{2} e^{2j\omega t}]}_{\text{time varying part}}$$

PROBLEM 8.18

Choose the surface shown in the figure.



$$f_1 = \int T_{1j}n_j da = \int_{3,4} T_{11}n_1 da + \int_{5,6} T_{12}n_2 da + \int_{1,2} T_{13}n_3 da \qquad \text{(a)}$$

Since the plates are perfectly conducting, $E_1 = 0$ at surfaces (5) and (6) and hence $T_{12} = 0$ on surfaces (5) and (6). Surfaces (1), (2), (3) and (4) are far from the body so

PROBLEM 8.18 (Continued)

$$\bar{E} = \frac{V_o}{d}\bar{i}_z \tag{b}$$

at each of them, and thus, on surfaces (1) and (3), $T_{13} = 0$. Therefore,

$$f_1 = -\int_{(3)} T_{11}da_3 + \int_{(4)} T_{11}da_4 \tag{c}$$

$$T_{11}^{(3)} = T_{11}^{(4)} = -\frac{\varepsilon_o}{2}\left(\frac{V_o}{d}\right)^2 \tag{d}$$

and $a_3 = a_4$ (areas). Hence,

$$f_1 = 0 \tag{e}$$

PROBLEM 8.19

Part a

Since the system is electrically linear,

$$\bar{B} = \bar{B}_\ell + \bar{B}_r \tag{a}$$

where $\bar{B}_\ell$ and $\bar{B}_r$ are respectively the fields from the left and right wires. The force on a unit length of the right wire is

$$\bar{f}_r = \int \bar{J}_r \times \bar{B}\,da = \int \bar{J}_r \times \bar{B}_\ell\,da + \int \bar{J}_r \times \bar{B}_r\,da \tag{b}$$

but,

$$\int \bar{J}_r \times \bar{B}_r\,da = 0 \tag{c}$$

ane hence,

$$\bar{f}_r = \bar{J}_r \times \bar{B}_\ell\,da \tag{d}$$

Since, we don't need the fields near the wire,

$$\bar{B}_\ell \simeq \frac{\mu_o I}{2\pi}\left[\frac{x_2\bar{i}_1 - (x_1+a)\bar{i}_2}{(x_1+a)^2 + x_2^2}\right] \tag{e}$$

$$\bar{B}_r \simeq \frac{\mu_o I}{2\pi}\left[\frac{-x_2\bar{i}_1 + (x_1-a)\bar{i}_2}{(x_1-a)^2 + x_2^2}\right] \tag{f}$$

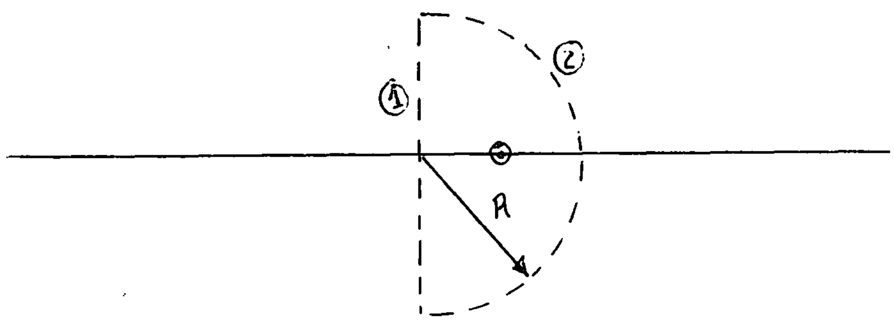
Hence,

$$\bar{f}_r = \bar{J}_r \times \bar{B}_\ell da \simeq I\bar{i}_3 \times \bar{B}_\ell\ (x_1=a,\ x_2=0) \tag{g}$$

$$\bar{f}_r = \frac{\mu_o I^2}{2\pi}\frac{(2a)\bar{i}_1}{(2a)^2} = \frac{\mu_o I^2}{4\pi a}\bar{i}_1 \tag{h}$$

PROBLEM 8.19 (Continued)

Part b



Along the symmetry plane of the surface shown in the figure

$$\bar{B} = \frac{\mu_o I}{2\pi} \frac{(-2a)}{(a^2+x_2^2)} \bar{i}_2 \tag{i}$$

The terms of T_{ij} go as B^2, but $B^2 \alpha \frac{1}{R^2}$ and the surface area goes as $2\pi R$ on surface (2), hence the contributions of the stress tensor will vanish on surface (2) as $R \to \infty$; we need only compute the integral on surface (1). Because $H_1 = 0$ in the plane $x_1 = 0$

$$f_1 = \int -T_{11} da = \int_{-\infty}^{+\infty} \frac{\mu_o H_2^2}{2} dx_2$$

$$= \frac{\mu_o}{2} \left(\frac{Ia}{\pi}\right)^2 \int_{-\infty}^{+\infty} \frac{dx_2}{(a^2+x_2^2)^2} \tag{j}$$

Solving this integral, we find

$$f_1 = \frac{\mu_o I^2}{4\pi a} \tag{k}$$

also

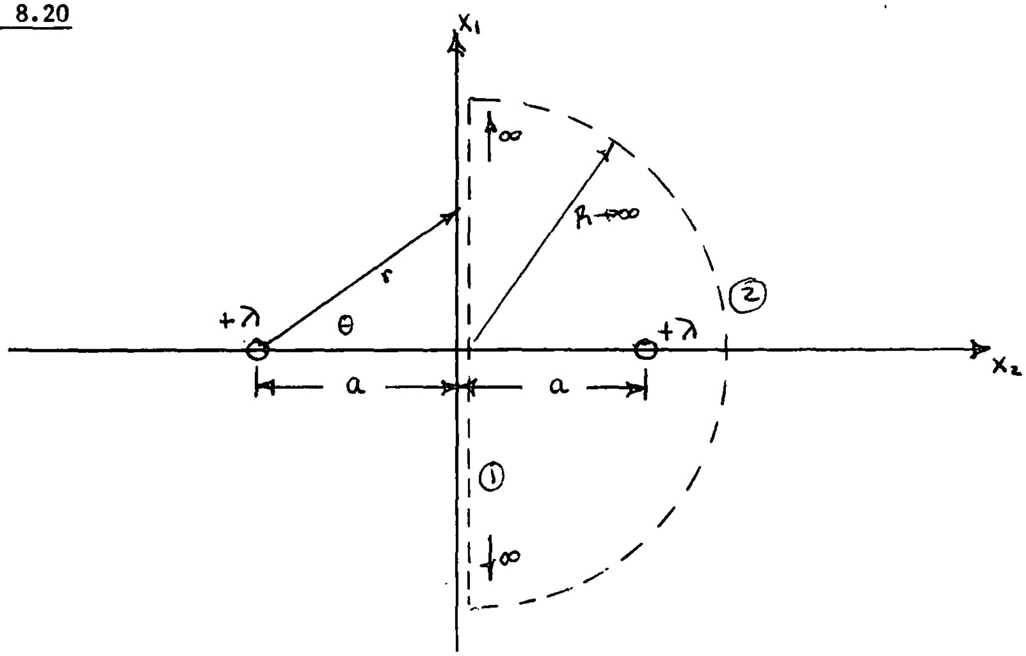
$$f_2 = f_3 = 0 \tag{ℓ}$$

since

$$T_{21} = T_{31} = 0 \tag{m}$$

and hence the total force is that of (k) and it agrees with that determined in part (a).

PROBLEM 8.20



Part a

Use the contour indicated in the figure. At infinity the fields will go to zero, and hence there will be no contribution to the force from the semicircular part of the area, i.e. surface (2).

Along the line $x_2 = 0$, $E_2 = 0$ by symmetry and

$$E_1 = \frac{2}{\varepsilon_o}\left(\frac{\lambda}{2\pi r}\right)\sin\theta \tag{a}$$

$$r^2 = a^2 + x_1^2 \tag{b}$$

$$\sin\theta = \frac{x_1}{r} = \frac{x_1}{\sqrt{a^2+x_1^2}} \tag{c}$$

Hence

$$E_1 = \frac{\lambda}{\varepsilon_o\pi}\ \frac{x_1}{a^2+x_1^2} \tag{d}$$

$$f_2 = \int T_{2j}n_j da = \int_{(1)} T_{21}n_1 da + \int_{(1)} T_{22}n_2 da + \int_{(1)} T_{23}n_3 da \tag{e}$$

first and last integrals = 0, $\bar{n}_1$ and $\bar{n}_3 = 0$ on surface 1

$$T_{22} = -\frac{\varepsilon_o}{2}E_1^2 = -\frac{\varepsilon_o}{2}\left(\frac{\lambda^2}{\varepsilon_o^2\pi^2}\right)\frac{x_1^2}{(a^2+x_1^2)^2} \tag{f}$$

PROBLEM 8.20 (Continued)

Thus

$$f_2 = -2\frac{\lambda^2}{2\varepsilon_o\pi^2}\int_o^\infty \frac{x_1^2\,dx_1}{(a^2+x_1^2)^2} \tag{g}$$

$$f_2 = \frac{\lambda^2}{4\pi\varepsilon_o a} \tag{h}$$

Part b

From electrostatics,

$$\bar{f} = \lambda\bar{E}$$

From the figure, we see that

$$\bar{E}(x_2=a) = \frac{\lambda}{2\pi\varepsilon_o(2a)}\,\bar{i}_2 \tag{i}$$

Hence,

$$\bar{F} = \frac{\lambda^2}{4\pi\varepsilon_o a}\,\bar{i}_2 \tag{j}$$

which is the same as we obtained using the stress tensor - (see equation (h)).

PROBLEM 8.21

Part a

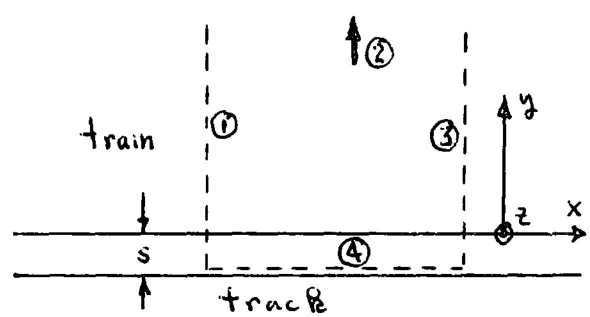
From Eq. 8.1.11,

$$T_{ij} = \begin{bmatrix} \frac{1}{2\mu_o}(B_x^2-B_y^2) & \frac{B_xB_y}{\mu_o} & 0 \\ \frac{B_xB_y}{\mu_o} & \frac{1}{2\mu_o}(-B_x^2+B_y^2) & 0 \\ 0 & 0 & \frac{1}{2\mu_o}(-B_x^2-B_y^2) \end{bmatrix} \tag{a}$$

where the components of $\bar{B}$ are given in the problem.

Part b

The appropriate surface of integration, which is fixed with respect to the fixed frame, is shown in the figure. We compute the time average force, and hence contributions from surfaces (1) and (3) cancel. Fields go to zero on surface (2), which is at $y\to\infty$. Thus, there remains the stress on surface (4). The time average value of the surface force density $\bar{T}$



PROBLEM 8.21 (Continued)

is independent of x. Hence,

$$T_y = - \langle T_{yy}(y=0) \rangle \tag{b}$$

$$T_y = - \frac{1}{2\mu_o} \langle -B_x^2 + B_y^2 \rangle \tag{c}$$

Observe that

$$\langle \text{Re}\, \hat{A}\, e^{-jkUt}\, \text{Re}\, \hat{B}\, e^{-jkUt} \rangle \equiv \frac{1}{2}\, \text{Re}\, \hat{A}\, \hat{B}^* \tag{d}$$

where $\hat{B}^*$ is complex conjugate of $\hat{B}$, and (c) becomes

$$T_y = - \frac{1}{4\mu_o}\, \text{Re}\Big\{-(\mu_o K_o e^{jkx})(\mu_o K_o e^{-jkx}) + \frac{(-jk\mu_o K_o)}{\alpha}\, e^{jkx}\, \frac{(jk\mu_o K_o)}{\alpha^*}\, e^{-jkx}\Big\} \tag{e}$$

$$= \frac{\mu_o K_o^2}{4}\left(1 - \frac{k^2}{\alpha\alpha^*}\right) \tag{f}$$

Finally, use the given definition of α to write (f) as

$$T_y = \frac{\mu_o K_o^2}{4}\left[1 - \frac{1}{\sqrt{1 + \left(\frac{\mu_o \sigma U}{k}\right)^2}}\right] \tag{g}$$

Note that T_y is positive so that the train is supported by the magnetic field. However, as $U \to 0$ (the train is stopped) the levitation force goes to zero.

Part c

For the force per unit area in the x direction;

$$T_x = - \frac{1}{2\mu_o} \langle B_x B_y(y=o) \rangle \tag{h}$$

$$= - \frac{1}{2\mu_o}\, \text{Re}\left[\mu_o K_o e^{jkx} \left(\frac{jk\mu_o}{\alpha^*}\right) K_o e^{-jkx}\right] \tag{i}$$

Thus

$$T_x = - \frac{\mu_o K_o^2}{2\left[1 + \left(\frac{\mu_o \sigma U}{k}\right)^2\right]^{1/2}}\, \text{Re}\, j \sqrt{1 - j\left(\frac{\mu_o \sigma U}{k}\right)} \tag{j}$$

As must be expected, the force on the train in the x directions vanishes as $U \to 0$. Note that in any case the force always tends to retard the motion and hence could hardly be used to propel the train.

The identity $\sin(\theta/2) = \pm\sqrt{(1 - \cos\theta)/2}$ is helpful in reducing (j) to the form

$$T_x = \frac{-\mu_o K_o^2}{2\left[1 + \left(\frac{\mu_o \sigma U}{k}\right)^2\right]^{1/2}} \sqrt{\frac{1}{2}\left(\sqrt{1 + \left(\frac{\mu_o \sigma U}{k}\right)^2} - 1\right)} \tag{k}$$

PROBLEM 8.22

This problem makes the same point as Probs. 8.16 and 8.17, with the additional effect of material motion included. Regardless of the motion, with the current constrained as given, the magnetic field intensity is zero to the right of the block and uniform into the paper (z direction) to the left of the block, where

$$\bar{H} = \bar{i}_z \frac{I_o}{d} \qquad \text{(a)}$$

The only contribution to an integration of the stress tensor over a surface enclosing the block is on the left surface. Thus

$$f_x = ds\, T_{xx} = -\, ds\, \frac{1}{2} \mu_o H_z^2 \qquad \text{(b)}$$

$$= -\frac{ds}{2} \mu_o \left(\frac{I_o}{d}\right)^2 \qquad \text{(c)}$$

The magnetic force is to the right and independent of the magnetic Reynolds number.

PROBLEM 8.23

In plane geometry, a knowledge of the charge on the upper plate is equivalent to knowing the electric field intensity on the surface of the plate. Thus, the surface charge density on the upper plate is

$$\sigma_f = \frac{1}{A} \int_o^t I_o \cos\omega t\, dt = \frac{I_o}{A\omega} \sin \omega t \qquad \text{(a)}$$

and

$$E_x(x=a) = -\frac{\sigma_f}{\varepsilon_o} = -\frac{I_o}{A\varepsilon_o \omega} \sin \omega t \qquad \text{(b)}$$

Now, we enclose the upper plate with a surface just outside the electrode surface. The only contribution to the integration of Eq. 8.1.17 using the stress tensor 8.3.10 is

$$f_x = -\, A T_{xx}(x=a) = -\frac{A\varepsilon_o}{2} E_x^2(x=a) \qquad \text{(c)}$$

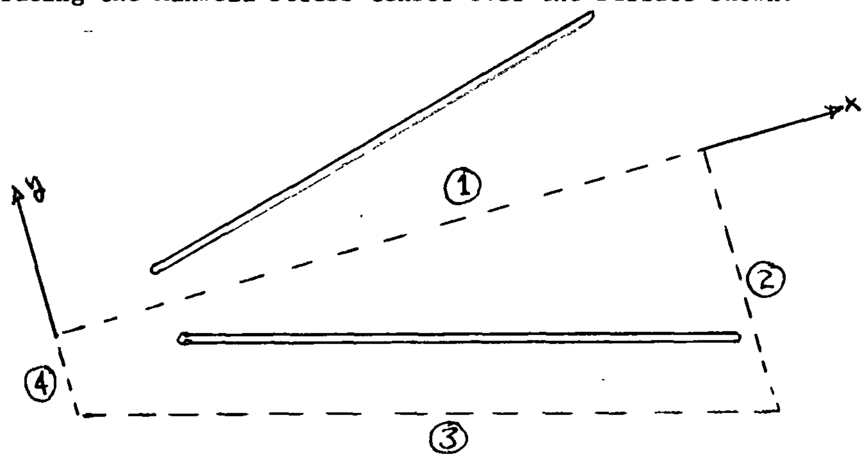
which we can evaluate from (b) as

$$f_x = -\frac{A\varepsilon_o}{2} \left(\frac{I_o}{A\varepsilon_o \omega}\right)^2 \sin^2 \omega t \qquad \text{(d)}$$

The force of attraction between the conducting slab and upper electrode is not dependent on σ_1 or σ_o.

PROBLEM 8.24

The force on the lower electrode in the x direction is zero, as can be seen by integrating the Maxwell stress tensor over the surface shown.



The fields are zero on surfaces (2), (3) and (4). Hence, the total force per unit depth into the paper is

$$f_x = \int_o^\infty T_{xy}dx \tag{a}$$

where contributions from surfaces in the plane of the paper cancel because the problem is two-dimensional. Moreover, by symmetry the electric field intensity on the surface (1), even in the fringing regions, is in the y direction only and $T_{xy} = \varepsilon_o E_x E_y$ in (a) is zero. Thus, the total x directed force is zero.

PROBLEM 8.25

The force density in the dielectric slab is Eq. 8.5.45. Not only is the first term zero, but because the block moves as a rigid body (we are interested only in the net force giving rise to a rigid body displacement) the last term, which originates in changes in volume of the material, does not give a contribution. Hence, the force density is

$$\bar{F} = -\frac{1}{2}\bar{E}\cdot\bar{E}\nabla\varepsilon \tag{a}$$

and the stress tensor is

$$T_{ij} = \varepsilon E_i E_j - \frac{\delta_{ij}}{2}\varepsilon E_k E_k \tag{b}$$

Note that, from (a), the force density in the x_1 direction is confined to the right edge of the block, where it acts as a surface force. Thus, we obtain the total force by simply integrating over a surface that encloses the right edge;

PROBLEM 8.25 (Continued)

$$f_1 = aD\left[-\frac{1}{2}\varepsilon_o(E_2^a)^2 + \frac{1}{2}\varepsilon(E_2^b)^2\right] \tag{c}$$

where a and b are to the right and left of the right edge of the slab. Also $E_2^a = E_2^b = -V_o/a$. Hence (c) becomes

$$f_1 = \frac{aD}{2}\left(\frac{V_o}{a}\right)^2(\varepsilon-\varepsilon_o) \tag{d}$$

The force acts to the right, as could be computed by the energy method.

PROBLEM 8.26

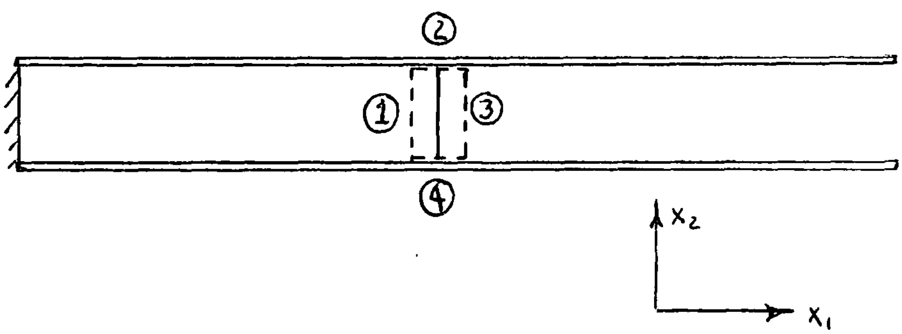
Part a

The force density for polarizable materials is:

$$\bar{F} = -\frac{1}{2}\bar{E}\cdot\bar{E}\,\nabla\varepsilon + \frac{1}{2}\nabla\left(\bar{E}\cdot\bar{E}\,\rho\frac{\partial\varepsilon}{\partial\rho}\right) \tag{a}$$

The second term on the right side represents electrostriction. Note that this is a case where the material volume must change, and hence the effect of electrostriction is important. Since free space and the elastic bulk are homogeneous, changes in permittivity and $\partial\varepsilon/\partial\rho$ occur only at the boundary where the permittivity is discontinuous. The upper and lower elastic bulk surfaces are constrained by the plates. Thus only the x_1 component of force is pertinent. Since the left-hand edge is fixed, any stress arising from the discontinuity in permittivity at that boundary is counterbalanced by the rigidity of the wall. Therefore, all of the force arises at the right-hand boundary which is free to move.

The closed surface of integration is shown in the figure.



PROBLEM 8.26 (Continued)

$$T_{ij} = \varepsilon E_i E_j - \frac{1}{2}\delta_{ij}\,[\varepsilon - \rho\frac{\partial\varepsilon}{\partial\rho}]E_k E_k \qquad \text{(b)}$$

Since $a/c \ll 1$ and $b \sim \frac{1}{2}a$, the field at the dielectric interface is essentially uniform.

$$\bar{E} = -\bar{i}_2\,\frac{V_o}{a} \qquad \text{(c)}$$

The relevant components of the stress tensor are:

$$T_{11} = -\frac{\varepsilon}{2}E_2^2 + \frac{1}{2}\rho\frac{\partial\varepsilon}{\partial\rho}E_2^2 \qquad \text{(d)}$$

$$T_{12} = \varepsilon E_1 E_2 = 0 \qquad \text{(e)}$$

$$f_1 = \int_{(1)(3)} T_{11}n_1\,da + \int_{\cancel{(2)}(4)} \cancelto{0}{T_{12}n_2}\,da \qquad \text{(f)}$$

Hence

$$f_1 = \int_{(3)} T_{11}\,da_3 - \int_{(1)} T_{11}\,da_1$$

$$= -\frac{\varepsilon_o}{2}\left(\frac{V_o}{a}\right)^2(aD) - \left[-\frac{\varepsilon_1}{2}\left(\frac{V_o}{a}\right)^2(aD) + \frac{\rho}{2}\frac{\partial\varepsilon}{\partial\rho}\left(\frac{V_o}{a}\right)^2(aD)\right] \qquad \text{(g)}$$

Thus;

$$f_1 = \frac{(\varepsilon_1 - \varepsilon_o)V_o^2 D}{2a} - \frac{\rho}{2}\frac{\partial\varepsilon}{\partial\rho}\left(\frac{V_o^2 D}{a}\right) \qquad \text{(h)}$$

Part b

In order to use lumped parameter energy methods, the charge on the upper plate will be found. The permittivity of the dielectric bulk is a junction of the displacement of the right-hand edge. That is, if mass conservation is to hold,

$$\rho_o\,abD = (\rho_o + \Delta\rho)aD(b+\xi) \qquad \text{(i)}$$

where

$$\rho = \rho_o + \Delta\rho,\ \Delta\rho = 0 \text{ if } \xi = 0 \qquad \text{(j)}$$

Thus, if $\Delta\rho \ll \rho_o$ and $\xi \ll b$, to first order

$$\Delta\rho = -\rho_o\frac{\xi}{b} \qquad \text{(k)}$$

(see Eqs. 8.5.9 and 8.5.10)

Furthermore, to first order, using a Taylor series,

PROBLEM 8.26 (Continued)

$$\varepsilon = \varepsilon_1 + \frac{\partial\varepsilon}{\partial\rho}\Delta\rho = \varepsilon_1 - \frac{\rho_o}{b}\frac{\partial\varepsilon}{\partial\rho}\xi \tag{ℓ}$$

Also, the electric field will be assumed as uniform everywhere between the plates. Hence; in the block

$$\bar{D} = -\bar{i}_2\left\{\frac{V_o}{2}\left[\varepsilon_1 + \frac{\partial\varepsilon}{\partial\rho}\Delta\rho\right]\right\} \tag{m}$$

to the right of the block

$$\bar{D} = -\bar{i}_2\frac{V_o}{a}\varepsilon_o \tag{n}$$

By employing Gauss's law, we find the charge on the upper plate as:

$$q = \left(\frac{V_o}{a}\right)\left\{\varepsilon_1 - \frac{\rho_o}{b}\frac{\partial\varepsilon}{\partial\rho}\xi\right\}(b+\xi)D + \varepsilon_o\left(\frac{V_o}{a}\right)(c-b-\xi)D \tag{o}$$

$$\int dw_e' = \int q\,dv + \int f_e\,dx \tag{p}$$

integrating we find

$$w_e' = \frac{1}{2}\left(\frac{V_o^2}{a}\right)\left[\varepsilon_1 - \frac{\rho_o}{b}\frac{\partial\varepsilon}{\partial\rho}\xi\right](b+\xi)D + \frac{1}{2}\frac{V_o^2}{a}(c-b-\xi)D \tag{q}$$

Thus,

$$f_e = \left.\frac{\partial w_e'}{\partial\xi}\right|_{v=V_o} = \frac{(\varepsilon_1-\varepsilon_o)V_o^2D}{2a} - \frac{1}{2}\frac{V_o^2}{a}\rho_o D\frac{\partial\varepsilon}{\partial\rho} \tag{r}$$

Second order terms have been dropped in the co-energy expression (alternatively, first order terms can be dropped in the force expression).

Part c

If the result of part (a) is written for $\rho = \rho_o + \Delta\rho$, where $\rho_o >> \Delta\rho$, then the answers to part (a) and (b) are identical to first order. This should be expected since the lumped parameter approach assumed a value for permittivity which was correct only to first order.

PROBLEM 8.27

The surface force density is

$$T_m = [T^a_{mn} - T^b_{mn}]n_n \tag{a}$$

For this problem, we require $m = 1$ and $\bar{n} = \bar{i}_2$. Thus

$$T_1 = (T^a_{12} - T^b_{12}) \tag{b}$$

From Eq. 8.5.46,

PROBLEM 8.27 (Continued)

$$T_1 = \varepsilon_o E_1^a E_2^a - \varepsilon E_1^b E_2^b \tag{c}$$

Note that $E_2^a = E_2^b$ (see Eq. 6.2.31). Moreover, because there is no free charge $\varepsilon_o E_1^a = \varepsilon E_1^b$ (see Eq. 6.2.33). Thus, (c) becomes

$$T_1 = E_2^a[\varepsilon_o E_1^a - \varepsilon E_1^b] = 0 \tag{d}$$

That the shear surface force density is zero in the x_3 direction follows the same reasoning.

PROBLEM 8.28

The force density, Eq. 8.5.45, written in component form, is

$$F_i = E_i \frac{\partial \varepsilon E_j}{\partial x_j} - \frac{1}{2} E_k E_k \frac{\partial \varepsilon}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\frac{1}{2} E_k E_k \frac{\partial \varepsilon}{\partial \rho}\rho\right) \tag{a}$$

The first term can be rewritten as two terms, one of which is in the desired form

$$F_i = \frac{\partial}{\partial x_j}(\varepsilon E_i E_j) - \varepsilon E_j \frac{\partial E_i}{\partial x_j} - \frac{1}{2} E_k E_k \frac{\partial \varepsilon}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\frac{1}{2} E_k E_k \frac{\partial \varepsilon}{\partial \rho}\rho\right) \tag{b}$$

Because $\nabla \times \bar{E} = 0$, $\partial E_i/\partial x_j = \partial E_j/\partial x_i$, so that the second term can be rewritten and combined with the third. (Note the j is a dummy summation variable.)

$$F_i = \frac{\partial}{\partial x_j}(\varepsilon E_i E_j) - \frac{1}{2}\frac{\partial}{\partial x_i}(\varepsilon E_k E_k) + \frac{\partial}{\partial x_i}\left(\frac{1}{2} E_k E_k \frac{\partial \varepsilon}{\partial \rho}\rho\right) \tag{c}$$

Finally, we introduce δ_{ij} (see Eq. 8.1.7) to write (c) in the required form

$$F_i = \frac{\partial T_{ij}}{\partial x_j} \tag{d}$$

where

$$T_{ij} = \varepsilon E_i E_j - \frac{1}{2}\delta_{ij} E_k E_k \left(\varepsilon - \rho \frac{\partial \varepsilon}{\partial \rho}\right) \tag{e}$$

This is identical to Eq. 8.5.46.