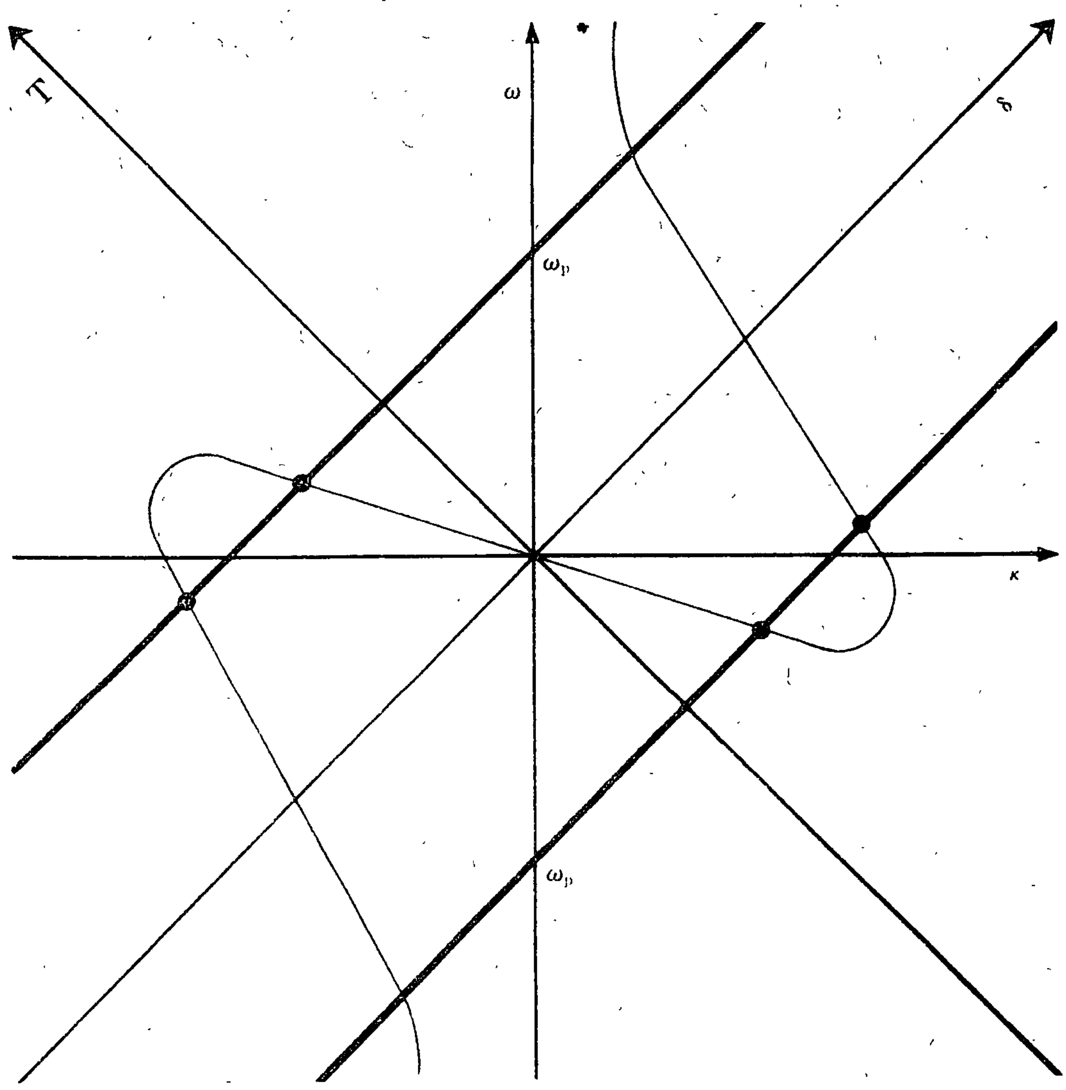
SOLUTIONS MANUAL FOR

ELECTROMECHANICAL DYNAMICS

PART II: Fields, Forces, and Motion

HERBERT H. WOODSON JAMES R. MELCHER



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ELECTROMECHANICAL DYNAMICS

Part II: Fields, Forces, and Motion

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PREFACE TO: SOLUTIONS MANUAL FOR
ELECTROMECHANICAL DYNAMICS, PART II:
FIELDS, FORCES, AND MOTION

This manual presents in an informal format solutions to the problems found at the ends of chapters in Part II of the book, Electromechanical Dynamics. It is intended as an aid for instructors, and in special circumstances for use by students. We have included a sufficient amount of explanatory material that solutions, together with problem statements, are in themselves a teaching aid. They are substantially as found in our records for the course 6.06, as taught at M.I.T. over a period of several years.

Typically, the solutions were originally written up by graduate student tutors, whose responsibility it was to conduct one-hour tutorials once a week with students in pairs. These tutorials focused on the homework, with the problem solutions reproduced and given to the students upon receipt of their own homework solutions.

It is difficult to give proper credit to all of those who contributed to these solutions, because the individuals involved range over teaching assistants, instructors, and faculty, who have taught the material over a period of more than four years. However, significant contributions were made by D.S. Guttman, Dr. K.R. Edwards, M. Zahn, F.A. Centanni, and T.B. Jones, Jr. The manuscript was typed by Mrs. Barbara Morton, whose patience and expertise were invaluable.

H.H. Woodson

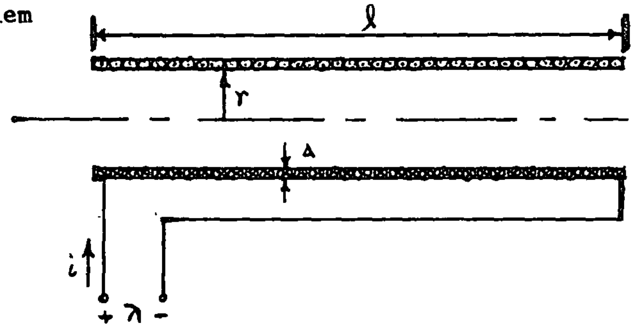
J.R. Melcher

Cambridge, Massachusetts

September, 1968

PROBLEM 7.1

It is the purpose of this problem to illustrate the limitations inherent to common conductors in achieving long magnetic time constants. (Diffusion times.) For convenience in making this point consider the solenoid shown with



ℓ = length

Δ = cross-sectional dimensions of single layer of wire (square-cross section).

r = radius ($r >> \Delta$) but $r << \ell$.

Then there are ℓ/Δ turns, each having a length $2\pi r$, and the total d-c resistance is directly proportional to the length and inversely proportional to the area and electrical conductivity σ.

$$R = \frac{2\pi r}{\sigma(\Delta^2)} \left(\frac{\ell}{\Delta}\right)$$

The H field in the axial direction, by Ampere's law, is $H = \frac{i}{\Delta}$ and the flux linked by one turn is $\mu_o H(\pi R^2)$ so that

$$\lambda = \mu_o H(\pi r^2)\left(\frac{\ell}{\Delta}\right) = \mu_o(\pi r^2)\, \frac{\ell}{\Delta^2}\, i$$

and it follows that

$$L = \mu_o(\pi r^2)\left(\frac{\ell}{\Delta^2}\right)$$

Finally, the time constant is

$$\frac{L}{R} = \frac{1}{2}\, \mu_o r\Delta\sigma$$

Thus, the diffusion time (see Eq. 7.1.28) is based on an equivalent length $\sqrt{r\Delta}$. Consider using copper with

σ = 5.9×10^7 mhos/m

Δ = 10 m

and find Δ required to give $L/R = 10^2$

$$\Delta = 2\left(\frac{L}{R}\right) \frac{1}{\mu_o r\sigma} = \frac{(200)}{(4\pi x 10^{-7})(10)(5.9x10^7)}$$

$$= 2.7 \times 10^{-1}\text{m or } 27 \text{ cm}$$

PROBLEM 7.1 (Continued)

Note that to satisfy the condition that $\ell >> r$, the length must be greater than 10 meters also. The coil is larger than the average class-room! Of course, if magnetic materials are used, the dimensions of the coil can be reduced considerably, but long L/R time constants are difficult to obtain on a laboratory scale with ordinary conductors.

PROBLEM 7.2

Part a

Our solution will parallel the one in the text, only now the $\bar{B}$ field will be trapped in the slab until it diffuses away. The fundamental equations are

$$\nabla\times\bar{B} = \mu\bar{J} = \mu\sigma\bar{E}; \quad \nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t}$$

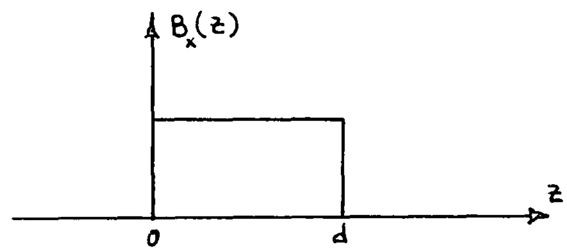
$$\nabla\times\nabla\times\bar{B} = \nabla(\nabla\cdot\bar{B}) - \nabla^2\bar{B} = \mu\sigma\nabla\times\bar{E} = -\mu\sigma\frac{\partial\bar{B}}{\partial t}$$

Because $\nabla\cdot\bar{B} = 0$,

$$\nabla^2\bar{B} = \mu\sigma\frac{\partial\bar{B}}{\partial t}$$

or in one dimension

$$\frac{1}{\mu\sigma}\frac{\partial^2 B_x}{\partial z^2} = \frac{\partial B_x}{\partial t} \qquad \text{(a)}$$



at $t = 0^+$

$$B_x = \begin{cases} 0 & z < 0 \\ B_o & 0 < z < d \\ 0 & z > d \end{cases}$$

This suggests that between 0 and d, we can write $B_x(z)$ as

$$B_x(z) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi z}{d}\right) \qquad 0 < z < d$$

To solve for the coefficients a_n, we take advantage of the orthogonal property of the sine functions.

$$\int_o^d B_x(z)\sin\left(\frac{m\pi z}{d}\right)dz = \sum_{n=1}^{\infty}\int_o^d a_n \sin\left(\frac{n\pi z}{d}\right)\sin\left(\frac{m\pi z}{d}\right)dz$$

But

$$\int_o^d B_x(z)\sin\left(\frac{m\pi z}{d}\right)dz = \int_o^d B_o \sin\left(\frac{m\pi z}{d}\right)dx = \begin{bmatrix} \dfrac{2dB_o}{m\pi} & m \text{ odd} \\ 0 & m \text{ even} \end{bmatrix}$$

PROBLEM 7.2 (Continued)

Also

$$\int_0^d a_n \sin\left(\frac{n\pi z}{d}\right)\sin\left(\frac{m\pi z}{d}\right)dz = \begin{bmatrix} \frac{a_m d}{2} & n = m \\ 0 & n \neq m \end{bmatrix}$$

Hence,

$$a_m = \begin{cases} \frac{4B_o}{m\pi} & m \text{ odd} \\ 0 & m \text{ even} \end{cases}$$

$$B_x(t=0,z) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} B_o \sin\left(\frac{n\pi z}{d}\right) \qquad 0 < z < d$$

We assume that for $t > 0$, $0 < z < d$

$$B_x(t,z) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} B_o \sin\left(\frac{n\pi z}{d}\right)e^{-\alpha_n t}$$

Plugging into (a) we find that $\frac{1}{\mu\sigma}\cdot\left(\frac{n\pi}{d}\right)^2 = \alpha_n$. Let's define $\tau = \mu\sigma\left(\frac{d}{\pi}\right)^2$ as the fundamental diffusion time. Then

$$B_x(t,z) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} B_o \sin\left(\frac{n\pi z}{d}\right)e^{-n^2 t/\tau} \qquad \begin{matrix} 0 < z < d \\ t > 0 \end{matrix}$$

Part b

$$\bar{J} = \frac{\nabla\times\bar{B}}{\mu} = \frac{1}{\mu}\frac{\partial B_x}{\partial z}\bar{i}_y = \frac{4B_o}{\mu_o d}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \cos\left(\frac{n\pi z}{d}\right)e^{-n^2 t/\tau}\,\bar{i}_y \qquad \begin{matrix} 0<z<d \\ t>0 \end{matrix}$$

PROBLEM 7.3

Part a

If we neglect the capacitance of the block, the current we put in at t=0 will have to return by means of the block. This can be seen from the magnetic field system equation

$$\nabla\times\bar{H} = \bar{J} \qquad \text{(a)}$$

which implies

$$\nabla\cdot\bar{J} = 0 \qquad \text{(b)}$$

or "what goes in must come out".

If the current penetrated the block at $t=0^+$ there would be a magnetic field within the block at $t=0^+$, a situation we cannot allow since some time must elapse (relative to the diffusion time) before the fields in the block can change significantly.

PROBLEM 7.3 (Continued)

We conclude that the source current returns as a surface current down the left side of the block. This current must be

$$K_y = -I_o/D \tag{c}$$

where y is the upwards vertical direction. The current loop between x = - L and x = 0 thus provides a magnetic field

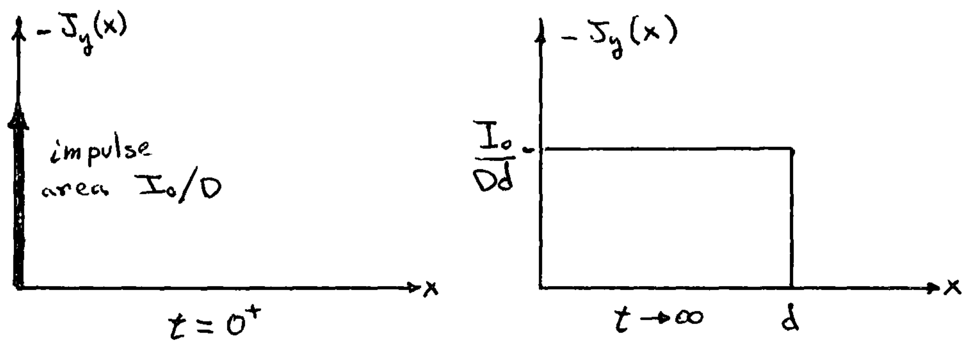
$$H_z(t=0^+) = \begin{cases} -I_o/D & -L<x<0 \\ 0 & 0<x \end{cases} \tag{d}$$

where z points out of sketch.

Part b

As $t \to \infty$ the system will reach a static state with input current I_o/D per unit length. The current will return uniformly through the block. Hence,

$$J_y(x) = -\frac{I_o}{Dd} \tag{e}$$



Part c

As a diffusion problem this system is very much like the system of Fig. 7.1.1 of the text except for the fact that here diffusion occurs on only one side of the block instead of two. This suggests a fundamental diffusion time constant of

$$\tau = \frac{\mu_o \sigma (2d)^2}{\pi^2} \tag{f}$$

where we have replaced the term d^2 by $(2d)^2$ in Eq. 7.1.28 of the text.

PROBLEM 7.4

Part a

This is a magnetic field system characterized by a diffusion equation. With $B_z = \text{Re}\hat{B}_z(x)e^{j\omega t}$,

PROBLEM 7.4 (Continued)

$$\frac{1}{\mu\sigma}\frac{d^2\hat{B}_z}{dx^2} = j\omega B_z \tag{a}$$

Let $\hat{B}_z(x) = \hat{B}_o e^{\alpha x}$, then

$$\alpha^2 = j\omega\mu\sigma \tag{b}$$

or

$$\alpha = \pm\frac{1}{\delta}(1 + j), \ \delta = \sqrt{\frac{2}{\omega\mu\sigma}} \tag{c}$$

The boundary conditions are

$$\hat{B}_z(x=0) = -\mu\hat{i}/D \tag{d}$$

$$\hat{B}_z(x\to\infty) \to 0$$

which means that we use only the (-) sign

$$\hat{B}_z(x,t) = -\text{Re}\ \frac{\mu\hat{i}}{D}\ e^{-x/\delta}\ e^{j(\omega t-\frac{x}{\delta})} \tag{e}$$

Part b

$$\nabla\times\bar{B} = \mu\bar{J} \tag{f}$$

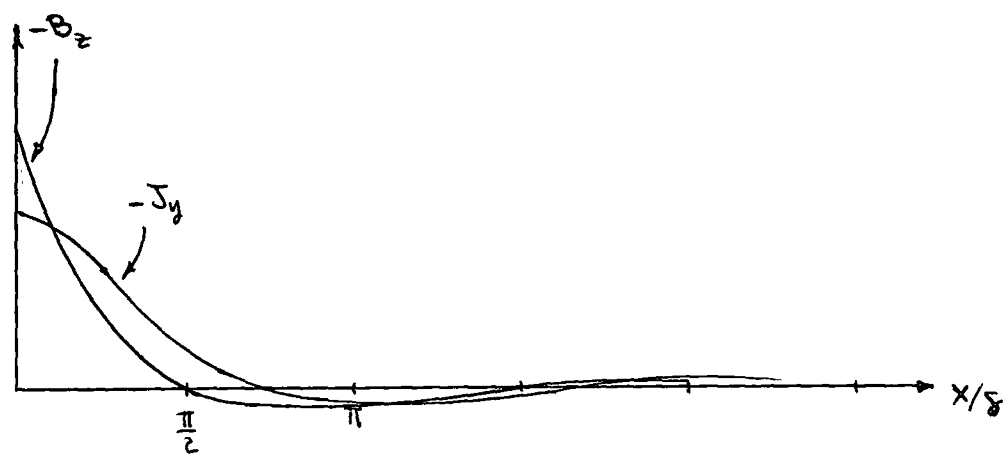
or

$$\frac{\partial B_z}{\partial x} = -\mu J_y \tag{g}$$

so that

$$J_y = -\text{Re}\frac{\hat{i}}{D}\ \frac{(1+j)}{\delta}\ e^{-\frac{x}{\delta}}\ e^{j(\omega t-\frac{x}{\delta})} \tag{h}$$

Part c



PROBLEM 7.4 (Continued)

Part d

The electric field is given by

$$\nabla\times\bar{E} = \vec{i}_z \frac{\partial E_y}{\partial x} = -j\omega\hat{B}_z\vec{i}_z \tag{i}$$

$$E_y(x=0) = -\frac{\omega\delta\mu}{2D}(1+j)\hat{i} \tag{j}$$

Faraday's law (Eq. 1.1.23, Table 1.2, Appendix E) written for a counter-clockwise contour through the source and left edge of the block, gives

$$\hat{V} + \hat{E}_y d = \frac{j\omega\mu_o(Ld)}{D}\hat{i} \tag{k}$$

where from (j)

$$\hat{E}_y d = -\frac{1}{\sigma}\left(\frac{d}{\delta}\right)\frac{1}{D}(1+j)\hat{i} \tag{ℓ}$$

Hence, assuming that $\hat{V} = \hat{i}[R(\omega)+j\omega L(\omega)]$, (don't confuse the L's)

$$R(\omega) = \frac{1}{\sigma}\left(\frac{d}{D}\right)\frac{1}{\delta} = \frac{d}{D}\sqrt{\frac{\omega\mu}{2\sigma}} \tag{m}$$

$$L(\omega) = \frac{\mu_o Ld}{D} + \frac{d}{D}\sqrt{\frac{\mu}{2\omega\sigma}} \tag{n}$$

Thus, as $\omega\to\infty$ the inductance becomes just that due to the free-space portion of the circuit between x=0 and x=-L. The resistance becomes infinite because the currents crowd to the left edge of the block.

In the opposite extreme, as $\omega\to 0$, the resistance approaches zero because the currents have an infinite x-z area of the block through which to flow. Similarly, the inductance becomes large because the x-y area enclosed by the current paths increases without limit. At low frequencies it would be necessary to include the finite extent of the block in the x direction in the analysis to obtain a realistic estimate of the resistance and inductance.

PROBLEM 7.5

Part a

This is a magnetic field system characterized by a diffusion equation. Place origin of coordinates at left edge of block, x to right and z out of paper. With $B_x = \mathrm{Re}\hat{B}_x(x)e^{j\omega t}$

$$\frac{1}{\mu\sigma}\frac{\partial^2 B_z}{\partial x^2} = j\omega\hat{B}_z \tag{a}$$

PROBLEM 7.5 (Continued)

Let $\hat{B}_z(x) = \hat{B}_o e^{\alpha x}$, then

$$\alpha^2 = j\omega\mu\sigma \tag{b}$$

$$\alpha = \pm \frac{1}{\delta}(1+j), \quad \delta = \sqrt{\frac{2}{\omega\mu\sigma}} \tag{c}$$

The boundary conditions are

$$\hat{B}_z(x=0) = -\mu\frac{\hat{I}}{D} \tag{d}$$

$$\hat{B}_z(x=\ell) = 0 \tag{e}$$

because all of the current $I_o(t)$ is returned through the block. Thus the appropriate linear combination of solutions to satisfy the boundary conditions is

$$B_z(x,t) = \text{Re}\,\frac{\mu\hat{I}}{D}\,\frac{\sinh[\alpha(x-\ell)]e^{j\omega t}}{\sinh(\alpha\ell)} \tag{f}$$

where α is a complex quantity, (c). The current is related to $\hat{B}_z$ by

$$\nabla\times\bar{B} = -\frac{\partial B_z}{\partial x}\bar{i}_y = \mu\bar{J} = \mu J_y\bar{i}_y \tag{g}$$

From (f) and (g),

$$J_y = -\frac{\hat{I}\alpha}{D}\,\frac{\cosh[\alpha(x-\ell)]e^{j\omega t}}{\sinh\alpha\ell} \tag{h}$$

Part b

The time average magnetic force on the block is given by

$$f_x = \text{Re}\left[Dd\int_o^\ell \frac{\hat{J}_y(x)\hat{B}_z^*(x)\,dx}{2}\right] \tag{i}$$

where we have taken advantage of the identity

$$\langle \text{Re}\,\hat{A}e^{j\omega t}\,\text{Re}\hat{B}e^{j\omega t}\rangle = \frac{1}{2}\text{Re}\,\hat{A}\hat{B}^*$$

to integrate the force density $(\bar{J}\times\bar{B})_x$ over the volume of the block. Note that a detailed calculation is required to complete (i), because α in (f) and (h) is complex.

This example is one where the total force is more easily computed using the Maxwell stress tensor. See Probs. 8.16, 8.17 and 8.22 for this approach.

PROBLEM 7.6

As an example of electromagnetic phenomena that occur in conductors at rest we consider the system of Fig. 7.1.1 with the constant-current source and switch replaced by an alternating current source.

$$i(t) = I \cos \omega t \tag{a}$$

We make all of the assumptions of Sec. 7.1.1 and adopt the coordinate system of Fig. 7.1.2. Interest is now confined to a steady-state problem.

The equation that describes the behavior of the flux density in this system is Eq. 7.1.15

$$\frac{1}{\mu_o \sigma} \frac{\partial^2 B_x}{\partial z^2} = \frac{\partial B_x}{\partial t} \tag{b}$$

and the boundary conditions are now, at $z = 0$ and $z = d$,

$$B_x = B_o \cos \omega t = \left[\mathrm{Re}\ B_o e^{j\omega t}\right] \tag{c}$$

where

$$B_o = \frac{\mu_o NI}{w} \tag{d}$$

The boundary condition of (c) coupled with the linearity of (b) lead us to assume a solution

$$B_x = \mathrm{Re}\left[\hat{B}(z) e^{j\omega t}\right] \tag{e}$$

We substitute this form of solution into (b), cancel the exponential factor, and drop the Re to obtain

$$\frac{d^2 \hat{B}}{dz^2} = j\, \mu_o \sigma\, \hat{B} \tag{f}$$

Solutions to this equation are of the form

$$\hat{B}(z) = e^{rz} \tag{g}$$

where substitution shows that

$$r = \pm \sqrt{j\omega\mu_o\sigma} = \pm \sqrt{\frac{\omega\mu_o\sigma}{2}}\ (1+j) \tag{h}$$

It is convenient to define the skin depth δ as (see Sec. 7.1.3a)

$$\delta = \sqrt{\frac{2}{\omega\mu_o\sigma}} \tag{i}$$

We use this definition and write the solution, (g) as

PROBLEM 7.6 (Continued)

$$\hat{B}(z) = c_1 e^{(1+j)\frac{z}{\delta}} + c_2 e^{-(1+j)\frac{z}{\delta}} \tag{j}$$

The boundary conditions at z = 0 and z = d (c) require that

$$B_o = c_1 + c_2$$
$$B_o = c_1 e^{(1+j)d/\delta} + c_2 e^{-(1+j)d/\delta}$$

Solution of these equations for c_1 and c_2 yields

$$C_1 = \frac{B_o\left[1 - e^{-(1+j)\frac{d}{\delta}}\right]}{D} \tag{k}$$

$$C_2 = \frac{-B_o\left[1 - e^{(1+j)\frac{d}{\delta}}\right]}{D} \tag{ℓ}$$

where $D = 2(\cos\frac{d}{\delta}\sinh\frac{d}{\delta} + j\sin\frac{d}{\delta}\cosh\frac{d}{\delta})$

We now substitute (k) and (ℓ) into (j); and, after manipulation, obtain

$$\hat{B}(z) = B_o[f(z) + j\,g(z)] \tag{m}$$

where

$$f(z) = \frac{M}{F}\cos\frac{d}{\delta}\sinh\frac{d}{\delta} + \frac{N}{F}\sin\frac{d}{\delta}\cosh\frac{d}{\delta}$$

$$g(z) = \frac{N}{F}\cos\frac{d}{\delta}\sinh\frac{d}{\delta} - \frac{M}{F}\sin\frac{d}{\delta}\cosh\frac{d}{\delta}$$

$$M = \cos\frac{z}{\delta}\sinh\frac{z}{\delta} + \cos\left(\frac{d-z}{\delta}\right)\sinh\left(\frac{d-z}{\delta}\right)$$

$$N = \sin\frac{z}{\delta}\cosh\frac{z}{\delta} + \sin\left(\frac{d-z}{\delta}\right)\cosh\left(\frac{d-z}{\delta}\right)$$

$$F = \cos^2\frac{d}{\delta}\sinh^2\frac{d}{\delta} + \sin^2\frac{d}{\delta}\cosh^2\frac{d}{\delta}$$

Substitution of (m) into (e) yields

$$B_x = B_m(z)\cos[\omega t + \theta(z)] \tag{n}$$

where

$$B_m(z) = B_o\sqrt{[f(z)]^2 + [g(z)]^2} \tag{o}$$

$$\theta(z) = \tan^{-1}\frac{g(z)}{f(z)} \tag{p}$$

It is clear from the form of (n) that both the amplitude and phase of the flux density vary as functions of z.

To illustrate the nature of the distribution of flux density predicted

PROBLEM 7.6 (Continued)

by this set of equations the maximum flux density is plotted as a function of position for several values of d/δ in the figure. Recalling the definition of the skin depth δ in (i), we realize that for a system of fixed geometry and fixed properties $\frac{d}{\delta} \propto \sqrt{\omega}$, thus, as $\frac{d}{\delta}$ increases, the frequency of the excitation increases. From the curves of the figure we see that as the frequency increases the flux density penetrates less and less into the specimen until at high frequencies ($\frac{d}{\delta} >> 1$) the flux density is completely excluded from the conductor. At very low frequencies ($\frac{d}{\delta} << 1$) the flux density penetrates completely and is essentially unaffected by the presence of the conducting material.

It is clear that at high frequencies ($\frac{d}{\delta} >> 1$) when the flux penetrates very little into the slab, the induced (eddy) currents flow near the surfaces. In this case it is often convenient, when considering electromagnetic phenomena external to the slab, to assume $\sigma \to \infty$ and treat the induced currents as surface currents.

It is informative to compare the flux distribution of the figure for a steady-state a-c problem with the distribution of Fig. 7.1.4 for a transient problem. We made the statement in Sec. 7.1.1 that when we deal with phenomena having characteristic times that are short compared to the diffusion time constant, the flux will not penetrate appreciably into the slab. We can make this statement quantitative for the steady-state a-c problem by defining a characteristic time as

$$\tau_c = \frac{1}{\omega}$$

We now take the ratio of the diffusion time constant given by Eq. 7.1.28 to this characteristic time and use the definition of skin depth in (i).

$$\frac{\tau}{\tau_c} = \omega\tau = \frac{2}{\pi^2}\left(\frac{d}{\delta}\right)^2 \tag{q}$$

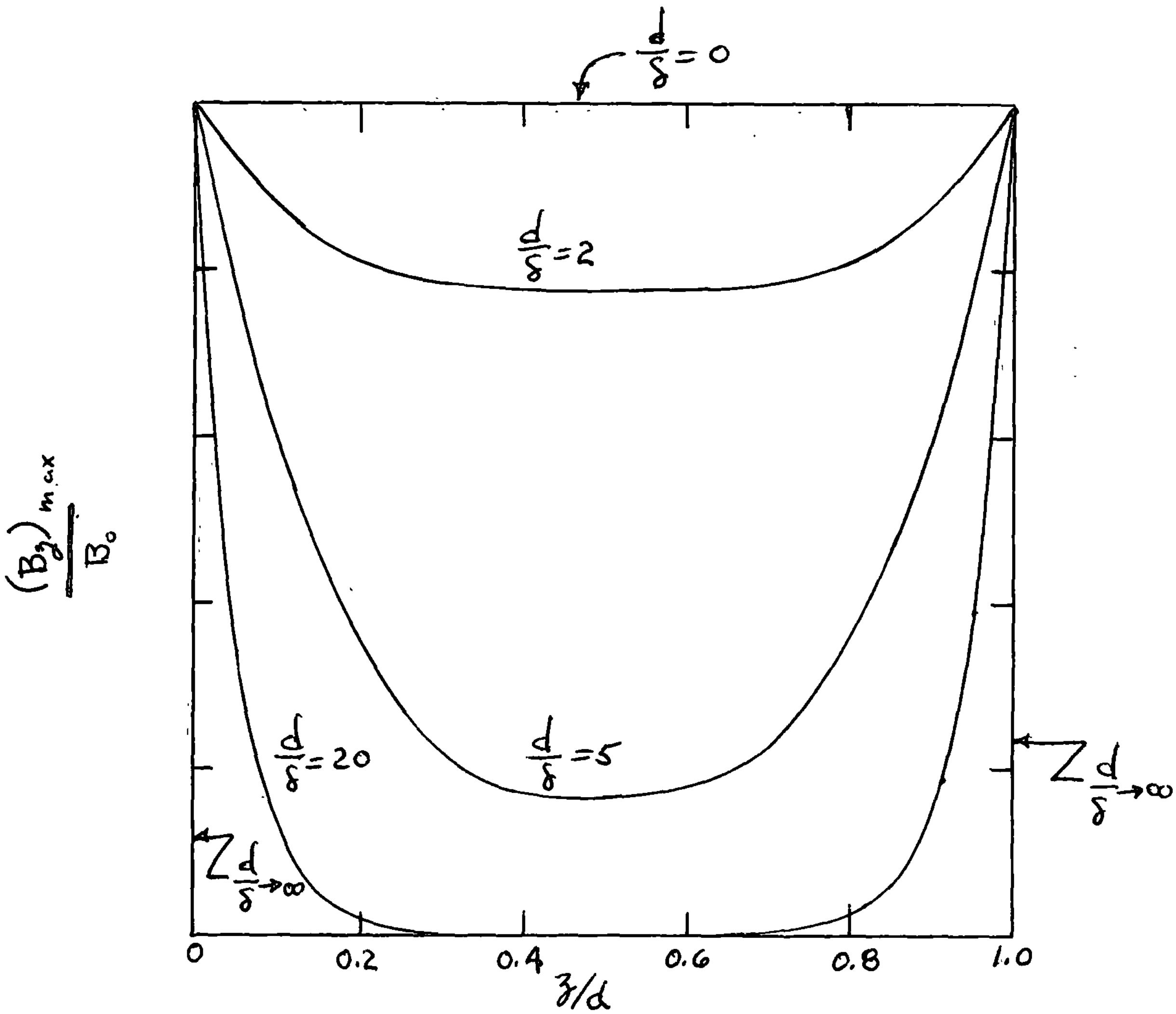
Thus, for our steady-state a-c problem, this statement that the diffusion time constant is long compared to a characteristic time is the same as saying that the significant dimension d is much greater than the skin depth δ.

The current distribution follows from the magnetic flux density by using Ampere's law;

$$J_y = \frac{1}{\mu_o}\frac{\partial B_x}{\partial z} \tag{r}$$

PROBLEM 7.6 (Continued)

Thus the distribution of $|J_y|$ is somewhat as shown in the figure for B_x. The instantaneous J_y has odd symmetry about $z = 0.5$ d.



DISTRIBUTION OF FlUX DENSITY WITH SKIN EFFECT

PROBLEM 7.7

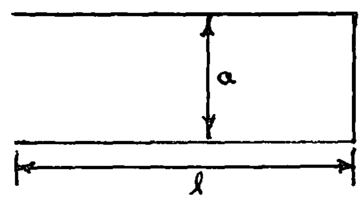
Part a

Assume the resistors in the circuit model each have approximately their D.C. resistance

$$R \sim R_{D.C.} = \frac{a}{\sigma \Delta D} \tag{a}$$

The inductance is the "loop" of metal

Depth D



$$L \approx \frac{\mu_o a \ell}{D} \tag{b}$$

Hence the time constant involved is

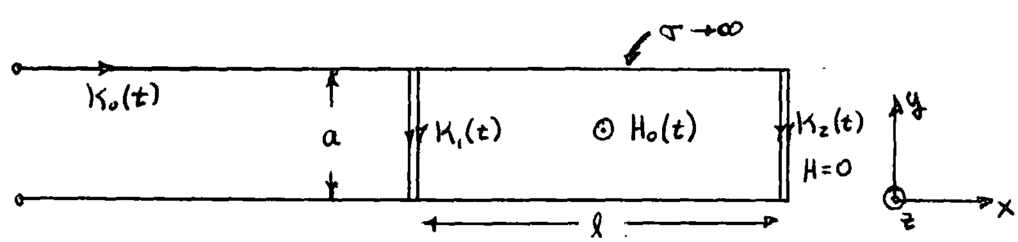
$$\tau \approx \frac{L}{2R} = \frac{\mu_o \Delta \ell \sigma}{2} \tag{c}$$

The equivalent length in the diffusion time is $\sqrt{\Delta \ell} >> \Delta$.

Part b

By adding the vacuum space of region 2 we have increased the amount of magnetic field that must be stored in the region before equilibrium is reached while the dissipation is confined to the two slabs. In the problem of Fig. 7.1.1, the slab stores a magnetic field only in a region of thickness Δ, the same region occupied by the currents , while here the magnetic field region is of thickness ℓ.

Part c



PROBLEM 7.7 (Continued)

Since diffusion in the slabs takes negligible time compared to the main problem, each slab could be modeled as a conducting sheet with

$$\bar{K} = (\sigma\Delta)\bar{E} \tag{d}$$

In region 2.

$$\nabla\times\bar{H} = 0 \quad \text{or } \bar{H} = H_o(t)\bar{i}_z = -K_2(t)\bar{i}_z \tag{e}$$

From

$$\oint\bar{E}\cdot d\bar{\ell} = -\frac{d}{dt}\int\bar{B}\cdot\bar{n}\,da \tag{f}$$

we learn that

$$\frac{a}{\sigma\Delta}[K_1(t)-K_2(t)] = +\frac{d}{dt}[\mu_o a\ell\, K_2(t)] \tag{g}$$

Since $K_o(t) = K_1(t) + K_2(t)$ we know that

$$K_o(t) = \frac{I}{D}u_{-1}(t) = 2K_2(t) + \sigma\mu_o\Delta\ell\frac{dK_2(t)}{dt} \tag{h}$$

The solution is therefore

$$K_2(t) = \frac{I}{2D}(1-e^{-t/\tau})u_{-1}(t);\ \tau = \frac{\sigma\mu_o\Delta\ell}{2} \tag{i}$$

and, because $K_2 = -H_o$, the magnetic field fills region (2) with the time constant τ.

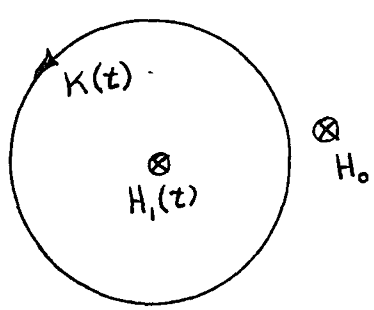
PROBLEM 7.8

As in Prob. 7.7, the diffusion time associated with the thin conducting shell is small compared to the time required for the field to fill the region $r < R$. Modeling the thin shell as having the property

$$\bar{K} = \Delta\sigma\,\bar{E} \tag{a}$$

and assuming that

$$H_1(t)\bar{i}_z = [H_o-K(t)]\bar{i}_z \tag{b}$$



PROBLEM 7.8 (Continued)

We can use the induction equation

$$\oint \bar{E}\cdot d\bar{\ell} = -\frac{d}{dt}\int \bar{B}\cdot\bar{n}\, da \tag{c}$$

to learn that, because H_o = constant for $t > 0$

$$\frac{2\pi R}{\Delta\sigma} K(t) = -\pi R^2 \mu_o \frac{dK(t)}{dt} \tag{d}$$

The solution to (d) is

$$K(t) = H_o e^{-t/\tau} u_{-1}(t); \quad \tau = \frac{\mu_o \sigma R \Delta}{2} \tag{e}$$

and from (b), it follows that

$$H_1(t) = H_o - K(t) = H_o(1-e^{-t/\tau})u_{-1}(t) \tag{f}$$

The $\bar{H}$ field is finally distributed uniformly for $r < a$, with a diffusion time based on the length $\sqrt{R\Delta}$.

PROBLEM 7.9

Part a

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \tag{a}$$

$$\nabla \times \bar{B} = \mu\sigma E \tag{b}$$

So

$$\nabla \times \nabla \times \bar{B} = -\mu\sigma \frac{\partial \bar{B}}{\partial t} \tag{c}$$

But

$$\nabla\times(\nabla\times\bar{B}) = \nabla(\nabla\cdot\bar{B}) - \nabla^2\bar{B} = -\nabla^2\bar{B} \tag{d}$$

So

$$\nabla^2\bar{B} = \mu\sigma \frac{\partial \bar{B}}{\partial t}$$

Part b

Since $\bar{B}$ only has a z component

$$\nabla^2 B_z = \mu\sigma \frac{\partial B_z}{\partial t} \tag{e}$$

In cylindrical coordinates

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{f}$$

Here $B_z = B_z(r,t)$ so

PROBLEM 7.9 (Continued)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \hat{B}}{\partial r}\right) + \mu\sigma\alpha\hat{B} = 0 \tag{g}$$

Part c

We want the magnetic field to remain finite at $r = 0$, hence $C_2 = 0$.

Part d

At $r = a$

$$B(a,t) = \mu_o H_o - C_1 J_o\left(\sqrt{\mu_o\sigma\alpha}\ a\right) = \mu_o H_o \tag{h}$$

Hence if $C_1 \neq 0$

$$J_o\left(\sqrt{\mu_o\sigma\alpha}\ a\right) = 0 \tag{i}$$

Part e

Multiply both sides of expression for $B(r,t=0) = 0$ by $rJ_o(\nu_j\ r/a)$ and integrate from 0 to a. Then,

$$\int_o^a \mu_o H_o\ r\ J_o(\nu_j\ r/a)dr = \mu_o H_o \frac{a^2}{\nu_j} J_1(\nu_j) \tag{j}$$

$$\int_o^a \sum_{i=1}^{\infty} C_i J_o(\nu_i\ r/a) r\ J_o(\nu_j\ r/a)dr = C_j \frac{a^2}{2} J_1^2(\nu_j) \tag{k}$$

from which it follows that

$$C_j = \frac{2\mu_o H_o}{\nu_j J_1(\nu_j)} \tag{ℓ}$$

The values of ν_j and $J_1(\nu_j)$ given in the table lead to the coefficients

$$\frac{C_1}{2\mu_o H_o} = .802;\ \frac{C_2}{2\mu_o H_o} = -\ .535;\ \frac{C_3}{2\mu_o H_o} = 0.425 \tag{m}$$

Part f

$$\alpha_1 = \frac{1}{\mu_o\sigma}\left(\frac{\nu_1}{a}\right)^2$$

$$\tau_1 = \frac{\mu_o\sigma a^2}{\nu_1^2} = 0.174\ \mu_o\sigma a^2 \tag{n}$$

$$\tau_1 = (0.174)(4\pi \times 10^{-7})\ \frac{10^4}{4\pi}\ (25) \times 10^{-4}$$

$$\simeq 4.35 \times 10^{-7} \text{ seconds} \tag{o}$$

PROBLEM 7.10

Part a

$$\nabla \times \bar{E} = \vec{i}_z \frac{\partial E_y}{\partial x} = -\frac{\partial}{\partial t} B_z \vec{i}_z = 0 \tag{a}$$

$$\nabla \times \bar{B} = -\vec{i}_y \frac{\partial B_z}{\partial x} = \mu_o \bar{J} = \mu_o \sigma (E_y - UB_z) \vec{i}_y \tag{b}$$

$$\nabla \times \nabla \times \bar{B} = \nabla(\nabla \cdot \bar{B}) - \nabla^2 \bar{B}$$

$$= -\frac{\partial^2 B_z}{\partial x^2} \vec{i}_z = \mu_o \sigma \left(\frac{\partial E_y}{\partial x} - U \frac{\partial B_z}{\partial x} \right) \vec{i}_z \tag{c}$$

But $\frac{\partial E_y}{\partial x} = 0$ from (a), so

$$\frac{\partial^2 B_z}{\partial x^2} = \mu_o \sigma U \frac{\partial B_z}{\partial x} \tag{d}$$

Part b

At

$$x = 0 \qquad B_z = -\mu_o K \tag{e}$$

at

$$x = L \qquad B_z = 0 \tag{f}$$

Part c

Let

$B_z(x) = C e^{\alpha x}$, then

$$\alpha(\alpha - \mu_o \sigma U) = 0 \tag{g}$$

$$\alpha = 0, \ \alpha = \mu_o \sigma U \tag{h}$$

Using the boundary conditions

$$B_z(x) = -\mu_o K \frac{(1 - e^{\mu_o \sigma U (x-L)})}{1 - e^{-\mu_o \sigma U L}} \tag{i}$$

Note that as $U \to 0$

$$B_z(x) = -\mu_o K \left(\frac{L-x}{L}\right) \tag{j}$$

as expected.

PROBLEM 7.11

Part a

$$\bar{F} = \bar{J}\times\bar{B} = -J_y B_x \bar{i}_z$$

$$= -\frac{\mu_o I^2}{w^2 \ell}\,\frac{R_m e^{R_m z/\ell}\,(e^{R_m z/\ell} - 1)}{(e^{R_m} - 1)^2}\,\bar{i}_z \tag{a}$$

Part b

$$f_z = \int -F_z\, wd\, dz = -\frac{\mu_o I^2 d}{2w} \tag{b}$$

This result can be found more simply by using the Maxwell Stress Tensor by methods similar to those used with Probs. 8.16 and 8.17.

Part c

The power supplied by the velocity source is

$$P_U = -f_z U = \frac{\mu_o I^2 dU}{2w} = \frac{I^2 d}{\sigma w \ell}\,\frac{R_m}{2} \tag{c}$$

The electric field at the current source is

$$E_y(z=\ell) = \frac{J_y(z=\ell)}{\sigma} - UB_x(z=\ell) \tag{d}$$

$$= \frac{I}{\sigma \ell w}\,\frac{R_m}{(e^{R_m} - 1)} \tag{e}$$

Power supplied by the current source is then

$$-V_g I = +E_y dI = +\frac{I^2 d}{\sigma \ell w}\left(\frac{R_m}{e^{R_m} - 1}\right) \tag{f}$$

Power dissipated in the moving conductor is then

$$P_d = P_U + V_g I = \frac{I^2 d}{\sigma \ell w}\,\frac{R_m}{2}\left(\frac{e^{R_m} + 1}{e^{R_m} - 1}\right) \tag{g}$$

which is just what is obtained from

$$P_d = wd \int_o^{\ell} \frac{J^2}{\sigma}\, dx \tag{h}$$

PROBLEM 7.12

If a point in the reference frame is outside the block it must satisfy

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \tag{a}$$

$$\bar{J} = 0 \quad \text{and} \quad \nabla \times \bar{B} = 0 \tag{b}$$

Since the points outside the block have $\bar{J}=0$, and uniform static fields (for differential changes in time), (a) and (b) are satisfied.

Points inside the block must satisfy

$$\frac{1}{\mu_o}\frac{\partial B_x}{\partial z} = J_y \tag{c}$$

$$-\frac{1}{\mu_o \sigma}\frac{\partial^2 B_x}{\partial z^2} + \frac{\partial B_x}{\partial t} = -V\frac{\partial B_x}{\partial z} \tag{d}$$

Since these points see

$$J_y = \sigma V B_o, \ \frac{\partial B_x}{\partial z} = \frac{B_o}{\ell}, \ \frac{\partial^2 B}{\partial z^2} = 0 \tag{e}$$

$$\frac{\partial B_x}{\partial t} = -V\frac{B_o}{\ell} \quad \text{and} \quad V\mu_o\sigma\ell = 1$$

these conditions are satisfied.

Points on the block boundaries are satisfied because the field quantities $\bar{E}$ and $\bar{B}$ are continuous.

PROBLEM 7.13

Part a

Because $\nabla\cdot\bar{B}=0$ the magnetic flux lines run in closed loops. The field lines prefer to run through the high μ material near the source, hence very few lines will close beyond the edge of the material at $z=0$. Currents in the slab will tend to remain between the pole pieces.

Part b

$$\frac{1}{\mu\sigma}\frac{\partial^2 B_y}{\partial z^2} = \frac{\partial B_y}{\partial t} + V\frac{\partial B_y}{\partial z} \tag{a}$$

Let $B_y(z,t) = C\,e^{j(\omega t - kz)}$, then

$$k^2 - j\mu\sigma V k + j\mu\sigma\omega = 0; \tag{b}$$

A quadratic equation with roots

PROBLEM 7.13 (Continued)

$$k = \mu\sigma\left[j\,\frac{V}{2} \pm j\,\sqrt{\left(\frac{V}{2}\right)^2 + j\,\frac{\omega}{\mu\sigma}}\right] \tag{c}$$

or in terms of $R_m = \mu\sigma VL$ and $\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$

$$(k^{\pm})L = j\,\frac{R_m}{2} \pm j\sqrt{\left(\frac{R_m}{2}\right)^2 + j2\,\left(\frac{L}{\delta}\right)^2} \tag{d}$$

From Fig. 7.1.16 of the text we see that

$$k^+ = k_r^+ + jk_i^+\;,\; k^- = k_r^- + jk_i^-$$

where

$$-k_r^- = k_r^+ > 0 \quad \text{and} \quad k_i^- > -k_i^+ > 0 \tag{e}$$

To meet the boundary condition of part (a) we must have

$$B_y(z,t) = C\left[e^{-jk^+z} - e^{-jk^-z}\right]e^{j\omega t} \tag{f}$$

Using the boundary condition at $z = -L$

$$B_y(z,t) = \frac{B_o}{\left(e^{jk^+L} - e^{jk^-L}\right)}\left(e^{-jk^+z} - e^{-jk^-z}\right)e^{j\omega t} \tag{g}$$

Part c

$$\nabla\times\bar{B} = -\,\bar{i}_x\,\frac{\partial B_y}{\partial z} = J_x\bar{i}_x/\mu_o \tag{h}$$

$$J_x = \frac{jB_o/\mu_o}{\left(e^{jk^+L} - e^{jk^-L}\right)}\left(k^+\,e^{-jk^+z} - K^-\,e^{-jk^-z}\right)e^{j\omega t} \tag{i}$$

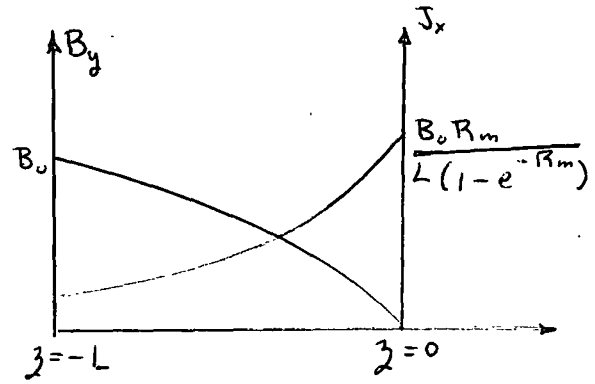
Part d

As $\omega \to 0 \quad k^+ \to 0,\; k^- \to j\,\frac{R_m}{L}$

$$B_y = \frac{B_o}{\left(1-e^{-R_m}\right)}\left(1-e^{(R_m/L)z}\right) \tag{j}$$

$$J_x = \frac{B_o/L}{\left(1-e^{-R_m}\right)}\,R_m\,e^{(R_m/L)z} \tag{k}$$

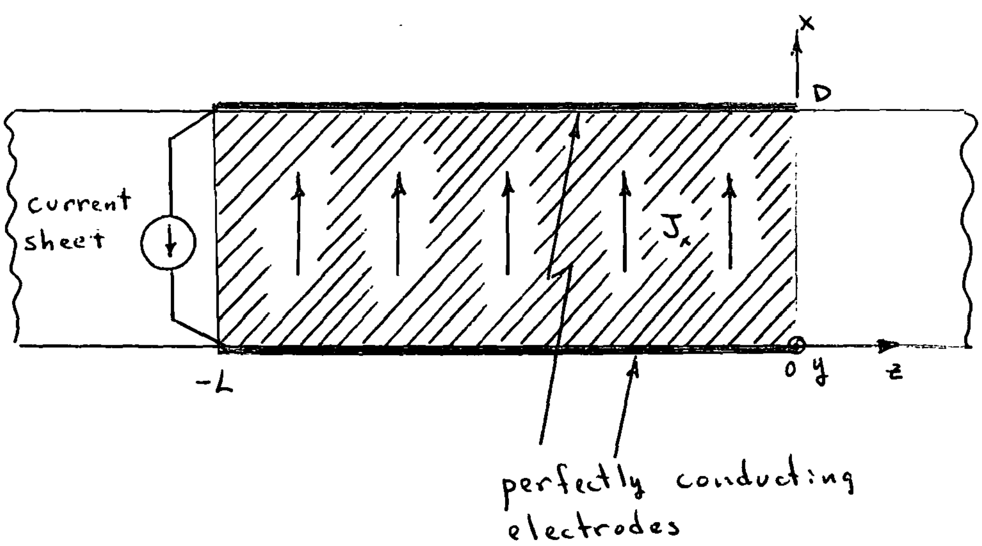
PROBLEM 7.13 (Continued)



As the sketch Fig. 7.1.9 of the text suggests, we could realize this problem by placing a current sheet source

$$\bar{K} = -\frac{B_o}{\mu_o}\bar{i}_x\, e^{j\omega t}$$

across the end z = - L and providing perfect conductors to slide against the slab at x = 0,D. The top view of the slab then appears as shown in the figure.



Note from (j) and (k) that as $R_m \to 0$, the current density J_x is uniform and B_y is a linear function of z. This limiting case is as would be obtained with the given driving arrangement.

PROBLEM 7.14

Part a

Since $\bar{J}' = \bar{J}$

PROBLEM 7.14 (Continued)

$$\bar{K} = \bar{i}_z K_o \cos(kUt-kx) \tag{a}$$

$$= \bar{i}_z K_o \cos(\omega t-kx); \quad \omega = kU$$

Part b

The track can be taken as large in the y direction when it is many skin depths thick

$$L = \text{track thickness} >> \delta = \sqrt{\frac{2}{\omega\mu_o\sigma}} = \sqrt{\frac{2}{kU\mu_o\sigma}} \tag{b}$$

In the track we have the diffusion equation

$$\frac{1}{\mu_o\sigma}\nabla^2\bar{B} = \frac{\partial\bar{B}}{\partial t} \tag{c}$$

or, with $\bar{B} = \text{Re}\,\hat{\bar{B}}\exp j(\omega t-kx)$,

$$\frac{1}{\mu_o\sigma}\left(\frac{\partial^2\hat{B}_x}{\partial y^2} - k^2\hat{B}_x\right) = j\omega\hat{B}_x \tag{d}$$

Let $\hat{B}_x(y) = C\,e^{\alpha y}$, then

$$\frac{1}{\mu_o\sigma}\alpha^2 = j\omega + \frac{k^2}{\mu_o\sigma} \tag{e}$$

$$\alpha = k\sqrt{1+jS}\;; \quad S = \frac{\omega\mu_o\sigma}{k^2} = \frac{U\mu_o\sigma}{k} \tag{f}$$

Since the track is modeled as infinitely thick

$$B_x = C\,e^{\alpha y}\,e^{j(\omega t-kx)} \tag{g}$$

The gap between track and train is very thin; thus,

$$-\bar{i}_y \times \frac{\bar{B}}{\mu_o} = \bar{K} = K_o\,e^{j(\omega t-kx)}\,\bar{i}_z \tag{h}$$

which yields

$$B_x(x,y,t) = \mu_o K_o\,e^{\alpha y}\,e^{j(\omega t-kx)} \tag{i}$$

We must also have $\nabla\cdot\bar{B} = \partial B_x/\partial x + \partial B_y/\partial y = 0$ or

$$B_y = \frac{jk}{\alpha}B_x(x,y,t) \tag{j}$$

PROBLEM 7.14 (Continued)

To compute the current in the track we note that

$$\nabla\times\bar{B} = \bar{i}_z \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) = \mu_o \bar{J} \tag{k}$$

$$\bar{J} = -\left(j\,\frac{S}{\alpha}\,k^2\right)\frac{B_x}{\mu_o}(x,y,t)\,\bar{i}_z \tag{ℓ}$$

Part c

The time average force density in the track is (see footnote, page 368)

$$\langle F_y\rangle = \frac{1}{2}\,\mathrm{Re}(J_z B_x^*) \tag{m}$$

Hence the time average lifting force per unit x-z area on the train is

$$\langle T_y\rangle = -\int_{-\infty}^{o}\langle F_y\rangle dy = -\,\mathrm{Re}\int_{-\infty}^{o}\frac{1}{2}\,J_z B_x^*\,dy \tag{n}$$

$$= \frac{1}{4}\,\mu_o K_o^2\left(\frac{\sqrt{1+S^2}-1}{\sqrt{1+S^2}}\right) > 0 \tag{o}$$

See Fig. 7.1.21 of the text for a plot of this lifting force.

Part d

The time average force density in the track in the x direction is

$$\langle F_x\rangle = -\frac{1}{2}\,\mathrm{Re}(J_z B_y^*) \tag{p}$$

The force on the train in the x direction is then

$$\langle T_x\rangle = -\int_{-\infty}^{o}\langle F_x\rangle dy = \frac{1}{2}\,\mathrm{Re}\int_{-\infty}^{o} J_z B_y^*\,dy \tag{q}$$

$$= -\frac{\mu_o K_o^2}{4}\;\frac{S}{\sqrt{1+S^2}\;\;\mathrm{Re}\sqrt{1+jS}} < 0$$

The problem is that this force drags the train instead of propelling it in the x direction. (See Fig. 7.1.20 of the text for a plot of the magnitude of this drag force). To make matters worse, if the train stops, the magnetic levitation force becomes zero.

PROBLEM 7.15

Part a

Let the current sheet lie in the plane $y = -s$. In the region $-s<y<0$ we have the "diffusion equation"

$$\nabla^2 B_z = 0 \tag{a}$$

If $B_z(x,y,t) = B_z(y)e^{j(\omega t-kx)}$ this equation yields

$$\frac{\partial^2 B_z}{\partial y^2} = k^2 B_z \tag{b}$$

Hence we can conclude that

$$B_z = [A \cosh k(y+s) + B \sinh k(y+s)]e^{j(\omega t-kz)} \tag{c}$$

At $y = -s$ we have the boundary condition

$$\bar{i}_y \times \bar{B}_z = \mu_o K_o \cos(\omega t-kz)\bar{i}_x \tag{d}$$

Thus

$$B_z = [\mu_o K_o \cosh k(y+s) + B \sinh k(y+s)]e^{j(\omega t-kz)} \tag{e}$$

Since $\nabla\cdot\bar{B} = \partial B_y/\partial y + \partial B_z/\partial z = 0$ we must have

$$B_y = [j(\mu_o K_o \sinh k(y+s) + B \cosh k(y+s))]e^{j(\omega t-kz)} \tag{f}$$

In the conductor the diffusion equation is

$$\frac{1}{\mu_o\sigma}\nabla^2\bar{B} = \frac{\partial\bar{B}}{\partial t} + V\frac{\partial\bar{B}}{\partial z} \tag{g}$$

Then

$$\frac{\partial^2 B_z}{\partial y^2} = (j\mu_o\sigma(\omega-kV) + k^2)B_z \tag{h}$$

which suggests a solution

$$B_z(y) = C e^{-\alpha y},\ \alpha = k\sqrt{1+jS},\quad S = \frac{\mu_o\sigma(\omega-kV)}{k^2} \tag{i}$$

Since $\nabla\cdot\bar{B} = 0$ in the conductor too, we must have

$$B_y = -j\frac{k}{\alpha}B_z \tag{j}$$

As the boundary $y = 0$ we must have

$$B_{y1} = B_{y2},\quad H_{z1} = H_{z2} \tag{k}$$

Note that

PROBLEM 7.15 (Continued)

$$\cosh ks\, B_{z2} + j \sinh ks\, B_{y2}$$
$$= \mu_o K_o(\cosh^2 ks - \sinh^2 ks) = \mu_o K_o \qquad (\ell)$$

Then we must also have

$$\mu_o K_o = \cosh ks\, B_{z1} + j \sinh ks\, B_{y1}$$
$$= C\,(\cosh ks + \frac{k}{\alpha} \sinh ks) \qquad (m)$$

It follows that the $\bar{B}$ field for y>0 is

$$\bar{B} = \frac{\mu_o K_o}{\cosh ks + \frac{k}{\alpha} \sinh ks} \left(-j \frac{k}{\alpha} \bar{i}_y + \bar{i}_z\right) e^{-\alpha y}\, e^{j(\omega t - kz)} \qquad (n)$$

Comparing with Eq. 7.1.91 of Sec. 7.1.4 of the text we see that it is only necessary to replace

$$K_o \text{ by } \frac{K_o}{\cosh ks + \frac{k}{\alpha} \sinh ks}$$

starting with Eq. 7.1.90. The average forces depend on the magnitude, not the phase, of K_o, which is reduced by this substitution.

Part b

We note that if ks << 1

$$\frac{K_o}{\cosh ks + \frac{k}{\alpha} \sinh ks} \simeq K_o \qquad (o)$$

which shows that the results of Sec. 7.1.4 are valid when ks << 1.

Part c

When $ks \to \infty$

$$\frac{K_o}{\cosh ks + \frac{k}{\alpha} \sinh ks} \longrightarrow 0$$

No fields will then be present in the conductor.

PROBLEM 7.16

Part a

Because the charge needs time to move through the conductor, at $t=0^+$ there is only free charge on the plates. The electric fields are directed in the negative vertical direction and satisfy

PROBLEM 7.16 (Continued)

$$E_\ell b + E_g a = V_o \tag{a}$$

at the interface at $t=0^+$

$$\varepsilon E_\ell = \varepsilon_o E_g \tag{b}$$

Hence at $t=0^+$

$$E_\ell = \frac{V_o}{b + \frac{\varepsilon}{\varepsilon_o} a}, \quad E_g = \frac{V_o}{\frac{\varepsilon_o}{\varepsilon} b + a} \tag{c}$$

Part b

As $t\to\infty$ the charge on the interface excludes the fields from the conducting liquid, hence

$$E_\ell = 0 \quad E_g = \frac{V_o}{a} \tag{f}$$

Part c

The charge on the interface at any time is

$$\sigma_f = \varepsilon E_\ell - \varepsilon_o E_g \tag{g}$$

Conservation of charge requires

$$\frac{d\sigma_f}{dt} = - \sigma E_\ell \tag{h}$$

The voltage across the plates is V_o for $t>0$

$$V_o = E_\ell b + E_g a \tag{i}$$

Solving g, h, i we find that the charge obeys

$$\frac{(\varepsilon + \varepsilon_o\, b/a)}{\sigma} \frac{d\sigma_f}{dt} + \sigma_f = - \frac{\varepsilon_o}{a} V_o \tag{j}$$

Let $\tau = \dfrac{\varepsilon+\varepsilon_o\, b/a}{\sigma}$, then

$$\sigma_f = - \frac{\varepsilon_o V_o}{a} (1 - e^{-t/\tau}), \; t \geq 0 \tag{k}$$

$$q_f = - \frac{\varepsilon_o A V_o}{a} (1 - e^{-t/\tau}), \; t \geq 0 \tag{ℓ}$$

PROBLEM 7.17

Part a

In the inner sphere

$$\frac{\sigma_i}{\varepsilon_o}\rho_f + \frac{\partial \rho_f}{\partial t} = 0 \tag{a}$$

So we find that

$$\rho_f(r,t) = \rho_o(r)e^{-\sigma_i/\varepsilon_o\, t}, \quad t \geq 0 \quad r < R_i \tag{b}$$

A similar equation holds for the charge in the outer sphere, but it has no initial charge distribution at t = 0, so

$$\rho_f(r,t) = 0, \quad t \geq 0 \quad R_i<r<R_o \tag{c}$$

Part b

Let

$$Q_o = \int_0^{R_i} 4\pi r^2\, \rho_o(r)dr \tag{d}$$

Also define

σ_A = the surface charge density at $r = R_i$

σ_B = the surface charge density at $r = R_o$

The field at R_o^+ is, by Gauss' law

$$E(R_o^+) = \frac{Q_o}{4\pi\varepsilon_o R_o^2} \tag{e}$$

Then, conservation of charge requires that the electric field at $r = R_o^-$ obey

$$\sigma_o\, E(R_o^-) + \varepsilon_o \frac{\partial E}{\partial t}(R_o^-) = 0 \tag{f}$$

$$E(R_o^-) = \frac{Q_o}{4\pi\varepsilon_o R_o^2}\, e^{-(\sigma_o/\varepsilon_o)t}, \quad t \geq 0 \tag{g}$$

We can thus conclude that

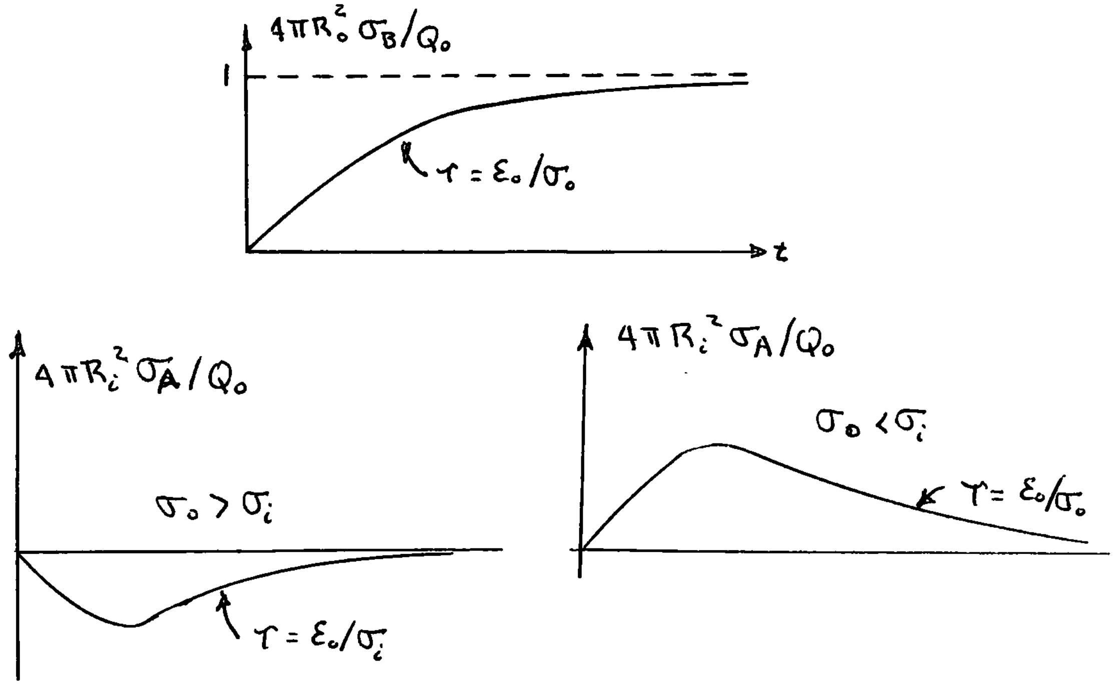
$$\sigma_B = \frac{Q_o}{4\pi R_o^2}\, (1 - e^{-(\sigma_o/\varepsilon_o)t}), \quad t \geq 0 \tag{h}$$

Since charge is conserved we now know that

$$\sigma_A = \frac{Q_o}{4\pi R_i^2}\, (e^{-(\sigma_o/\varepsilon_o)t} - e^{-(\sigma_i/\varepsilon_o)t}), \quad t \geq 0 \tag{i}$$

PROBLEM 7.17 (Continued)

Part c



PROBLEM 7.18

Part a

At the radius b

$$\varepsilon[E(b^+) - E(b^-)] = \sigma_f \tag{a}$$

$$\sigma[E(b^+)-E(b^-)] = -\frac{\partial\sigma_f}{\partial t} = -\varepsilon\frac{\partial}{\partial t}[E(b^+)-E(b^-)] \tag{b}$$

For t < 0 when the system has come to rest

$$\nabla\cdot\bar{J} = (\sigma/\varepsilon)\nabla\cdot\bar{E} = -\frac{\partial\rho_f}{\partial t} = 0 \tag{c}$$

For cylindrical geometry this has the solution

$$\bar{E} = +\frac{A}{r}\bar{i}_r; \quad V_o = +\int_a^b E_r dr = A\ln(b/a) \tag{d}$$

PROBLEM 7.18 (Continued)

then

$$\left.\begin{aligned} E(r=b^-) &= +\frac{V_o}{\ln(b/a)}\frac{1}{b} \\ E(r=b^+) &= 0 \end{aligned}\right\} \quad t = 0^- \tag{e}$$

Since $E(b^+) - E(b^-) = \sigma_f/\varepsilon$ it cannot change instantaneously, so

$$E(b^+) - E(b^-) = -\frac{V_o}{b\ \ln(b/a)}\ e^{-(\sigma/\varepsilon)t}, \quad t \geq 0 \tag{f}$$

Because there is no initial charge between the shells, there will be no charge between the shells for $t \geq 0$, thus

$$E_r = \begin{cases} +\dfrac{C_1(t)}{r} & a<r<b \\ +\dfrac{C_2(t)}{r} & b<r<c \end{cases} \qquad t \geq 0 \tag{g}$$

The battery adds the constraint

$$V_o = C_1\ln(b/a) + C_2\ln(c/b) \tag{h}$$

while (f) becomes

$$C_1 - C_2 = \frac{V_o}{\ln(b/a)}\ e^{-(\sigma/\varepsilon)t} \tag{i}$$

Solving (h) and (i) for C_1, C_2

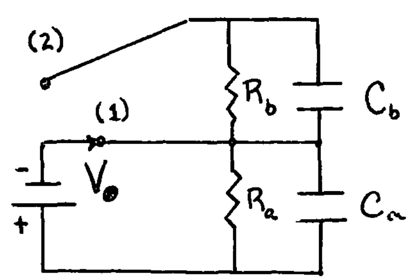
$$C_2 = \frac{V_o}{\ln\ c/a}\ (1 - e^{-(\sigma/\varepsilon)\ t}) \tag{j}$$

$$C_1 = \frac{V_o}{\ln\ c/a}\ (1 + \frac{\ln(c/b)}{\ln(b/a)}\ e^{-(\sigma/\varepsilon)t}) \tag{k}$$

Part b

$$\sigma_f = \varepsilon(E(b^+) - E(b^-)) = -\frac{\varepsilon}{b\ \ln(b/a)}\ V_o\ e^{-(\sigma/\varepsilon)t} \tag{ℓ}$$

Part c



$$R_b = \frac{\ln c/b}{2\pi\sigma}, \quad C_b = \frac{2\pi\varepsilon}{\ln\ c/b}$$

$$R_a = \frac{\ln(b/a)}{2\pi\sigma}, \quad C_a = \frac{2\pi\varepsilon}{\ln\ b/a}$$

PROBLEM 7.19

While the potential v is applied the system reaches an equilibrium. During this time

$$\nabla\cdot\bar{J} = \frac{\sigma}{\varepsilon}\,\rho_f = -\frac{\partial\rho_f}{\partial t} \tag{a}$$

in the bulk of the liquid. If the potential V is applied for many time constants ($\tau=\varepsilon/\sigma$) any charge in the fluid decays away. For $t>0$ if the fluid is incompressible ($\nabla\cdot\bar{v} = 0$) and $\bar{J} = \sigma\bar{E} + \rho_f\bar{v}$ we know that

$$\nabla\cdot\bar{J} = (\sigma/\varepsilon)\rho_f + \bar{v}\cdot\nabla\rho_f = -\frac{\partial\rho_f}{\partial t} \tag{b}$$

But in a frame moving with the particles of fluid

$$\frac{d}{dt}\rho_f = \frac{\partial\rho_f}{\partial t} + \bar{v}\cdot\nabla\rho_f = -(\sigma/\varepsilon)\rho_f \tag{c}$$

$$\rho_f(t) = \rho_f(t=0)e^{-(\sigma/\varepsilon)\,t} \quad t \geq 0 \tag{d}$$

where $\rho_f(t)$ is the local charge seen by a moving particle. But for all fluid particles

$$\rho_f(t = 0) = 0 \tag{e}$$

Hence the charge remains zero everywhere for $t \geq 0$.

Now draw a volume around the upper sphere big enough to enclose it for a few seconds even though it is moving.

$$\oint_S \bar{J}\cdot d\bar{a} = -\frac{d}{dt}\int_V \rho_f\, dV \tag{f}$$

Now because $\rho_f = 0$ in the fluid

$$\bar{J} = \sigma\bar{E},\ \oint_S \bar{J}\cdot d\bar{a} = (\sigma/\varepsilon)\oint_S \varepsilon\bar{E}\cdot d\bar{a} = (\sigma/\varepsilon)Q(t) \tag{g}$$

Then

$$(\sigma/\varepsilon)Q(t) = -\frac{d}{dt}\int_V \rho_f\, dV = -\frac{d}{dt}Q(t) \tag{h}$$

which has solution

$$Q(t) = Q\, e^{-t/\tau};\tau = \varepsilon/\sigma$$

PROBLEM 7.20

Part a

We can use Gauss' law

$$\oint_S \varepsilon_o \bar{E}\cdot d\bar{a} = \int_V \rho_f \, dV \tag{a}$$

to determine the electric field if we note that there is no net charge in the system, which means that

$$\bar{E} = E_x \bar{i}_x = 0 \quad x<0 \text{ and } x>3d \tag{b}$$

$$\varepsilon_o E_x(x) = \int_o^x \frac{Q}{D^2 d}\, dx = \frac{Q}{D^2}\frac{x}{d} \quad \begin{cases} 0<x<d \\ t = 0 \end{cases} \tag{c}$$

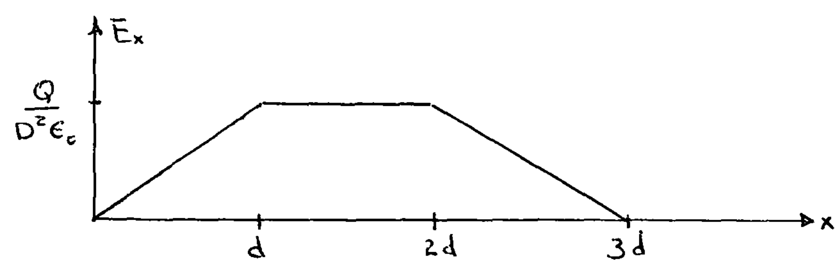
There is no charge in the middle region so

$$E_x = \frac{Q}{D^2 \varepsilon_o} \quad d<x<2d;\ t = 0 \tag{d}$$

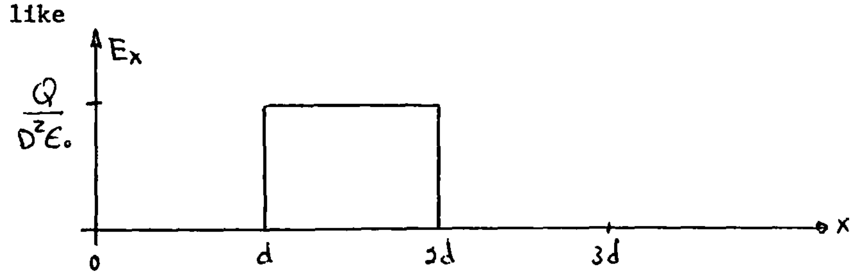
In the region 2d<x<3d

$$\varepsilon_o(E_x(x) - E_x(2d)) = \int_{2d}^x \frac{-Q}{D^2 d}\, dx = -\frac{Q}{D^2}\frac{(x-2d)}{d} \tag{e}$$

$$E_x(x) = \frac{Q}{D^2\varepsilon_o}\frac{(3d-x)}{d} \quad \begin{cases} 2d<d<3d \\ t = 0 \end{cases} \tag{f}$$



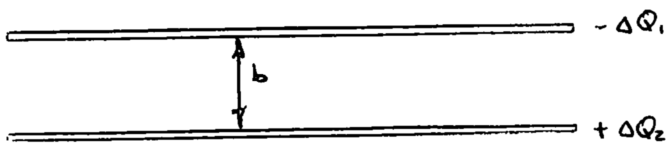
As t→∞ all the charge on the lower plate relaxes to the surface x = d, while the charge on the upper plate relaxes to the surface x = 2d. The electric field then looks like



PROBLEM 7.20 (Continued)

Part b

Each charge distribution can be thought of as made up of many thin charges sheets; any two such sheets,



one located somewhere in the top conductor, one located somewhere in the bottom conductor, attract each other with a force

$$\Delta F = \frac{\Delta Q_1 \ \Delta Q_2}{2\varepsilon_o \ D^2} \tag{g}$$

which is independent of their separation, hence the net attractive force between plates does not change with time. At $t\to\infty$ there is a surface charge

$$\sigma_T = -\frac{Q}{D^2} \qquad x = 2d$$
$$\sigma_B = +\frac{Q}{D^2} \qquad x = d \tag{h}$$

and the force per unit area T_x is simply that found for a pair of capacitor plates having separation d and supporting surface charge densities $\pm$ Q. (See Sec. 3.1.2b).

$$T_x = \frac{Q^2}{2\varepsilon_o D^2} \qquad t \geq 0 \tag{i}$$

This force can be easily seen to be constant from the viewpoint taken in Chapter 8, where the force on the lower plate can be found from the Maxwell Stress Tensor. The only contribution comes from $T_{xx} = \frac{1}{2}\,\varepsilon_o E_x^2$ evaluated at x = d, and thus $T_{xx}(x = d) = T_x$ as given by (i) regardless of t. Problem 8.23 is worked out following the stress-tensor approach.

PROBLEM 7.21

Part a

If the electric field beyond the plates is zero the conservation of charge equation

$$\oint_S \bar{J}\cdot d\bar{a} = -\frac{\partial}{\partial t}\int_V \rho_f \ dV = -\frac{\partial}{\partial t}\oint_S \varepsilon\bar{E}\cdot d\bar{a} \tag{a}$$

PROBLEM 7.21 (Continued)

becomes

$$\sigma\hat{E}_x(x) - \frac{\hat{I}}{A} = -j\omega\varepsilon\,\hat{E}_x(x) \tag{b}$$

That is, the equation for $\hat{E}_x$ is as given by (f) of Example 7.2.3, with ε now a function of x.

$$\hat{E}_x(x) = \frac{\hat{I}}{A(\sigma+j\omega\varepsilon)} = \frac{\hat{I}/A}{[\sigma_1 + \frac{\sigma_2}{\ell}x + j\omega(\varepsilon_1 + \frac{\varepsilon_2}{\ell}x)]} \tag{c}$$

From Coulomb's law

$$\hat{\rho}_f = \frac{d}{dx}(\varepsilon\hat{E}_x) = -\frac{\varepsilon\hat{I}}{A}\frac{\left(j\omega\frac{d\varepsilon}{dx} + \frac{d\sigma}{dx}\right)}{(j\omega\varepsilon + \sigma)^2} + \frac{\hat{I}\,\frac{d\varepsilon}{dx}}{A(j\omega\varepsilon+\sigma)} \tag{d}$$

$$\frac{\hat{\rho}_f A}{\hat{I}} = -\frac{(\varepsilon_1 + \frac{\varepsilon_2}{\ell}x)(j\omega\frac{\varepsilon_2}{\ell} + \frac{\sigma_2}{\ell})}{[(\sigma_1 + \frac{\sigma_2}{\ell}x) + j\omega(\varepsilon_1 + \frac{\varepsilon_2}{\ell}x)]^2} + \frac{\frac{\varepsilon_2}{\ell}}{[(\sigma_1 + \frac{\sigma_2}{\ell}x) + j\omega(\varepsilon_1 + \frac{\varepsilon_2}{\ell}x)]} \tag{e}$$

Part b

Consider the effect of a small change in ε alone

$$\sigma_2 = 0;\ \varepsilon_2/\varepsilon_1 << 1$$

then

$$\rho_f \simeq \frac{\sigma_1\varepsilon_2\,\hat{I}}{A\ell(j\omega\varepsilon_1+\sigma_1)^2} \tag{f}$$

It is seen from (f) that in the presence of conduction the gradient of ε causes free charge to be stored in the bulk of the fluid. This effect is highly dependent on frequency, being greatest at zero frequency and disappearing when the cycle time is short compared to the relaxation time of the material.

PROBLEM 7.22

Part a

In the fluid the consitutive law for conduction is

$$\bar{J} = \sigma\bar{E} + \rho_f\bar{v} \tag{a}$$

Since the given velocity distribution has the property

PROBLEM 7.22 (Continued)

$$\nabla\cdot\bar{v} = 0 \tag{b}$$

$$\nabla\cdot\bar{J} = \frac{\sigma}{\varepsilon}\nabla\cdot(\varepsilon E) + \bar{v}\cdot\nabla\rho_f = \frac{\sigma}{\varepsilon}\rho_f + U\frac{\partial}{\partial x}\rho_f = -\frac{\partial\rho_f}{\partial t} \tag{c}$$

or

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\rho_f = -\frac{\sigma}{\varepsilon}\rho_f \tag{d}$$

The charge is relaxing in the frame of the moving fluid. The solution has the form

$$\rho_f = \text{Re}\,\hat{\rho}_o\, e^{-\frac{\sigma}{\varepsilon}\frac{x}{U}}\, e^{j\omega(t-\frac{x}{U})};\quad -\frac{\Delta}{2} < y < \frac{\Delta}{2} \tag{e}$$

$$= 0 \text{ elsewhere in the channel}$$

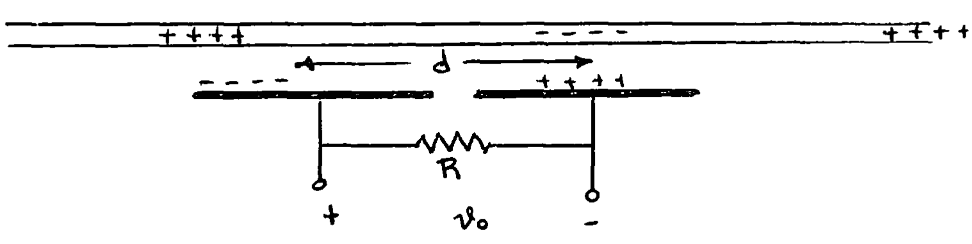
where y = 0 is the channel center. Note that (e) satisfies the boundary condition at x = 0 and states that a charge at x at time t has been decaying $\frac{x}{U}$ seconds (since it left the source) and was dumped in the channel at time

$$t' = t - \frac{x}{U}$$

Substitution of (e) into (d) verifies that it is a solution.

Part b

From (e) it is clear that the wave length of the sinusoidally (and decaying) charge stream is $2\pi U/\omega$. Thus, the wave length can be altered simply by changing ω. One technique for measuring the flow velocity would consist in measuring the voltage induced across the resistance R (as shown in the figure) as a function of the frequency. With the distance between electrode centers d equal to 1/2 wave length, a peak in the output signal would be expected. If we call the frequency at which this peak occurs ω_p, then



PROBLEM 7.22 (Continued)

$$\frac{2\pi U}{\omega_p} = 2d$$

or

$$U = \frac{d\omega_p}{\pi}$$

Thus, a determination of ω_p gives U. There are, of course, problems with this approach. For example, there would be lesser peaks in the output at harmonic frequencies that could be mistaken for the desired peak. Alternatives are to use the decay rate, but such techniques are vulnerable to conductivity variations which are likely to be large.

PROBLEM 7.23

Part a

Current is carried by the conductor because of normal conduction and also because of convection of a net charge.

$$\bar{J} = \sigma\bar{E} + \rho_f\bar{v}$$

Also

$$\nabla\cdot\bar{E} = \rho_f/\varepsilon = \nabla\cdot\frac{(\bar{J}-\rho_f\bar{v})}{\sigma}$$

But

$$\nabla\cdot\bar{J} = -\frac{\partial\rho_f}{\partial t} = 0 \text{ in steady state}$$

$$\nabla\cdot\bar{v} = \nabla\cdot(U\,\bar{i}_x) = 0 \text{ also, so that}$$

$$\rho_f/\varepsilon = -\frac{\bar{v}\cdot\nabla\rho_f}{\sigma} = -\frac{U}{\sigma}\frac{\partial\rho_f}{\partial x}$$

The solution to this last equation is

$$\rho = \rho_o\, e^{-(\frac{\sigma}{\varepsilon})(\frac{x}{U})}$$

i.e., the charge relaxes in the conductor; the time $\tau = \frac{x}{U}$ is a measure of how long since the charge left the source at the first screen.

Part b

Let

$$E_x(x=0) = E_o$$

$$\frac{\partial E_x}{\partial x} = \frac{\rho(x)}{\varepsilon} = \frac{\rho_o}{\varepsilon}\, e^{-(\frac{\sigma}{\varepsilon})(\frac{x}{U})}$$

PROBLEM 7.23 (Continued)

$$E_x(x) = E_o + \int_o^x \frac{\rho_f(x)}{\varepsilon}\,dx = E_o + \frac{\rho_o U}{\sigma}(1-e^{-\frac{\sigma}{\varepsilon}\frac{x}{U}})$$

Note that since $J_x(x=0) = \sigma E_o + \rho_o U = \frac{V}{RA}$

$$E_x(x) = \frac{V}{RA\sigma} - \frac{\rho_o U}{\sigma} e^{-(\frac{\sigma}{\varepsilon})(\frac{x}{U})}$$

We must finish the problem to know V

Part c

$$V = -\int_o^{\ell} E_x(x)dx = -\frac{V\ell}{RA\sigma} + \rho_o\varepsilon_o(\frac{U}{\sigma})^2 (1-e^{-(\frac{\sigma}{\varepsilon})\frac{\ell}{U}})$$

$$V = (\frac{1}{1+\frac{\ell}{RA\sigma}})\rho_o\varepsilon\ (\frac{U}{\sigma})^2\ (1-e^{-(\frac{\sigma}{\varepsilon})(\frac{\ell}{U})}$$

PROBLEM 7.24

Part a

The model for this problem is similar to that used in Example 7.2.6 of the text. Each ring induces a charge on the stream having opposite polarity to its potential. Thus, conservation of charge for the can at potential v_3 (under the ring at potential v_1) is

$$-C_i\, nv_1 = C\frac{dv_3}{dt} + \frac{v_3}{R} \tag{a}$$

Similarly, for the other two cans,

$$-C_i\, nv_2 = C\frac{dv_1}{dt} + \frac{v_1}{R} \tag{b}$$

$$-C_i\, nv_3 = C\frac{dv_2}{dt} + \frac{v_2}{R} \tag{c}$$

To solve these three equations, we assume solutions of the form

$$v_i = \hat{v}_i\, e^{st} \tag{d}$$

and the complex amplitudes $\hat{v}_i$ are governed by the conditions that follow from substitution of (d) into (a)-(c)

$$\begin{bmatrix} C_i n & 0 & (Cs+\frac{1}{R}) \\ (Cs+\frac{1}{R}) & C_i n & 0 \\ 0 & (Cs+\frac{1}{R}) & C_i n \end{bmatrix} = 0 \tag{e}$$

PROBLEM 7.24 (Continued)

The solution for s is

$$s = -\frac{1}{RC} + \frac{C_1 n}{C}\left[\frac{1}{2} \pm \frac{j\sqrt{3}}{2}, -1\right] \tag{f}$$

Part b

Thus, the system is unstable if

$$\frac{1}{RC} < \frac{C_1 n}{2C} \tag{g}$$

Part c

In particular, from (g), the system is self-excited as

$$\frac{1}{R} = \frac{C_1 n}{2} \tag{h}$$

Part d

The frequency of oscillation under condition (h) follows from (f) and (h) as

$$\omega = \frac{C_1 n\sqrt{3}}{2C} = \frac{\sqrt{3}}{RC} \tag{i}$$

PROBLEM 7.25

The crucial quantities in the respective systems are the magnetic diffusion time (Eq. 7.1.28) and the charge relaxation time (Eq. 7.2.11) relative to the period of excitation $T = 1/f$. The conductivities required to make these respective times equal to the excitation period T are

$$\sigma = \pi^2\, T/\mu_o\, d^2 \tag{a}$$

$$\sigma = \varepsilon/T \tag{b}$$

In terms of the given numbers,

$$\begin{aligned}\sigma &= (3.14)^2(10^{-5})/(4)(3.14 \times 10^{-7})(10^{-4}) \\ &= 7.85 \times 10^7 \text{ mhos/m}\end{aligned} \tag{c}$$

and

$$\sigma = (81)(8.85 \times 10^{-12})/10^{-5} = 7.16 \times 10^{-5} \text{ mhos/m} \tag{d}$$

For the change in depth to have a large effect on the inductance, the conductivity must be greater than that given by (c). Thus, the magnetic device would not be satisfactory. By contrast, (d) indicates that the conductivity of the electric apparatus is more than sufficient to make a change in capacitance with liquid depth apparent even if $\varepsilon = \varepsilon_o$. Both devices would be attractive for this application only if the conductivity exceeded that given by (c).

PROBLEM 7.26

This problem depends on the same physical reasoning as used in connection with Prob. 7.25. There are two modes in which either device can operate. Consider configuration (a): the inductance can change either because of the magnetization of the water, or because of currents induced in the water. However, water is only weakly magnetic and so the first mode of operation is not attractive. Moreover, the frequency is too low to induce appreciable currents, as can be seen by comparing the magnetic diffusion time to the period of excitation. Hence, configuration (a) does not represent an attractive approach to the engineering problem.

On the other hand, configuration (b) can operate either because of a change in capacitance between the electrodes due to the change in position of the polarized liquid (at high frequencies) or due to a change in position of a perfectly conducting liquid (low frequencies). As the calculations of Prob. 7.26 show, it is this last mode of operation that is appropriate in this case.

PROBLEM 7.27

Part a

Because we have changed only a boundary condition, the potentials in regions (a) and (b) are still of the general form

$$\begin{aligned}\hat{\phi}_a &= A \sinh kx + B \cosh kx \\ \hat{\phi}_b &= C \sinh kx + D \cosh kx\end{aligned} \tag{a}$$

There are now four boundary conditions:

$$\hat{\phi}_a(d) = \hat{V} \tag{b}$$

$$\hat{\phi}_a(0) = \hat{\phi}_b(o) \tag{c}$$

$$-\sigma \frac{\partial\phi_b(0)}{\partial x} = \left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial z}\right)\left(-\varepsilon_o \frac{\partial\phi_a(0)}{\partial x} + \varepsilon \frac{\partial\phi_b(0)}{\partial x}\right) \tag{d}$$

$$\phi_b(-f) = 0 \tag{e}$$

Only boundary condition (e) is new; it has replaced the assumption that ϕ_b must go to zero as $x \to -\infty$.

Solving for A, B, C and D we find that

PROBLEM 7.27 (Continued)

$$\phi_a = \text{Re}\,\frac{\hat{v}}{\Delta}[(1+jS\ \varepsilon/\varepsilon_o)\sinh kx + jS\tanh kf\cosh kx]e^{j(\omega t-kz)} \quad (f)$$

$$\phi_b = \text{Re}\,\frac{\hat{v}}{\Delta}[jS\sinh kx + jS\tanh kf\cosh kx]e^{j(\omega t-kz)} \quad (g)$$

where $\Delta = (1+jS\ \varepsilon/\varepsilon_o)\sinh kd + jS\tanh kf\cosh kd.$

Part b

If

$$|fk| >> 1$$

$$\tanh kf \to 1 \quad (h)$$

A comparison shows that in this limit the results agree with Sec. 7.2.4 if we note that

$$e^{kx} = \cosh kx + \sinh kx \quad (i)$$

PROBLEM 7.28

Part a

The regions between the traveling wave electrodes and the moving sheet are free space, and therefore the fields are governed by

$$\nabla^2\phi = 0 \quad (a)$$

where

$$\bar{E} = -\nabla\phi \quad (b)$$

Moreover, solutions that have the same (z-t) dependence as the imposed traveling wave potentials, and that satisfy (a) are

$$\phi_a = \text{Re}[A_1\cosh kx + A_2\sinh kx]e^{j(\omega t-kx)} \quad (c)$$

$$\phi_b = \text{Re}[B_1\cosh kx + B_2\sinh kx]e^{j(\omega t-kz)} \quad (d)$$

The constants A_1, A_2, B_1, B_2 must be adjusted to make these solutions satisfy the boundary conditions

$$\hat{\phi}_a = \hat{V}_o \quad \text{at} \quad x = c \quad (e)$$

$$\hat{\phi}_b = \hat{V}_o \quad \text{at} \quad x = -c \quad (f)$$

$$\hat{\phi}_a = \hat{\phi}_b \quad \text{at} \quad x = 0 \quad (g)$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\varepsilon_a E_x^a - \varepsilon_b E_x^b\right) + \sigma_s\frac{\partial E_z^a}{\partial z} = 0 \quad (h)$$

Part b

The symmetry requires that

PROBLEM 7.28 (Continued)

$$\phi_a(x,z,t) = \phi_b(-x,z,t) \tag{i}$$

and this implies that $A_1 = B_1$, $A_2 = -B_2$. The boundary conditions become

$$A_1 \cosh kc + A_2 \sinh kc = \hat{V}_o \tag{j}$$

$$jS\,(2A_2) = A_1 \tag{k}$$

where

$$S = (\omega - kU)\varepsilon_o/k\sigma_s \tag{ℓ}$$

Thus,

$$A_1 = B_1 = 2j\,S\hat{V}_o/(\sinh kc + 2j\,S\cosh kc) \tag{m}$$

and

$$A_2 = -B_2 = \hat{V}_o/(\sinh kc + 2j\,S\cosh kc) \tag{n}$$

Part b

A section of the sheet can be enclosed by a thin volume of small area in the y-z plane to give the force per unit area as

$$T_z = 2T^a_{zx}\ (x = 0) \tag{o}$$

where the symmetry has been used to set

$$T^a_{zx} = -T^b_{zx} \tag{p}$$

Thus, the time average force per unit area is

$$\langle T_z\rangle = \mathrm{Re}[\varepsilon_o \hat{E}^a_x(0)\hat{E}^{a*}_z(0)] \tag{q}$$

and from (m) and (n),

$$\langle T_z\rangle = \mathrm{Re}[\varepsilon_o(-jk)A_1^*(-k)A_2] \tag{r}$$

$$= \mathrm{Re}\left[\frac{2\varepsilon_o k^2|\hat{V}_o|^2 S}{\sinh^2 kc + 4S^2\cosh^2 kc}\right] \tag{s}$$

$$= \frac{2\varepsilon_o k^2|V_o|^2 S}{(\sinh^2 kc + 4S^2\cosh^2 kc)} \tag{t}$$

It follows from (t) that the maximum occurs as

$$S' = \frac{1}{2}\tanh kc \tag{u}$$

or

$$\omega = kU + \frac{\sigma_s k}{2\varepsilon_o}\tanh kc \tag{v}$$

PROBLEM 7.28 (Continued)

Part c

Note that if S is held fixed at the value given by (u), the force per unit area remains fixed. Thus, as $\sigma_s \to 0$, the velocities of the potential wave and the sheet must become equal to retain the force at a constant value

$$\omega \to kU \tag{w}$$

PROBLEM 8.1

The identity to be verified is

$$\nabla\cdot(\psi\bar{A}) = \psi\nabla\cdot\bar{A} + \bar{A}\cdot\nabla\psi \tag{a}$$

First express the identity in index notation.

$$\frac{\partial}{\partial x_m}[\psi A_m] = \psi\frac{\partial A_m}{\partial x_m} + A_m\frac{\partial\psi}{\partial x_m} \tag{b}$$

The repeated subscript indicates summation. Thus, expanding the first term on the left yields:

$$\psi\frac{\partial A_m}{\partial x_m} + A_m\frac{\partial\psi}{\partial x_m} \equiv \psi\nabla\cdot\bar{A} + \bar{A}\cdot\nabla\psi \tag{c}$$

PROBLEM 8.2

We wish to show that

$$\bar{B}\cdot\nabla(\psi\bar{A}) = \psi\bar{B}\cdot\nabla\bar{A} + \bar{A}\bar{B}\cdot\nabla\psi \tag{a}$$

First, the identity is expressed in index notation, considering the m^{th} component of this vector equation. Note that the equation relates two vectors.

$$(\bar{B}\cdot[\nabla(\psi\bar{A})])_m = (\psi\bar{B}\cdot[\nabla\bar{A}])_m + A_m\bar{B}\cdot\nabla\psi \tag{b}$$

Now, consider each term separately

$$(\bar{B}\cdot[\nabla(\psi\bar{A})])_m = B_k\frac{\partial}{\partial x_k}(\psi A_m) = A_m B_k\frac{\partial\psi}{\partial x_k} + \psi B_k\frac{\partial A_m}{\partial x_k} \tag{c}$$

$$(\psi\bar{B}\cdot[\nabla\bar{A}])_m = \psi B_k\frac{\partial A_m}{\partial x_k} \tag{d}$$

$$A_m\bar{B}\cdot\nabla\psi = A_m B_k[\nabla\psi]_k = A_m B_k\frac{\partial\psi}{\partial x_k} \tag{e}$$

The sum of (d) and (e) give (c) so that the identity is verified.

PROBLEM 8.3

Part a

a_{ik} is the cosine of the angle between the x_i' axis and the x_k axis (see page 435). Thus for our geometry

$$a_{ik} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{a}$$

PROBLEM 8.3 (Continued)

Now, we may apply the transformation law for vectors (Eq. 8.2.10)

$$A_i' = a_{ik}A_k \tag{b}$$

where the components of $\bar{A}$ in the (x_1,x_2,x_3) system are given as

$$A_1 = 1;\ A_2 = 2;\ A_3 = -1 \tag{c}$$

Thus:

$$A_1' = a_{1k}A_k = a_{11}A_1 + a_{12}A_2 + a_{13}A_3 \tag{d}$$

$$A_1' = 1/2 + \sqrt{3} \tag{e}$$

$$A_2' = a_{2k}A_k = -\frac{\sqrt{3}}{2} + 1 \tag{f}$$

$$A_3' = a_{3k}A_k = -1 \tag{g}$$

Using matrix alegbra, we can write a more concise solution. That is:

$$\begin{bmatrix} A_1' \\ A_2' \\ A_3' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} (\frac{1}{2} + \sqrt{3}) \\ (-\frac{\sqrt{3}}{2} + 1) \\ (-1) \end{bmatrix} \tag{h}$$

Part b

The tensor a_{ik} is associated with coordinate transforms involving the direction of force while the tensor $a_{j\ell}$ is associated with coordinate transforms involving the direction of the area normal vectors. The tensor transformation is (Eq. 8.2.17), page 437;

$$T_{ij}' = a_{ik}\,a_{j\ell}T_{k\ell} \tag{i}$$

For example,

$$\begin{aligned} T_{11}' = a_{1k}a_{1\ell}T_{k\ell} = {} & a_{11}a_{11}T_{11} + a_{12}a_{11}T_{21} + a_{13}a_{11}T_{31} \\ & + a_{11}a_{12}T_{12} + a_{12}a_{12}T_{22} + a_{13}a_{12}T_{32} \\ & + a_{11}a_{13}T_{13} + a_{12}a_{13}T_{23} + a_{13}a_{13}T_{33} \end{aligned} \tag{j}$$

Thus:

$$T_{11}' = \frac{7}{4} + \frac{6\sqrt{3}}{4} \tag{k}$$

PROBLEM 8.3 (Continued)

Similarly

$$T'_{12} = -\frac{3}{2} + \frac{\sqrt{3}}{4} \qquad (\ell)$$

$$T'_{13} = 0 \qquad (m)$$

$$T'_{21} = -\frac{3}{2} + \frac{\sqrt{3}}{4} \qquad (n)$$

$$T'_{22} = \frac{5}{4} - \frac{3\sqrt{3}}{2} \qquad (o)$$

$$T'_{23} = 0 \qquad (p)$$

$$T'_{31} = 0 \qquad (q)$$

$$T'_{32} = 0 \qquad (r)$$

$$T'_{33} = 1 \qquad (s)$$

Written in matrix algebra, the problem is solved below:

$$\begin{bmatrix} T'_{11} & T'_{12} & T'_{13} \\ T'_{21} & T'_{22} & T'_{23} \\ T'_{31} & T'_{32} & T'_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \qquad (t)$$

Note that the third matrix on the right is the transpose of a_{ij}. Matrix multiplication of (t) gives

$$T'_{ij} = \begin{bmatrix} (\frac{7}{4} + \frac{6\sqrt{3}}{2}) & (-\frac{3}{2} + \frac{\sqrt{3}}{4}) & 0 \\ (-\frac{3}{2} + \frac{\sqrt{3}}{4}) & (\frac{5}{4} - \frac{3\sqrt{3}}{2}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (u)$$

PROBLEM 8.4

The m^{th} component of the force density at a point is (Eq. 8.1.10)

$$F_i = \frac{\partial T_{ij}}{\partial x_j} \qquad (a)$$

Thus in the $\vec{i}_1$ direction,

PROBLEM 8.4 (Continued)

$$F_1 = \left(\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3}\right) = \left(\frac{P_o^2}{a^2} x_1 - \frac{P_o^2}{a^2} x_1 + 0\right) = 0 \qquad \text{(b)}$$

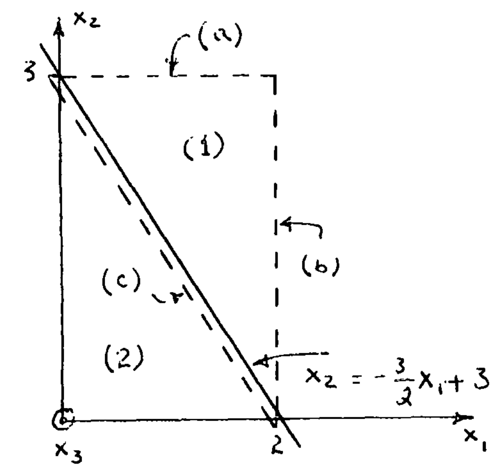
Similarly in the $\vec{i}_2$ and $\vec{i}_3$ directions we find

$$F_2 = \left(\frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3}\right) = 0 \qquad \text{(c)}$$

$$F_3 = \left(\frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3}\right) = 0 \qquad \text{(d)}$$

Hence, the total volume force density resulting from the given stress tensor is zero.

PROBLEM 8.5



in region (1) $\bar{E}=E_o(\frac{3}{2}\vec{i}_1 + \vec{i}_2)$

in region (2) $\bar{E} = 0$

$$T_{ij} = \varepsilon E_i E_j - \frac{\delta_{ij}}{2} \varepsilon_o E_k E_k \qquad \text{(a)}$$

Thus in region (2)

$$T_{ij} = [0] \qquad \text{(b)}$$

in region (1)

PROBLEM 8.5 (Continued)

$$T_{ij} = \begin{bmatrix} \frac{5}{8}\varepsilon_o E_o^2 & \frac{3}{2}\varepsilon_o E_o^2 & 0 \\ \frac{3}{2}\varepsilon_o E_o^2 & -\frac{5}{8}\varepsilon_o E_o^2 & 0 \\ 0 & 0 & -\frac{13}{8}\varepsilon_o E_o^2 \end{bmatrix} \tag{c}$$

The total contribution to the forces found by integrating the stress tensor over surface (c) is zero, because surface (c) lies in region (2) where the stress tensor is zero. By symmetry the sum of contributions to the force resulting from integrations over the two surfaces perpendicular to the x_3 axis is zero.

Now let us note the fact that:

$$\text{area (a)} = 2 \tag{d}$$

$$\text{area (b)} = 3 \tag{e}$$

Thus:

$$f_i = \oint T_{ij}\, n_j\, da \tag{f}$$

$$f_1 = \int_{(b)} T_{11} da + \int_{(a)} T_{12} da + \int T_{13} da$$

$$= \frac{5}{8}\varepsilon_o E_o^2(3) + \frac{3}{2}\varepsilon_o E_o^2(2) \tag{g}$$

$$f_1 = 4\,\frac{7}{8}\,\varepsilon_o E_o^2 \tag{h}$$

$$f_2 = \int_{(b)} T_{21} da + \int_{(a)} T_{22} da + \int T_{23} da$$

$$= \frac{3}{2}\varepsilon_o E_o^2(3) - \frac{5}{8}\varepsilon_o E_o^2(2) \tag{i}$$

$$f_2 = 3\,\frac{1}{4}\,\varepsilon_o E_o^2 \tag{j}$$

$$f_3 = \int T_{31} da + \int T_{32} da + \int T_{33} da$$

$$= 0 \tag{k}$$

Hence, the total force is:

$$\bar{f} = 4\,\frac{7}{8}\,\varepsilon_o E_o^2\, \vec{i}_1 + 3\,\frac{1}{4}\,\varepsilon_o E_o^2\, \vec{i}_2 \tag{ℓ}$$

PROBLEM 8.6

Part a

At point A, the electric field intensity is a superposition of the imposed field and the field due to the surface charges; $\bar{E} = (\sigma_f/\varepsilon_o)\bar{i}_y$. Thus at A,

$$\bar{E} = \bar{i}_x(E_o) + \bar{i}_y(E_o + \frac{\sigma_f}{\varepsilon_o}) \tag{a}$$

while at B,

$$\bar{E} = \bar{i}_x(E_o) + \bar{i}_y(E_o) \tag{b}$$

Thus, from Eq. 8.3.10, at A,

$$T_{ij} = \begin{bmatrix} \frac{1}{2}\varepsilon_o[E_o^2 - (E_o + \frac{\sigma_f}{\varepsilon_o})^2] & \varepsilon_o E_o(E_o + \frac{\sigma_f}{\varepsilon_o}) & 0 \\ \varepsilon_o E_o(E_o + \frac{\sigma_f}{\varepsilon_o}) & \frac{1}{2}\varepsilon_o[(E_o + \frac{\sigma_f}{\varepsilon_o})^2 - E_o^2] & 0 \\ 0 & 0 & -\frac{1}{2}\varepsilon_o[E_o^2 + (E_o + \frac{\sigma_f}{\varepsilon_o})^2] \end{bmatrix} \tag{c}$$

while at B the components are given by (c) with $\sigma_f \to 0$.

Part b

In the x direction, because the fields are independent of x and z,

$$f_x = (b-a)\left[(T_{xy})\Big|_A - (T_{xy})\Big|_B\right]D = (b-a)DE_o\sigma_f \tag{d}$$

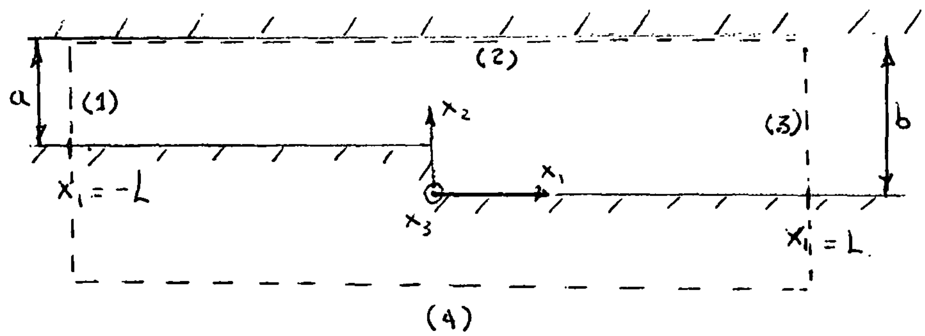
or simply the area multiplied by the surface charge density and x component of electric field intensity.

In the y direction

$$f_y = (b-a)(T_{yy}\Big|_A - T_{yy}\Big|_B)D = (b-a)D[E_o\sigma_f + \frac{\sigma_f^2}{2\varepsilon_o}] \tag{e}$$

Note that both (d) and (e) could be found by multiplying the surface charge density by the average electric field intensity and the area, as shown by Eq. 8.4.8.

PROBLEM 8.7



Before finding the force, we must calculate the $\bar{H}$ field at $x_1 = L$. To find this field let us use

$$\oint \bar{B}\cdot\bar{n}da = 0 \tag{a}$$

over the dotted surface. At $x_1 = + L$,

$$\bar{H}(x_1=L) = H_o\bar{i}_1 \tag{b}$$

over surface (4) $\bar{H} = 0$, and over surface (2), $\bar{H}$ is in the $\bar{i}_1$ direction, where $\bar{n} = \bar{i}_2$. Thus over surface (2) $\bar{B}\cdot\bar{n} = 0$.

Hence, the integral in (a) reduces to

$$-\int_{(1)} \mu_o H_o da + \int_{(3)} \mu_o H(x_1 = + L)da = 0$$

$$-\mu_o H_o a + \mu_o Hb = 0 \qquad \text{per unit depth} \tag{d}$$

Thus:

$$\bar{H}(x_1 = + L) = (a/b)H_o\bar{i}_1 \tag{e}$$

$$T_{ij} = \mu_o H_i H_j - \frac{\delta_{ij}}{2}\mu_o H_k H_k \tag{f}$$

Hence, the stress tensor over surfaces (1), (2) and (3) is:

$$T_{ij} = \begin{bmatrix} \frac{\mu_o}{2}H_1^2 & 0 & 0 \\ 0 & -\frac{\mu_o}{2}H_1^2 & 0 \\ 0 & 0 & -\frac{\mu_o}{2}H_1^2 \end{bmatrix} \tag{g}$$

PROBLEM 8.7 (Continued)

over surface (4)

$$T_{ij} = [0] \tag{h}$$

Thus the force in the 1 direction is

$$f_1 = \int T_{ij}\, n_j\, da \tag{i}$$

$$f_1 = -\int_{(1)} T_{11} da + \int_{(3)} T_{11} da + \int_{(2)} T_{12} da \tag{j}$$

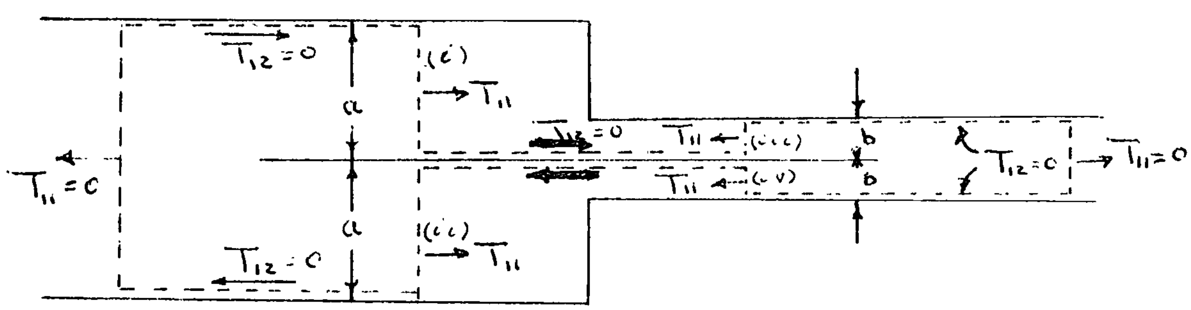
Thus, since the last integral makes no contribution,

$$f_1 = -\frac{\mu_o}{2} H_o^2 (a) + \frac{\mu_o}{2} H_o^2 \left(\frac{a}{b}\right)^2 \cdot b = \frac{\mu_o}{2} H_o^2\, a \left\{\frac{a}{b} - 1\right\} \tag{k}$$

Since $T_{ij} = 0$ over surface (4) there is no contribution to the force from this surface and by symmetry, there is no contribution to the force from the surfaces perpendicular to the x_3 axis. Thus, the force per unit depth in 1 direction is (k).

PROBLEM 8.8

The appropriate surface of integration is shown in the figure



The stresses acting in the x direction on the respective surfaces are as shown. Because the plates are perfectly conducting, all shear stresses required to complete the integration of Eq. 8.1.17 vanish. The only contributions are from surfaces (i), (ii), (iii) and (iv), where the fields

PROBLEM 8.8 (Continued)

are known to be

$$\bar{E} = -\frac{V_o}{a}\bar{i}_y \quad (i) \quad ; \quad \bar{E} = -\frac{V_o}{b}\bar{i}_y \quad (iii)$$

$$\bar{E} = \frac{V_o}{a}\bar{i}_y \quad (ii) \quad ; \quad \bar{E} = \frac{V_o}{b}\bar{i}_y \quad (iv) \qquad (a)$$

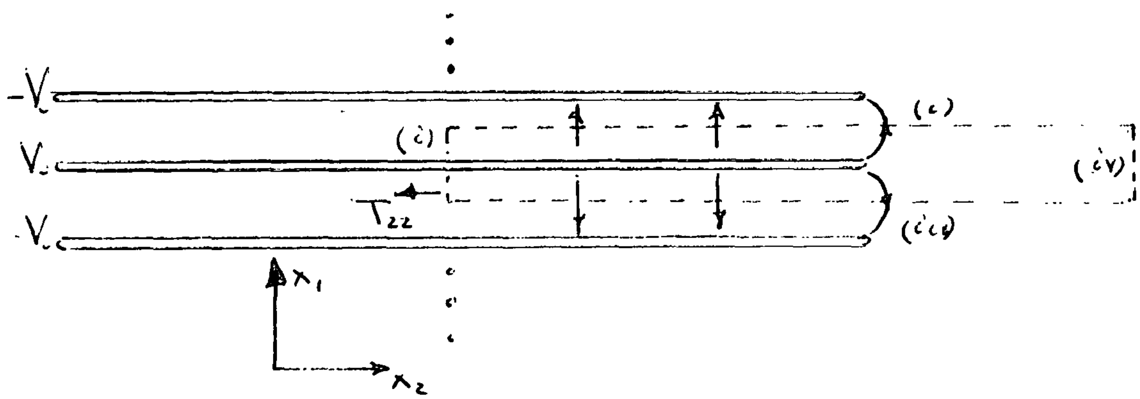
Thus,

$$f_1 = (T_{11})_i\, ad + (T_{11})_{ii}\, ad - (T_{11})_{iii}\, bd - (T_{11})_{iv}\, bd \qquad (b)$$

$$= dV_o^2\varepsilon_o[\frac{1}{b} - \frac{1}{a}] \qquad (c)$$

The plate tends to be drawn to the right, where the fields are greater.

PROBLEM 8.9



The volume enclosing the half of the plate is arbitrary so long as it is defined so that it does not include additional charge. Thus the volume shown in the figure encloses no more than the desired distribution of charge. Moreover, surfaces (i) and (iii) pass through the fringing fields half way between the plates where by symmetry there is no x_2 component of $\bar{E}$. Thus surfaces (i) and (iii) support no shear stress T_{21}. There is no field at surface (iv) and hence the only contribution is from surface (i), where the square of the field is known to be

$$E_1^2 = \frac{4V_o^2}{s^2} \qquad (a)$$

and it follows that because T_{22} on (i) is $-\frac{1}{2}\varepsilon_o E_1^2$ and the normal vector is negative

$$f_2 = \frac{4ws\varepsilon_o}{2}\frac{V_o^2}{s^2} \qquad (b)$$

PROBLEM 8.9 (Continued)

The fringing field tends to pull the end of the plate in the $+x_2$ direction.

PROBLEM 8.10

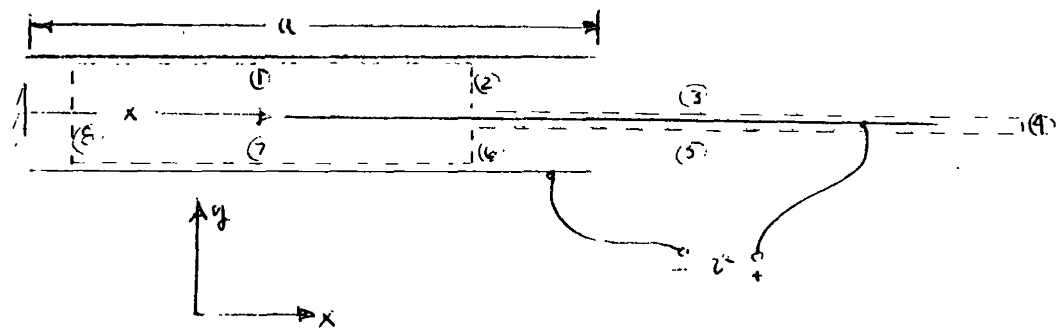


Figure 1

Part a

Consider the surface shown in Figure 1. The total force in the x direction is:

$$f_x = \int_{1,3} T_{xy}\, da - \int_{5,7} T_{xy}\, da + \int_{4} T_{xx}\, da - \int_{8} T_{xx}\, da + \int_{2,6} T_{xx}\, da \tag{a}$$

The first four integrals disappear because:

$T_{xy} = \varepsilon E_x E_y = 0$ on 1, 3, 5 and 7 because we are next to the conducting plates ($E_x = 0$)

$T_{xx} = 0$ an 4 and 8 because the $\bar{E}$ field $= 0$ there

Hence

$$f_x = \int_{2,6} T_{xx} da = \int_{2,6} -\frac{1}{2}\, E_y^2\, da \tag{b}$$

where T_{ij} is evaluated using Eq. 8.3.10.

$$E_y = \frac{v}{s} \tag{c}$$

and hence:

PROBLEM 8.10 (Continued)

$$f_x = \int_{2,6} -\frac{\varepsilon}{2}\left(\frac{v}{s}\right)^2 da = -\frac{\varepsilon d}{s} v^2 \tag{d}$$

Part b

The coenergy of the system is

$$W' = \frac{1}{2} C(x) v^2 \tag{e}$$

where $$C(x) = \frac{2(a-x)d\varepsilon}{s} \tag{f}$$

Thus, (see Sec. 3.1.2b)

$$f_x = \frac{\partial W'}{\partial x} = \frac{1}{2}\frac{\partial C(x)}{\partial x} v^2 = -\frac{d\varepsilon}{s} v^2 \tag{g}$$

which is the same value determined in part (a).

Part c

The equation of motion of the plate is:

$$M\frac{d^2x}{dt^2} + K(x-a) = f_x = -\frac{d\varepsilon}{s} V_o^2 \tag{h}$$

When the system reaches equilibrium with the switch closed,

$$K(X_o - a) = -\frac{d\varepsilon}{s} V_o^2 \tag{i}$$

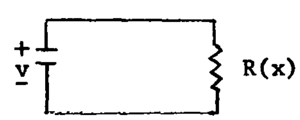
thus

$$X_o = a - \frac{d\varepsilon}{sK} V_o^2 \tag{j}$$

After the switch is opened,

$$M\frac{d^2x}{dt^2} + K(x-a) = -\frac{d\varepsilon}{s} v^2(t) \tag{k}$$

The electrical circuit is like an R-C circuit with time varying elements



$$v + R(x) i(t) = 0 \tag{ℓ}$$

$$v + R(x)\frac{d}{dt}[C(x)v] = 0 \tag{m}$$

$$v + R(x)C(x)\frac{dv}{dt} + R(x)\frac{dC(x)}{dx}\frac{dx}{dt} v = 0 \tag{n}$$

PROBLEM 8.10 (Continued)

where:

$$R(x) = \frac{s}{2\sigma d(a-x)} \qquad \text{and} \quad C(x)\ \frac{2d(a-x)\varepsilon}{s} \tag{o}$$

Hence

$$v + \frac{\varepsilon}{\sigma}\frac{dv}{dt} - \left[\frac{\varepsilon}{\sigma}\ \frac{1}{(a-x)}\ \frac{dx}{dt}\right] v = 0 \tag{p}$$

Part d

Dropping the inertial term from (h) leaves:

$$K(x-a) = -\frac{d\varepsilon}{s} v^2(t) \qquad \text{from (k)} \tag{q}$$

But we may write the identity

$$-\frac{1}{(a-x)}\ \frac{dx}{dt} = \frac{1}{K(x-a)}\ \frac{d}{dt}\ K(x-a) \tag{r}$$

and then, from (q)

$$-\frac{1}{(a-x)}\frac{dx}{dt} = -\frac{s}{d\varepsilon v^2(t)}\ \frac{d}{dt}\cdot\left[-\frac{d\varepsilon}{s} v^2(t)\right]$$

$$= \frac{1}{v^2(t)}\ \frac{d}{dt}\ v^2(t) = \frac{2}{v}\frac{dv}{dt} \tag{s}$$

Substituting back into (p) we have

$$v + \frac{\varepsilon}{\sigma}\frac{dv}{dt} + \frac{2\varepsilon}{\sigma}\frac{dv}{dt} = 0 \tag{t}$$

Solving we find

$$v = V_o\ e^{-(\sigma/3\varepsilon)t} \tag{u}$$

and substituting back into (q),

$$x = a - \frac{d\varepsilon}{sK} V_o^2\ e^{-\frac{2}{3}\frac{\sigma}{\varepsilon}t} \tag{v}$$

A long relaxation time is consistent with neglecting the inertial terms, as then x(t) varies slowly.

Part e

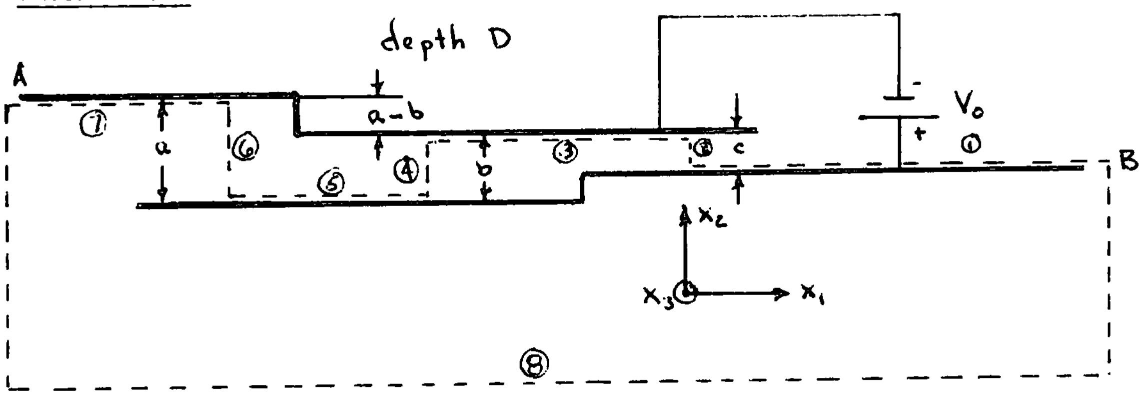
Proceed as in (c), and record the time constant τ of $a-x(t)$ by measuring the mechanical displacement. Then,

$$\frac{\varepsilon}{\sigma} = \frac{2}{3}\ \tau \tag{w}$$

PROBLEM 8.10 (Continued)

This problem should raise questions as to the appropriate form of T_{ij} used in (b). Note that the surface of integration encloses liquid as well as the plate. We want only the force on the plate, so our calculation is correct only if there is no net force on the enclosed liquid. The electrical force density in the liquid is given by Eq. 8.5.45. There is no free charge or gradient of permittivity in the bulk of the liquid and hence the first two of the three contributions to this force density vanish in the liquid. However, there remains the electrostriction force density. Note that it is ignored in our calculation because the electrostriction term was not included in the stress tensor (we used Eq. 8.3.10 rather than 8.5.46). Our reason for ignoring the electrostriction is this: it gives rise to a force density that takes the form of the gradient of a pressure. Hence, it simply alters the distribution of liquid pressure around the plate. Because each element of the liquid is in static equilibrium and can give way to motions of the plate without changing its volume, the "hydrostatic pressure" of the liquid is altered by the electric field so as to exactly cancel the effect of the electrostriction force density. Hence, to correctly include the effect of electrostriction in integrating the stresses over the surface, we must also include the hydrostatic pressure of the liquid. If this is done, the effect of the electrostriction will cancel out, leaving the force on the plate we have derived by two alternative methods here.

PROBLEM 8.11



PROBLEM 8.11 (Continued)

First, let us note the $\bar{E}$ fields on each of the surfaces of the figure over surfaces (1), (3), (5) and (7), $E_1 = 0$ (a)

over surface

$$(6)\ E_2 = \frac{V_o}{a} \quad E_1 = 0 \tag{b}$$

$$(4)\ E_2 = \frac{V_o}{b} \quad E_1 = 0 \tag{c}$$

$$(2)\ E_2 = \frac{V_o}{c} \quad E_1 = 0 \tag{d}$$

From Eq. 8.3.10,

$$T_{ij} = \varepsilon_o E_i E_j - \frac{\delta_{ij}}{2} \varepsilon_o E_k E_k \tag{e}$$

Hence, over surfaces (1), (3), (5) and (7)

$$T_{12} = 0 \tag{f}$$

and over surfaces

$$(6)\quad T_{11} = -\frac{\varepsilon_o}{2}\left(\frac{V_o}{a}\right)^2 \tag{g}$$

$$(4)\quad T_{11} = -\frac{\varepsilon_o}{2}\left(\frac{V_o}{b}\right)^2 \tag{h}$$

$$(2)\quad T_{11} = -\frac{\varepsilon_o}{2}\left(\frac{V_o}{c}\right)^2 \tag{i}$$

Now;

$$f_1 = \int T_{ij} n_j da = \int T_{11} n_1 da + \int T_{12} n_2 da + \int T_{13} n_3 da \tag{j}$$

$\int T_{13} n_3 da = 0$ because the problem is two dimensional. (k)

Let us consider each of the other integrals:

$$\int T_{12} n_2 da = 0 \tag{ℓ}$$

because the surfaces which have normal n_2 are (1), (3), (5) and (7) and by (f) we have shown that $T_{12} = 0$ over these surfaces. Also, we get no contribution to the force over surface (8), because $\bar{E} \to 0$ faster than the area $\to \infty$.

Hence the calculation of the force reduces tc

PROBLEM 8.11 (Continued)

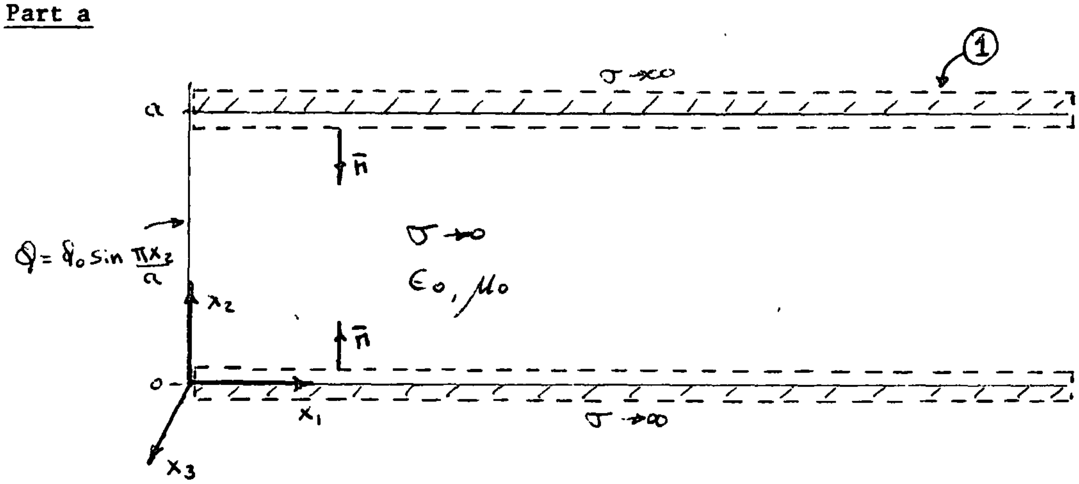
$$f_1 = \int_{(6)} T_{11}^{(6)}\, da_6 - \int_{(4)} T_{11}^{(4)}\, da_4 - \int_{(2)} T_{11}^{(2)}\, da_2 \tag{m}$$

$$f_1 = -\frac{\varepsilon_o D V_o^2}{2}\left\{\frac{1}{a} - \frac{1}{b} + \frac{1}{c}\right\} \tag{n}$$

Note: by symmetry, there is no contribution to the force from the surfaces perpendicular to the x_3 axis.

PROBLEM 8.12

Part a



From elementary field theory, we find that

$$\phi = \phi_o \sin\frac{\pi x_2}{a}\, e^{-\pi x_1/a} \tag{a}$$

satisfies $\nabla^2\phi = 0$ in the region between the plates and the required boundary conditions. The distribution of $\bar{E}$ follows from

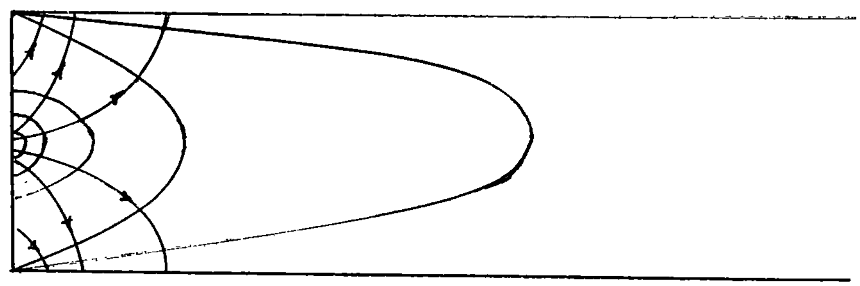
$$\bar{E} = -\nabla\phi \tag{b}$$

Hence,

$$\bar{E} = \frac{\pi\phi_o}{a}\, e^{-\pi x_1/a}\left[\sin\frac{\pi x_2}{a}\,\bar{i}_1 - \cos\frac{\pi x_2}{a}\,\bar{i}_2\right] \tag{c}$$

The sketch of the $\bar{E}$ field is obtained by recognizing that $\bar{E}$ is directed perpendicular to contours of constant ϕ.

PROBLEM 8.12 (Continued)



Part b

To find the force as the bottom plate, we use surface (2). $\bar{E} = 0$ everywhere except on the upper side where the normal $\bar{n} = \bar{i}_2$ (d)

and the field is

$$\bar{E} = -\frac{\pi\phi_o}{a} e^{-\pi x_1/a} \bar{i}_2 \tag{e}$$

Hence,

$$f_1 = \int T_{1j} n_j \, da = 0 \tag{f}$$

$$f_2 = \int T_{2j} n_j \, da = \int T_{22} n_2 \, da_2 \tag{g}$$

per unit x_3, this reduces to

$$f_2 = \int_o^\infty T_{22} dx_1 \tag{h}$$

but, $T_{22} = \frac{1}{2}\varepsilon_o E_2 E_2 = \frac{1}{2}\varepsilon_o \frac{\pi^2\phi_o^2}{a^2} e^{-\frac{2\pi x_1}{a}}$ (i)

and thus

$$f_2 = \frac{\varepsilon_o \pi^2 \phi_o^2}{2a^2} \int_o^\infty e^{-\frac{2\pi x_1}{a}} dx_1 \tag{j}$$

$$f_2 = \frac{\varepsilon_o \pi \phi_o^2}{4a} \tag{k}$$

PROBLEM 8.12 (Continued)

Part c

On the top plate, use surface (1). Only the sign of the normal changes, and the result is

$$f_1 = 0 \tag{ℓ}$$

$$f_2 = -\frac{\varepsilon_o \pi \phi_o^2}{4a} \tag{m}$$

or the force is equal and opposite to that on the bottom plate.

PROBLEM 8.13

Part a

$$T_{ij} = \varepsilon E_i E_j - \frac{\delta_{ij}}{2} \varepsilon E_k E_k \tag{a}$$

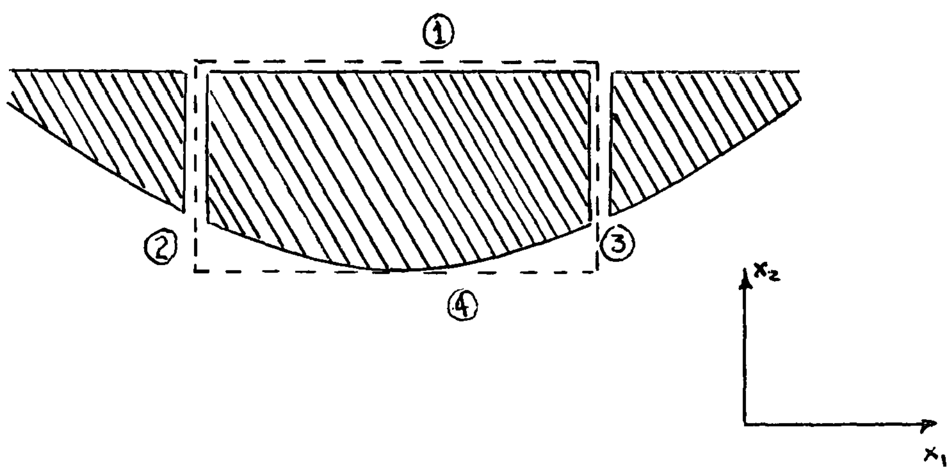
Hence:

$$T_{22} = \frac{1}{2} \varepsilon_o \left(\frac{2V_o}{3a^2}\right)^2 \left(x_2^2 - x_1^2\right) \tag{b}$$

$$T_{21} = \varepsilon_o E_2 E_1 = -\varepsilon_o \left(\frac{2V_o}{3a^2}\right)^2 x_1 x_2 \tag{c}$$

Part b

Consider the surface of integration shown in the figure.



$$f_2 = \int T_{2j} n_j da = \underset{(2)(3)}{\int T_{21} n_1 da} + \underset{(1)(4)}{\int T_{22} n_2 da} + \underset{\text{by symmetry}}{\int \cancelto{0}{T_{23} n_3 da}} \tag{d}$$

PROBLEM 8.13 (Continued)

Let us look at each of these integrals separately

$$\int_{(1)(4)} T_{22}n_2 da = \int_{(1)} T_{22}da_1 - \int_{(4)} T_{22}da_4 \tag{e}$$

over surface (1), $\bar{E} \simeq 0$; $T_{22} \simeq 0$ and hence, the integral is merely:

$$-\int_{(4)} T_{22}da_4 = -\int_{\substack{x_1=-a \\ x_2 = 2a}}^{x_1=a} \frac{\varepsilon_o}{2}\left(\frac{2V_o}{3a^2}\right)^2 \left(x_2^2 - x_1^2\right) wdx_1$$

$$= -\frac{44}{27}\frac{\varepsilon_o V_o^2 w}{a} \tag{f}$$

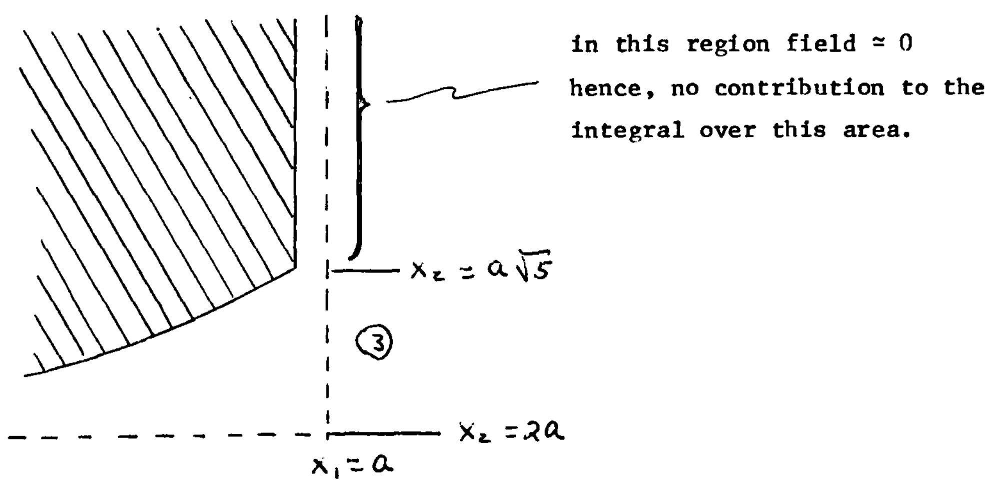
Thus,

$$\int_{(1)(4)} T_{22}n_2 da = -\frac{44}{27}\frac{\varepsilon_o V_o^2 w}{a} \tag{g}$$

Let us now evaluate:

$$\int_{(2)(3)} T_{21}n_1 da$$

Consider the surface shown.



PROBLEM 8.13 (Continued)

Thus;

$$\int_{(3)} T_{21}da_3 = \int_{\substack{x_2=2a \\ x_1=a}}^{x_2=a\sqrt{5}} -\varepsilon_o\left(\frac{2V_o}{3a^2}\right)^2 awx_2dx_2$$

$$= -\frac{2}{9}\frac{\varepsilon_o V_o^2 w}{a} \tag{h}$$

Over surface (2), we have essentially the same thing, except $\bar{n} = -\bar{i}_1$ and $x_1 = -a$. Hence:

$$\int_{(2)} T_{21}da_2 = -\frac{2}{9}\frac{\varepsilon_o V_o^2 w}{a} \tag{i}$$

Therefore, the total force in the f_2 direction is

$$f_2 = -\frac{56}{27}\frac{\varepsilon_o V_o^2 w}{a} \tag{j}$$

Part c

$$f_1 = \int_{(2)(3)} T_{11}n_1da + \int_{(1)(4)} T_{12}n_2da + \underbrace{\int \cancel{T_{13}n_3}^{0}da}_{\text{by symmetry}} \tag{k}$$

$$\int_{(1)(4)} T_{12}n_2da = -\int_{(4)} T_{12}da_4$$ over (1) we get 0 as before

$$= \varepsilon_o\left(\frac{2V_o}{3a^2}\right)^2 \int_{-a}^{a} x_1x_2wdx_1 = 0 \qquad x_2=2a \tag{ℓ}$$

Now, over surfaces, 2 and 3

$$\int_{(2)(3)} T_{11}n_1da = -\int_{(2)} T_{11}da_2 + \int_{(3)} T_{11}da_3 = 0 \tag{m}$$

because,

$$T_{11}\big|_2 = T_{11}\big|_3$$

hence $f_1 = 0$. (n)

PROBLEM 8.13 (Continued)

Part d

$$\sigma_f = \bar{n} \cdot \varepsilon_o \bar{E} \tag{o}$$

at the lower surface of the movable conductor. The functional relation, $f(x_1 x_2)$, for the lower surface if the movable conductor is given as

$$f(x_1 x_2) = \sqrt{4a^2 + x_1^2} - x_2 = 0 \tag{p}$$

the outward unit normal to this surface is

$$\bar{n} = \frac{\nabla f(x_1 x_2)}{|\nabla f(x_1 x_2)|} = \left[\frac{x_1}{x_2}\bar{i}_1 - \bar{i}_2\right]\left[\frac{1}{\sqrt{\left(\frac{x_1}{x_2}\right)^2 + 1}}\right] \tag{q}$$

at $x_2 = \sqrt{4a^2 + x_1^2}$,

$$\sigma_f = \varepsilon_o[n_1 E_1 + n_2 E_2] = \frac{2V_o\varepsilon_o}{3a^2}\left[\frac{x_1^2}{x_2} + x_2\right]\left[\frac{4a^2 + x_1^2}{4a^2 + 2x_1^2}\right]^{1/2} \tag{r}$$

The surface force density (see Eq. 8.4.8) is equal to:

$$\bar{T} = \sigma_f \frac{\bar{E}^b + \bar{E}^a}{2} \tag{s}$$

where, $\bar{E}^b$ = field just below the charge sheet

$\bar{E}^a$ = field just above the charge sheet

Since

$$\bar{E}^a = 0, \quad \bar{T} = \frac{1}{2}\sigma_f \bar{E}^b \tag{t}$$

thus

$$\bar{T} = \frac{\varepsilon_o}{2}\left(\frac{2V_o}{3a^2}\right)^2\left[\frac{x_1^2}{x_2} + x_2\right]\left[x_1\bar{i}_1 - x_2\bar{i}_2\right]\left[\frac{4a^2 + x_1^2}{4a^2 + 2x_1^2}\right]^{1/2} \tag{u}$$

To find the total force, the surface force density must be integrated over the surface. Hence, we find

$$f_2 = -2\varepsilon_o\left(\frac{V_o}{3a^2}\right)^2 \int_{-a}^{+a} \{2x_1^2 + 4a^2\}^{1/2}\{x_1^2 + 4a^2\}^{1/2}\, dx_1 \tag{v}$$

If the student wishes, he may **carry** out this integral, but the complexity of the integration shows the value of the stress tensor in calculating such a

PROBLEM 8.13 (Continued)

force. We realize that by using the stress tensor, we have essentially carried out this difficult integral by an integration by parts.

PROBLEM 8.14

Part a

$$\phi = \frac{V_o}{a^2} x_1 x_2 \tag{a}$$

$$\bar{E} = -\nabla\phi \tag{b}$$

hence,

$$\bar{E} = \bar{i}_1 \left(-\frac{V_o}{a^2} x_2\right) + \bar{i}_2 \left(-\frac{V_o}{a^2} x_1\right) \tag{c}$$

and, from Eq. 8.3.10

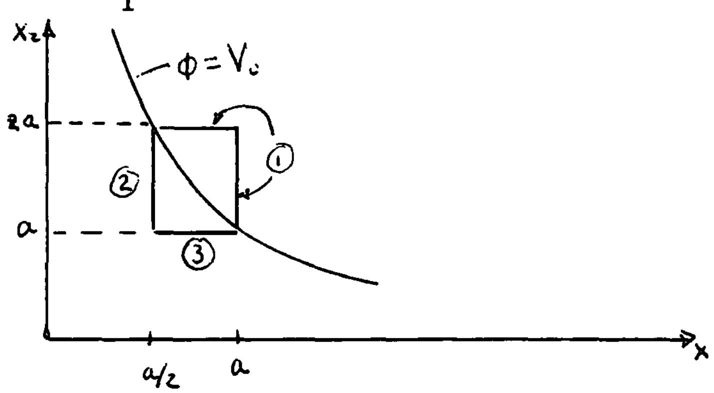
$$T_{ij} = \varepsilon E_i E_j - \delta_{ij} \frac{\varepsilon}{2} E_k E_k \tag{d}$$

Thus, the stress tensor becomes:

$$T_{ij} = \begin{bmatrix} \left(\frac{V_o}{a^2}\right)^2 \frac{\varepsilon_o}{2} (x_2^2 - x_1^2) & \left(\frac{V_o}{a^2}\right)^2 \varepsilon_o (x_1 x_2) & 0 \\ \left(\frac{V_o}{a^2}\right) \varepsilon_o (x_1 x_2) & \left(\frac{V_o}{a^2}\right)^2 \frac{\varepsilon_o}{2} (x_1^2 - x_2^2) & 0 \\ 0 & 0 & -\left(\frac{V_o}{a^2}\right)^2 \frac{\varepsilon_o}{2} (x_1^2 + x_2^2) \end{bmatrix} \tag{e}$$

Part b

Consider the surface shown, bounded by the line segment $x_2 = 2a$, $x_2 = a$, and $x_1 = a/2$ and $x_1 = a$.



PROBLEM 8.14 (Continued)

As before, because the geometry and fields are two-dimensional, the force in the $\bar{i}_3$ direction is zero. Also, since along surface (1) ϕ = constant, then the $\bar{E}$ field = 0, and hence $T_{ij} = 0$ along this surface. Thus the calculation of the force on AB reduces to:

$$f_1 = -\int_{(2)} T_{11}da - \int_{(3)} T_{12}da \tag{f}$$

$$f_2 = -\int_{(2)} T_{21}da - \int_{(3)} T_{22}da \tag{g}$$

$$f_1 = -\left(\frac{V_o}{a^2}\right)^2 \varepsilon_o D\left[\frac{1}{2}\int_a^{2a}[x_2^2 - (\tfrac{a}{2})^2]dx_2 + \int_{a/2}^{a} x_1 a dx_1\right] \tag{h}$$

and hence

$$f_1 = -\varepsilon_o \left(\frac{V_o}{a^2}\right)^2 Da^3 \left[\frac{17}{12}\right] \tag{i}$$

Similarly:

$$f_2 = -\left(\frac{V_o}{a^2}\right)^2 D\varepsilon_o\left[\int_a^{2a}\frac{a}{2}x_2dx_2 + \frac{1}{2}\int_{a/2}^{a}(x_1^2-a^2)dx_1\right] \tag{j}$$

and hence

$$f_2 = -\varepsilon_o D\, a^3 \left(\frac{V_o}{a^2}\right)^2 \left[\frac{31}{48}\right] \tag{k}$$

Thus,

$$\bar{f} = -\varepsilon_o \frac{V_o^2}{a} D\left[\bar{i}_1\frac{17}{12} + \bar{i}_2\frac{31}{48}\right] \tag{ℓ}$$

PROBLEM 8.15

Part a

The $\bar{E}$ field in the laboratory frame is zero since the two perfectly conducting plates are shorted. This can be seen by integrating $\bar{E}$ around a fixed contour through the block and short and recognizing that the enclosed flux is constant. Hence,

$$\bar{E}' = \bar{E} + \bar{v}\times\bar{B}\,, \quad \bar{E} = 0 \tag{a}$$

and thus

$$\bar{E}' = \bar{v}\times\bar{B} = -V\mu_o H_o \bar{i}_2 \tag{b}$$

Therefore we may now calculate $\bar{J}$ in the moving block.

PROBLEM 8.15 (Continued)

$$\bar{J}' = \sigma \bar{E}' = -\sigma \mu_o V H_o \bar{i}_2 \tag{c}$$

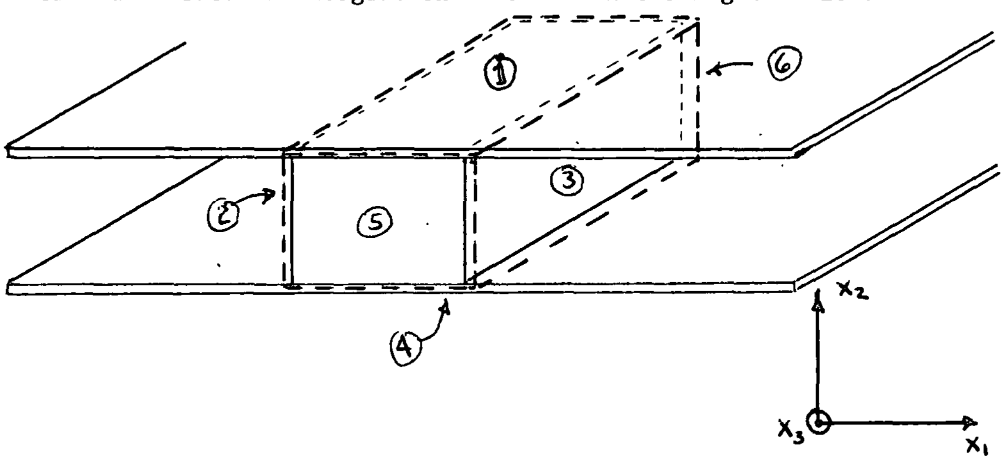
Thus:

$$\bar{F} = \bar{J} \times \bar{B} = -\sigma \mu_o^2 V H_o^2 \bar{i}_1 \tag{d}$$

$$\bar{f} = \int_{\text{volume}} (\bar{J} \times \bar{B}) dV = -\mu_o^2 \sigma V H_o^2 (abD) \bar{i}_1 \tag{e}$$

Part b

The closed surface of integration is shown in the figure below.



Since the field is uniform everywhere, the only non-zero components of the stress tensor are the diagonal elements

$$T_{11} = T_{22} = -\frac{1}{2} \mu_o H_o^2 \qquad T_{33} = \frac{1}{2} \mu_o H_o^2 \tag{f}$$

Thus

$$f_1 = \int_{(3)} T_{11} da_3 - \int_{(2)} T_{11} da_2$$

$$= \frac{\mu_o}{2} H_o^2 \, bD - \frac{\mu_o}{2} H_o^2 bD = 0 \tag{g}$$

Similarly

$$f_2 = \int_{(1)} T_{22} da_1 - \int_{(4)} T_{22} da_4 = 0 \tag{h}$$

$$f_3 = \int_{(5)} T_{33} da_5 - \int_{(6)} T_{33} da_6 = 0 \tag{i}$$

Hence:

$$\bar{f} = 0 \tag{j}$$

PROBLEM 8.15 (Continued)

Part c

The magnetic field strength and the current density are inconsistant. The quasi-static magnetic field cannot be uniform and irrotational in a region where a finite current density exists. The Maxwell stress tensor was developed with the aid of Ampere's Law (quasi-static) which relates current density and magnetic field rotation.

$$\bar{J} = \nabla \times \bar{H} \tag{k}$$

$$\bar{F} = \mu_o \bar{J} \times \bar{H} = \mu_o(\nabla \times \bar{H}) \times \bar{H} \tag{ℓ}$$

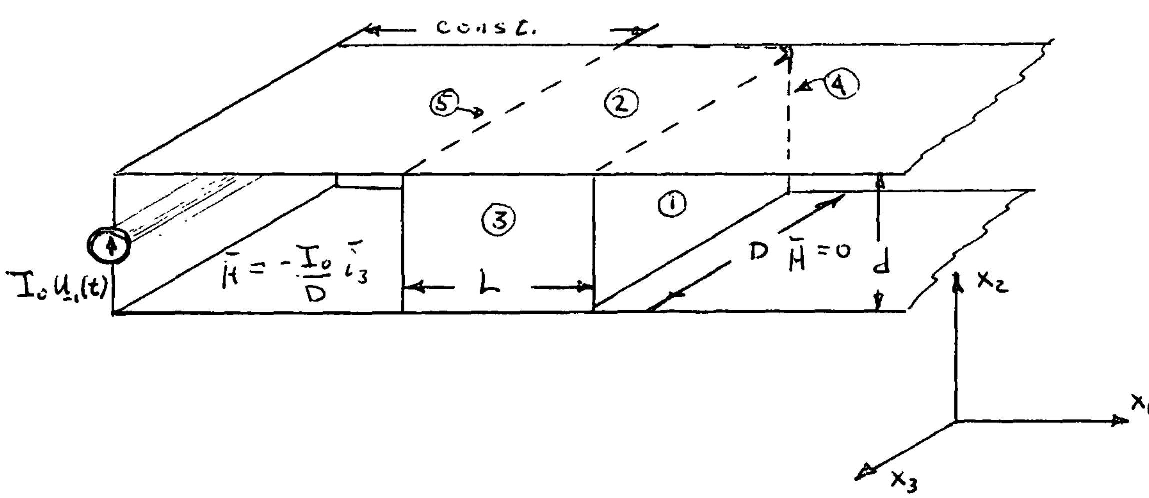
For this case, we have assumed that

$$\nabla \times \bar{H} = 0 \tag{m}$$

In the limit of small magnetic Reynold's number, ($R_m << 1$), the motion does not appreciably affect the field, and the answer found in part a is a good approximation. There are some problems more easily handled with the stress tensor. This problem illustrates that in other cases it is easiest to use the force density $\bar{J} \times \bar{B}$ directly. Note that we could compute the field induced by $\bar{J}$ and then use the Maxwell stress tensor and the self-consistent fields to find the same force as given by (e).

PROBLEM 8.16

To find the force on the block, we will use the stress tensor over the surface shown in the figure. Note that the surface is just outside the block.



PROBLEM 8.16 (Continued)

In the region to the left of the block

$$\bar{H} = -\frac{I_o}{D}\,\bar{i}_3 \text{ , and to the right } \bar{H}=0$$

Thus:

$$f_1 = \int T_{11}n_1 da + \int T_{12}n_2 da + \int T_{13}n_3 da \tag{a}$$

but, since

$$H_1 = H_2 = 0;\; T_{12} = T_{13} = 0 \tag{b}$$

hence,

$$f_1 = -\int_{(5)} T_{11} da_5 + \int_{(1)} T_{11} da_1 \tag{c}$$

on surface (5),

$$T_{11} = -\frac{\mu_o}{2}\frac{I_o^2}{D^2} \tag{d}$$

on surface (1)

$$T_{11} = 0 \tag{e}$$

therefore

$$f_1 = +\frac{\mu_o}{2}\frac{I_o^2}{D^2}\cdot Dd = +\frac{\mu_o I_o^2 d}{2D} \tag{f}$$

Similarly, f_2 reduces to

$$f_2 = \int_2 T_{22} da_2 - \int_6 T_{22} da_6 \tag{g}$$

But, since T_{22} is a function of x_1 alone ($\bar{H}$ is a function of x_1 alone) the two surface integrals are identical, and hence $f_2 = 0$. Similar reasoning shows that $f_3 = 0$ and thus the total force is

$$\bar{f} = \frac{\mu_o I_o^2 d}{2D}\,\bar{i}_1$$

PROBLEM 8.17

Part a

$$\nabla^2\bar{H} = \mu_o\sigma\,\frac{\partial\bar{H}}{\partial t} \tag{a}$$

Assume a solution of the form:

$$\bar{H} = R_e[\underline{H}_z(x)e^{j\omega t}\bar{i}_z] \tag{b}$$

$$\frac{\partial^2\underline{H}_z}{\partial x^2} = j\omega\sigma\mu_o\,\underline{H}_z \tag{c}$$

PROBLEM 8.17 (Continued)

Try

$$\underline{H}_z(x) = \underline{H}_o\, e^{Kx} \tag{d}$$

where

$$K^2 = j\omega\sigma\mu_o \tag{e}$$

and hence

$$K = \pm\sqrt{\frac{\omega\mu_o\sigma}{2}}\,(1+j) \tag{f}$$

Let us define the skin depth as:

$$\delta \equiv \sqrt{\frac{2}{\omega\mu_o\sigma}} \tag{g}$$

And thus

$$\bar{H} = \left[\underline{H}_1\, e^{-\frac{x}{\delta}(1+j)} + \underline{H}_2\, e^{+\frac{x}{\delta}(1+j)}\right] e^{j\omega t}\, \bar{i}_z \tag{h}$$

Because the skin depth δ is assumed to be small, and the excitation is on the left,

$$H_z(\text{large } x) \rightarrow 0 \qquad \text{which implies } \underline{H}_2 = 0$$

Hence,

$$\bar{H}(x_1 t) = \underline{H}_1\, e^{-\frac{x}{\delta}(1+j)}\, e^{j\omega t}\bar{i}_z \tag{i}$$

But, our boundary condition at $x = 0$ is

$$\underline{H}(x=0,t) = \text{Re}H_1 e^{-j\omega t}\, \bar{i}_z = -\,\text{Re}\,\frac{I}{D}\, e^{j\omega t}\, \bar{i}_z \tag{j}$$

and thus

$$\bar{H}(x_1 t) = -\frac{I}{D}\, e^{-\frac{x}{\delta}(1+j)}\, e^{j\omega t}\, \bar{i}_z \tag{k}$$

$$\underline{J} = \nabla \times \bar{H} = -\left(\frac{\partial H_z}{\partial x}\right)\bar{i}_y = -\frac{I}{D}\,\frac{(1+j)}{\delta}\, e^{-\frac{x}{\delta}(1+j)}\, e^{j\omega t}\, \bar{i}_y \tag{ℓ}$$

Part b

$$\bar{f} = \int \bar{J} \times \bar{B}\, dV = \int \bar{J} \times \mu_o \bar{H} dV \tag{m}$$

$$\bar{f} = \text{Re}\left[\frac{\int \bar{J}\times\mu_o\bar{H}^*}{2}\, dV\right] + \text{Re}\left[\frac{\int \bar{J}\times\mu_o\bar{H}}{2}\, e^{2j\omega t}\, dV\right] \tag{n}$$

Now, solving each of these integrals:

$$\frac{\int \bar{J}\times\mu_o\bar{H}^*}{2}\, dV = \frac{1}{2}\,\mu_o a D\left(\frac{I}{D}\right)^2 \frac{1}{\delta}\,(1+j)\int_o^\infty e^{-\frac{2x}{\delta}}\, dx\, \bar{i}_x$$

$$= \frac{1}{4}\,\frac{\mu_o a}{D}\, I^2(1+j)\bar{i}_x \tag{o}$$

PROBLEM 8.17 (Continued)

$$\frac{\int \bar{J}\times\mu_o\bar{H}}{2}\, e^{2j\omega t} dV = \frac{1}{2}\,\mu_o aD\, \left(\frac{I}{D}\right)^2 \frac{1}{\delta}(1+j)e^{2j\omega t}\int_o^\infty e^{-\frac{2x}{\delta}(1+j)} dx\ \bar{i}_x$$

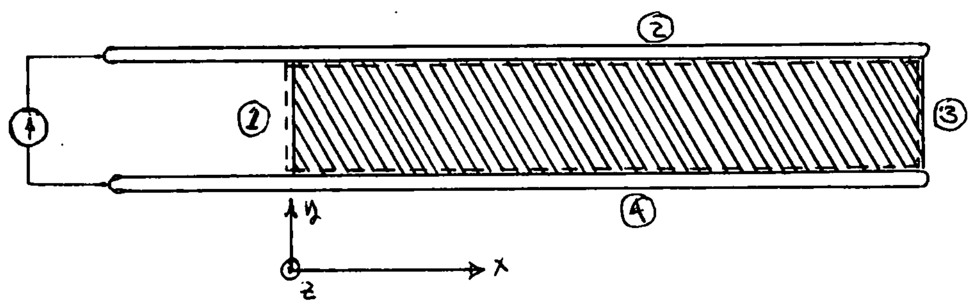
$$= \frac{1}{4}\,\frac{\mu_o a}{D}\, I^2\, e^{2j\omega t}\ \bar{i}_x \qquad (p)$$

Hence, taking the real part, the force as in equation (n) is:

$$\bar{f} = \frac{1}{4}\,\frac{\mu_o a}{D}\, I^2(1 + \cos\, 2\omega t)\bar{i}_x \qquad (q)$$

Part c

Using the Maxwell stress tensor, we choose the surface shown in the figure,



$$f_x = \int T_{xj}n_j da = \int_{(1)(3)} T_{xx}n_x da + \int_{(2)(4)} T_{xy}n_y da \qquad (r)$$

Along surfaces (2) and (4), $H_x = 0$ along the interface between the perfect conductors and the finite conductivity block. Thus,

$$T_{xy} = \mu_o H_x H_y = 0 \qquad (s)$$

At surface (3), the field is zero since all current filaments complete a closed loop circuit with the source through the block. Hence

$$T_{xx} = 0 \text{ on surface (3)} \qquad (t)$$

Therefore the calculation of the force reduces to

$$f_x = -\int_1 T_{xx} da \qquad (u)$$

$$T_{xx} = -\frac{\mu_o}{2} H_z^2 \qquad (v)$$

And thus,

$$f_x = \frac{aD\mu_o}{2} H_z^2 \qquad (w)$$

PROBLEM 8.17 (Continued)

where the field H_z is evaluated on surface 1, i.e. $x = 0$, and is simply given by the boundary condition (j). Thus it follows

$$\bar{f} = \frac{a\mu_o}{4D} I^2 \{1 + \cos 2\omega t\}\bar{i}_x \tag{x}$$

which checks with (q). Note that the distribution of $\bar{J}$ and $\bar{H}$, as found in part (a), are not required to find the total force in this problem. Even more, (x) is not limited to δ << x block dimension, while the detailed integration is.
Note: We have made use of the rule for products, namely of:

$$a(t) = Re[Ae^{j\omega t}] = \frac{Ae^{j\omega t} + A^*e^{-j\omega t}}{2}$$

$$b(t) = Re[Be^{j\omega t}] = \frac{Be^{j\omega t} + B^*e^{-j\omega t}}{2}$$

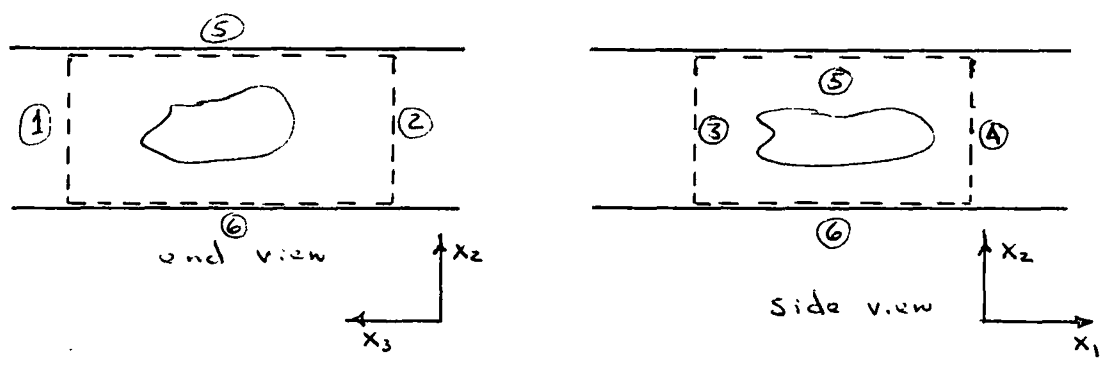
then

$$a(t)b(t) = \frac{AB^* + A^*B}{4} + \frac{ABe^{2j\omega t} + A^*B^*e^{-2j\omega t}}{4}$$

$$= \underbrace{Re[\frac{AB^*}{2}]}_{\text{avg. value}} + \underbrace{Re[\frac{AB}{2} e^{2j\omega t}]}_{\text{time varying part}}$$

PROBLEM 8.18

Choose the surface shown in the figure.



$$f_1 = \int T_{1j}n_j da = \int_{3,4} T_{11}n_1 da + \int_{5,6} T_{12}n_2 da + \int_{1,2} T_{13}n_3 da \tag{a}$$

Since the plates are perfectly conducting, $E_1 = 0$ at surfaces (5) and (6) and hence $T_{12} = 0$ on surfaces (5) and (6). Surfaces (1), (2), (3) and (4) are far from the body so

PROBLEM 8.18 (Continued)

$$\bar{E} = \frac{V_o}{d}\,\bar{i}_z \tag{b}$$

at each of them, and thus, on surfaces (1) and (3), $T_{13} = 0$. Therefore,

$$f_1 = -\int_{(3)} T_{11}da_3 + \int_{(4)} T_{11}da_4 \tag{c}$$

$$T_{11}^{(3)} = T_{11}^{(4)} = -\frac{\varepsilon_o}{2}\left(\frac{V_o}{d}\right)^2 \tag{d}$$

and $a_3 = a_4$ (areas). Hence,

$$f_1 = 0 \tag{e}$$

PROBLEM 8.19

Part a

Since the system is electrically linear,

$$\bar{B} = \bar{B}_\ell + \bar{B}_r \tag{a}$$

where $\bar{B}_\ell$ and $\bar{B}_r$ are respectively the fields from the left and right wires. The force on a unit length of the right wire is

$$\bar{f}_r = \int \bar{J}_r \times \bar{B}\,da = \int \bar{J}_r \times \bar{B}_\ell\,da + \int \bar{J}_r \times \bar{B}_r\,da \tag{b}$$

but,

$$\int \bar{J}_r \times \bar{B}_r\,da = 0 \tag{c}$$

ane hence,

$$\bar{f}_r = \bar{J}_r \times \bar{B}_\ell\,da \tag{d}$$

Since, we don't need the fields near the wire,

$$\bar{B}_\ell \simeq \frac{\mu_o I}{2\pi}\left[\frac{x_2\bar{i}_1 - (x_1+a)\bar{i}_2}{(x_1+a)^2 + x_2^2}\right] \tag{e}$$

$$\bar{B}_r \simeq \frac{\mu_o I}{2\pi}\left[\frac{-x_2\bar{i}_1 + (x_1-a)\bar{i}_2}{(x_1-a)^2 + x_2^2}\right] \tag{f}$$

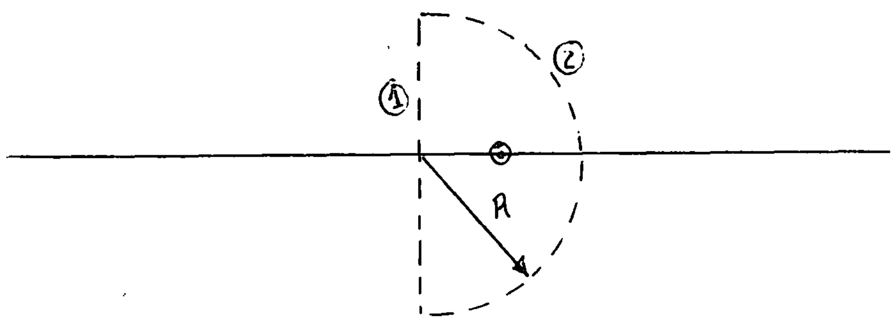
Hence,

$$\bar{f}_r = \bar{J}_r \times \bar{B}_\ell da \simeq I\bar{i}_3 \times \bar{B}_\ell\ (x_1=a,\ x_2=0) \tag{g}$$

$$\bar{f}_r = \frac{\mu_o I^2}{2\pi}\frac{(2a)\bar{i}_1}{(2a)^2} = \frac{\mu_o I^2}{4\pi a}\,\bar{i}_1 \tag{h}$$

PROBLEM 8.19 (Continued)

Part b



Along the symmetry plane of the surface shown in the figure

$$\bar{B} = \frac{\mu_o I}{2\pi} \frac{(-2a)}{(a^2+x_2^2)} \bar{i}_2 \tag{i}$$

The terms of T_{ij} go as B^2, but $B^2 \alpha \frac{1}{R^2}$ and the surface area goes as $2\pi R$ on surface (2), hence the contributions of the stress tensor will vanish on surface (2) as $R \to \infty$; we need only compute the integral on surface (1). Because $H_1 = 0$ in the plane $x_1 = 0$

$$f_1 = \int -T_{11} da = \int_{-\infty}^{+\infty} \frac{\mu_o H_2^2}{2} dx_2$$

$$= \frac{\mu_o}{2} \left(\frac{Ia}{\pi}\right)^2 \int_{-\infty}^{+\infty} \frac{dx_2}{(a^2+x_2^2)^2} \tag{j}$$

Solving this integral, we find

$$f_1 = \frac{\mu_o I^2}{4\pi a} \tag{k}$$

also

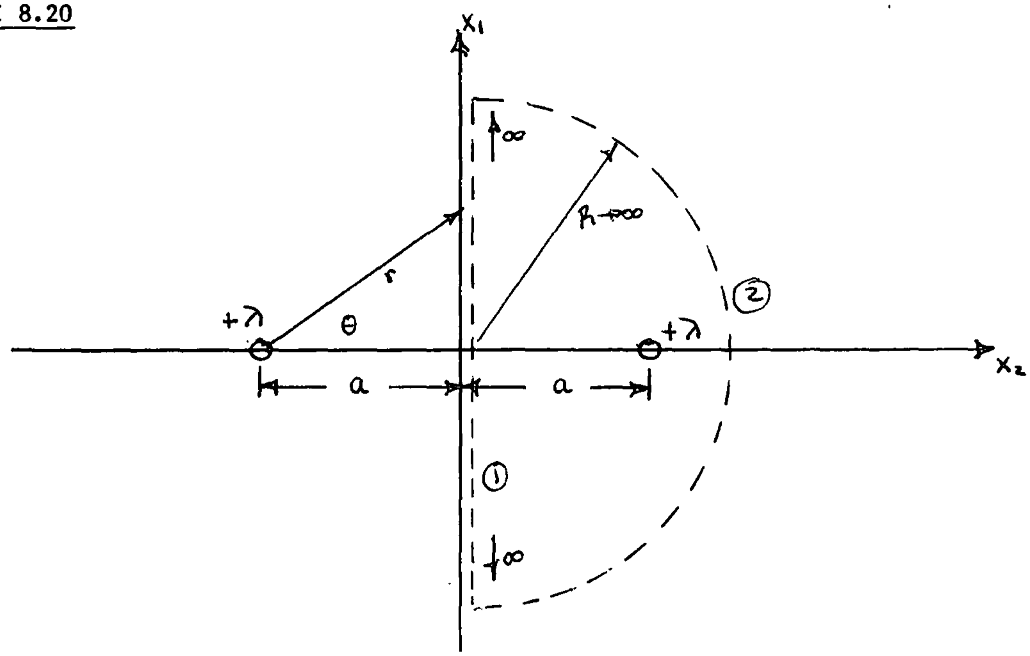
$$f_2 = f_3 = 0 \tag{ℓ}$$

since

$$T_{21} = T_{31} = 0 \tag{m}$$

and hence the total force is that of (k) and it agrees with that determined in part (a).

PROBLEM 8.20



Part a

Use the contour indicated in the figure. At infinity the fields will go to zero, and hence there will be no contribution to the force from the semi-circular part of the area, i.e. surface (2).

Along the line $x_2 = 0$, $E_2 = 0$ by symmetry and

$$E_1 = \frac{2}{\varepsilon_o}\left(\frac{\lambda}{2\pi r}\right)\sin\theta \tag{a}$$

$$r^2 = a^2 + x_1^2 \tag{b}$$

$$\sin\theta = \frac{x_1}{r} = \frac{x_1}{\sqrt{a^2+x_1^2}} \tag{c}$$

Hence

$$E_1 = \frac{\lambda}{\varepsilon_o \pi}\ \frac{x_1}{a^2+x_1^2} \tag{d}$$

$$f_2 = \int T_{2j} n_j da = \int_{(1)} T_{21} n_1 da + \int_{(1)} T_{22} n_2 da + \int_{(1)} T_{23} n_3 da \tag{e}$$

first and last integrals = 0, $\bar{n}_1$ and $\bar{n}_3 = 0$ on surface 1

$$T_{22} = -\frac{\varepsilon_o}{2} E_1^2 = -\frac{\varepsilon_o}{2}\left(\frac{\lambda^2}{\varepsilon_o^2 \pi^2}\right)\frac{x_1^2}{(a^2+x_1^2)^2} \tag{f}$$

PROBLEM 8.20 (Continued)

Thus

$$f_2 = -2\frac{\lambda^2}{2\varepsilon_o\pi^2}\int_o^\infty \frac{x_1^2\,dx_1}{(a^2+x_1^2)^2} \tag{g}$$

$$f_2 = \frac{\lambda^2}{4\pi\varepsilon_o a} \tag{h}$$

Part b

From electrostatics,

$$\bar{f} = \lambda\bar{E}$$

From the figure, we see that

$$\bar{E}(x_2=a) = \frac{\lambda}{2\pi\varepsilon_o(2a)}\,\bar{i}_2 \tag{i}$$

Hence,

$$\bar{F} = \frac{\lambda^2}{4\pi\varepsilon_o a}\,\bar{i}_2 \tag{j}$$

which is the same as we obtained using the stress tensor - (see equation (h)).

PROBLEM 8.21

Part a

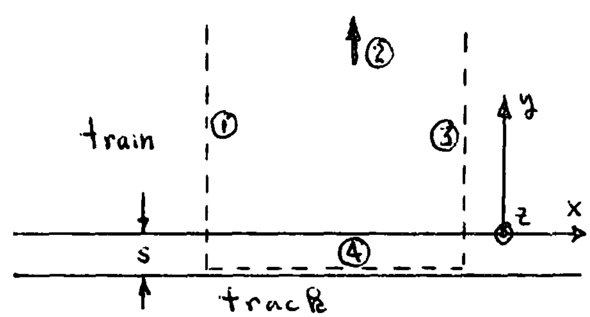
From Eq. 8.1.11,

$$T_{ij} = \begin{bmatrix} \frac{1}{2\mu_o}(B_x^2-B_y^2) & \frac{B_xB_y}{\mu_o} & 0 \\ \frac{B_xB_y}{\mu_o} & \frac{1}{2\mu_o}(-B_x^2+B_y^2) & 0 \\ 0 & 0 & \frac{1}{2\mu_o}(-B_x^2-B_y^2) \end{bmatrix} \tag{a}$$

where the components of $\bar{B}$ are given in the problem.

Part b

The appropriate surface of integration, which is fixed with respect to the fixed frame, is shown in the figure. We compute the time average force, and hence contributions from surfaces (1) and (3) cancel. Fields go to zero on surface (2), which is at $y\to\infty$. Thus, there remains the stress on surface (4). The time average value of the surface force density $\bar{T}$



PROBLEM 8.21 (Continued)

is independent of x. Hence,

$$T_y = -\langle T_{yy}(y=0)\rangle \tag{b}$$

$$T_y = -\frac{1}{2\mu_o}\langle -B_x^2 + B_y^2\rangle \tag{c}$$

Observe that

$$\langle \text{Re}\,\hat{A}\,e^{-jkUt}\,\text{Re}\,\hat{B}\,e^{-jkUt}\rangle \equiv \frac{1}{2}\,\text{Re}\,\hat{A}\,\hat{B}^* \tag{d}$$

where $\hat{B}^*$ is complex conjugate of $\hat{B}$, and (c) becomes

$$T_y = -\frac{1}{4\mu_o}\,\text{Re}\left\{-(\mu_oK_oe^{jkx})(\mu_oK_oe^{-jkx}) + \frac{(-jk\mu_oK_o)}{\alpha}\,e^{jkx}\,\frac{(jk\mu_oK_o)}{\alpha^*}\,e^{-jkx}\right\} \tag{e}$$

$$= \frac{\mu_oK_o^2}{4}\left(1 - \frac{k^2}{\alpha\alpha^*}\right) \tag{f}$$

Finally, use the given definition of α to write (f) as

$$T_y = \frac{\mu_oK_o^2}{4}\left[1 - \frac{1}{\sqrt{1 + \left(\frac{\mu_o\sigma U}{k}\right)^2}}\right] \tag{g}$$

Note that T_y is positive so that the train is supported by the magnetic field. However, as $U\to 0$ (the train is stopped) the levitation force goes to zero.

Part c

For the force per unit area in the x direction;

$$T_x = -\frac{1}{2\mu_o}\langle B_xB_y(y=o)\rangle \tag{h}$$

$$= -\frac{1}{2\mu_o}\,\text{Re}\left[\mu_oK_oe^{jkx}\left(\frac{jk\mu_o}{\alpha^*}\right)K_oe^{-jkx}\right] \tag{i}$$

Thus

$$T_x = -\frac{\mu_oK_o^2}{2\left[1 + \left(\frac{\mu_o\sigma U}{k}\right)^2\right]^{1/2}}\,\text{Re}\,j\,\sqrt{1 - j\left(\frac{\mu_o\sigma U}{k}\right)} \tag{j}$$

As must be expected, the force on the train in the x directions vanishes as $U\to 0$. Note that in any case the force always tends to retard the motion and hence could hardly be used to propel the train.

The identity $\sin(\theta/2) = \pm\sqrt{(1-\cos\theta)/2}$ is helpful in reducing (j) to the form

$$T_x = \frac{-\mu_oK_o^2}{2\left[1 + \left(\frac{\mu_o\sigma U}{k}\right)^2\right]^{1/2}}\sqrt{\frac{1}{2}\left(\sqrt{1 + \left(\frac{\mu_o\sigma U}{k}\right)^2} - 1\right)} \tag{k}$$

PROBLEM 8.22

This problem makes the same point as Probs. 8.16 and 8.17, with the additional effect of material motion included. Regardless of the motion, with the current constrained as given, the magnetic field intensity is zero to the right of the block and uniform into the paper (z direction) to the left of the block, where

$$\bar{H} = \bar{i}_z \frac{I_o}{d} \qquad \text{(a)}$$

The only contribution to an integration of the stress tensor over a surface enclosing the block is on the left surface. Thus

$$f_x = ds\ T_{xx} = -\ ds\ \frac{1}{2}\mu_o H_z^2 \qquad \text{(b)}$$

$$= -\frac{ds}{2}\mu_o\left(\frac{I_o}{d}\right)^2 \qquad \text{(c)}$$

The magnetic force is to the right and independent of the magnetic Reynolds number.

PROBLEM 8.23

In plane geometry, a knowledge of the charge on the upper plate is equivalent to knowing the electric field intensity on the surface of the plate. Thus, the surface charge density on the upper plate is

$$\sigma_f = \frac{1}{A}\int_o^t I_o \cos\omega t\ dt = \frac{I_o}{A\omega}\sin\omega t \qquad \text{(a)}$$

and

$$E_x(x{=}a) = -\frac{\sigma_f}{\epsilon_o} = -\frac{I_o}{A\epsilon_o\omega}\sin\omega t \qquad \text{(b)}$$

Now, we enclose the upper plate with a surface just outside the electrode surface. The only contribution to the integration of Eq. 8.1.17 using the stress tensor 8.3.10 is

$$f_x = -\ AT_{xx}(x{=}a) = -\frac{A\epsilon_o}{2}E_x^2(x{=}a) \qquad \text{(c)}$$

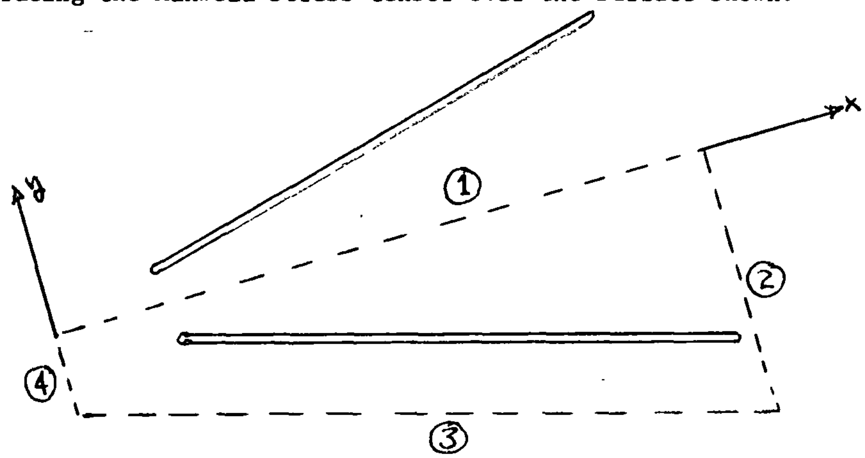
which we can evaluate from (b) as

$$f_x = -\frac{A\epsilon_o}{2}\left(\frac{I_o}{A\epsilon_o\omega}\right)^2\sin^2\omega t \qquad \text{(d)}$$

The force of attraction between the conducting slab and upper electrode is not dependent on σ_1 or σ_o.

PROBLEM 8.24

The force on the lower electrode in the x direction is zero, as can be seen by integrating the Maxwell stress tensor over the surface shown.



The fields are zero on surfaces (2), (3) and (4). Hence, the total force per unit depth into the paper is

$$f_x = \int_0^\infty T_{xy} dx \tag{a}$$

where contributions from surfaces in the plane of the paper cancel because the problem is two-dimensional. Moreover, by symmetry the electric field intensity on the surface (1), even in the fringing regions, is in the y direction only and $T_{xy} = \varepsilon_o E_x E_y$ in (a) is zero. Thus, the total x directed force is zero.

PROBLEM 8.25

The force density in the dielectric slab is Eq. 8.5.45. Not only is the first term zero, but because the block moves as a rigid body (we are interested only in the net force giving rise to a rigid body displacement) the last term, which originates in changes in volume of the material, does not give a contribution. Hence, the force density is

$$\bar{F} = -\frac{1}{2} \bar{E}\cdot\bar{E}\nabla\varepsilon \tag{a}$$

and the stress tensor is

$$T_{ij} = \varepsilon E_i E_j - \frac{\delta_{ij}}{2} \varepsilon E_k E_k \tag{b}$$

Note that, from (a), the force density in the x_1 direction is confined to the right edge of the block, where it acts as a surface force. Thus, we obtain the total force by simply integrating over a surface that encloses the right edge;

PROBLEM 8.25 (Continued)

$$f_1 = aD\left[-\frac{1}{2}\,\varepsilon_o(E_2^a)^2 + \frac{1}{2}\,\varepsilon(E_2^b)^2\right] \qquad (c)$$

where a and b are to the right and left of the right edge of the slab. Also $E_2^a = E_2^b = -V_o/a$. Hence (c) becomes

$$f_1 = \frac{aD}{2}\left(\frac{V_o}{a}\right)^2(\varepsilon-\varepsilon_o) \qquad (d)$$

The force acts to the right, as could be computed by the energy method.

PROBLEM 8.26

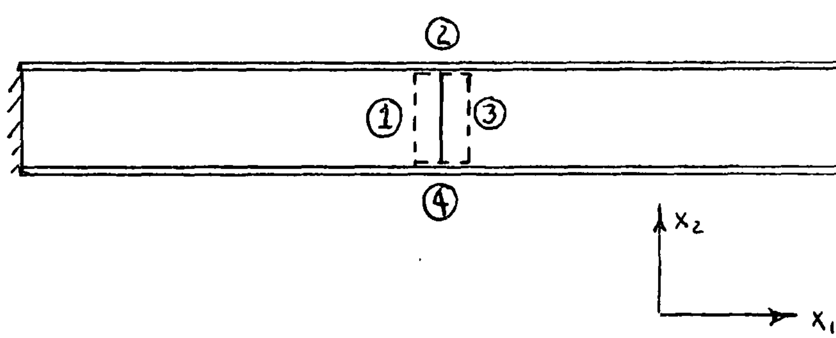
Part a

The force density for polarizable materials is:

$$\bar{F} = -\frac{1}{2}\,\bar{E}\cdot\bar{E}\,\nabla\varepsilon + \frac{1}{2}\,\nabla\left(\bar{E}\cdot\bar{E}\,\rho\,\frac{\partial\varepsilon}{\partial\rho}\right) \qquad (a)$$

The second term on the right side represents electrostriction. Note that this is a case where the material volume must change, and hence the effect of electrostriction is important. Since free space and the elastic bulk are homogeneous, changes in permittivity and $\partial\varepsilon/\partial\rho$ occur only at the boundary where the permittivity is discontinuous. The upper and lower elastic bulk surfaces are constrained by the plates. Thus only the x_1 component of force is pertinent. Since the left-hand edge is fixed, any stress arising from the discontinuity in permittivity at that boundary is counterbalanced by the rigidity of the wall. Therefore, all of the force arises at the right-hand boundary which is free to move.

The closed surface of integration is shown in the figure.



PROBLEM 8.26 (Continued)

$$T_{ij} = \varepsilon E_i E_j - \frac{1}{2}\delta_{ij}\left[\varepsilon - \rho\frac{\partial\varepsilon}{\partial\rho}\right]E_k E_k \tag{b}$$

Since $a/c \ll 1$ and $b \sim \frac{1}{2}a$, the field at the dielectric interface is essentially uniform.

$$\bar{E} = -\bar{i}_2\frac{V_o}{a} \tag{c}$$

The relevant components of the stress tensor are:

$$T_{11} = -\frac{\varepsilon}{2}E_2^2 + \frac{1}{2}\rho\frac{\partial\varepsilon}{\partial\rho}E_2^2 \tag{d}$$

$$T_{12} = \varepsilon E_1 E_2 = 0 \tag{e}$$

$$f_1 = \int_{(1)(3)} T_{11}n_1 da + \cancelto{0}{\int_{(2)(4)} T_{12}n_2 da} \tag{f}$$

Hence

$$f_1 = \int_{(3)} T_{11}da_3 - \int_{(1)} T_{11}da_1$$

$$= -\frac{\varepsilon_o}{2}\left(\frac{V_o}{a}\right)^2(aD) - \left[-\frac{\varepsilon_1}{2}\left(\frac{V_o}{a}\right)^2(aD) + \frac{\rho}{2}\frac{\partial\varepsilon}{\partial\rho}\left(\frac{V_o}{a}\right)^2(aD)\right] \tag{g}$$

Thus;

$$f_1 = \frac{(\varepsilon_1 - \varepsilon_o)V_o^2 D}{2a} - \frac{\rho}{2}\frac{\partial\varepsilon}{\partial\rho}\left(\frac{V_o^2 D}{a}\right) \tag{h}$$

Part b

In order to use lumped parameter energy methods, the charge on the upper plate will be found. The permittivity of the dielectric bulk is a junction of the displacement of the right-hand edge. That is, if mass conservation is to hold,

$$\rho_o\, abD = (\rho_o + \Delta\rho)aD(b+\xi) \tag{i}$$

where

$$\rho = \rho_o + \Delta\rho,\ \Delta\rho = 0 \text{ if } \xi = 0 \tag{j}$$

Thus, if $\Delta\rho \ll \rho_o$ and $\xi \ll b$, to first order

$$\Delta\rho = -\rho_o\frac{\xi}{b} \tag{k}$$

(see Eqs. 8.5.9 and 8.5.10)

Furthermore, to first order, using a Taylor series,

PROBLEM 8.26 (Continued)

$$\varepsilon = \varepsilon_1 + \frac{\partial\varepsilon}{\partial\rho}\Delta\rho = \varepsilon_1 - \frac{\rho_o}{b}\frac{\partial\varepsilon}{\partial\rho}\xi \tag{ℓ}$$

Also, the electric field will be assumed as uniform everywhere between the plates. Hence; in the block

$$\bar{D} = -\bar{i}_2\left\{\frac{V_o}{2}\left[\varepsilon_1 + \frac{\partial\varepsilon}{\partial\rho}\Delta\rho\right]\right\} \tag{m}$$

to the right of the block

$$\bar{D} = -\bar{i}_2\frac{V_o}{a}\varepsilon_o \tag{n}$$

By employing Gauss's law, we find the charge on the upper plate as:

$$q = \left(\frac{V_o}{a}\right)\left\{\varepsilon_1 - \frac{\rho_o}{b}\frac{\partial\varepsilon}{\partial\rho}\xi\right\}(b+\xi)D + \varepsilon_o\left(\frac{V_o}{a}\right)(c-b-\xi)D \tag{o}$$

$$\int dw_e' = \int q\,dv + \int f_e\,dx \tag{p}$$

integrating we find

$$w_e' = \frac{1}{2}\left(\frac{V_o^2}{a}\right)\left[\varepsilon_1 - \frac{\rho_o}{b}\frac{\partial\varepsilon}{\partial\rho}\xi\right](b+\xi)D + \frac{1}{2}\frac{V_o^2}{a}(c-b-\xi)D \tag{q}$$

Thus,

$$f_e = \left.\frac{\partial w_e'}{\partial\xi}\right|_{v=V_o} = \frac{(\varepsilon_1-\varepsilon_o)V_o^2D}{2a} - \frac{1}{2}\frac{V_o^2}{a}\rho_o D\frac{\partial\varepsilon}{\partial\rho} \tag{r}$$

Second order terms have been dropped in the co-energy expression (alternatively, first order terms can be dropped in the force expression).

Part c

If the result of part (a) is written for $\rho = \rho_o + \Delta\rho$, where $\rho_o >> \Delta\rho$, then the answers to part (a) and (b) are identical to first order. This should be expected since the lumped parameter approach assumed a value for permittivity which was correct only to first order.

PROBLEM 8.27

The surface force density is

$$T_m = [T_{mn}^a - T_{mn}^b]n_n \tag{a}$$

For this problem, we require $m = 1$ and $\bar{n} = \bar{i}_2$. Thus

$$T_1 = (T_{12}^a - T_{12}^b) \tag{b}$$

From Eq. 8.5.46,

PROBLEM 8.27 (Continued)

$$T_1 = \varepsilon_o E_1^a E_2^a - \varepsilon E_1^b E_2^b \tag{c}$$

Note that $E_2^a = E_2^b$ (see Eq. 6.2.31). Moreover, because there is no free charge $\varepsilon_o E_1^a = \varepsilon E_1^b$ (see Eq. 6.2.33). Thus, (c) becomes

$$T_1 = E_2^a[\varepsilon_o E_1^a - \varepsilon E_1^b] = 0 \tag{d}$$

That the shear surface force density is zero in the x_3 direction follows the same reasoning.

PROBLEM 8.28

The force density, Eq. 8.5.45, written in component form, is

$$F_i = E_i \frac{\partial \varepsilon E_j}{\partial x_j} - \frac{1}{2} E_k E_k \frac{\partial \varepsilon}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\frac{1}{2} E_k E_k \frac{\partial \varepsilon}{\partial \rho}\rho\right) \tag{a}$$

The first term can be rewritten as two terms, one of which is in the desired form

$$F_i = \frac{\partial}{\partial x_j}(\varepsilon E_i E_j) - \varepsilon E_j \frac{\partial E_i}{\partial x_j} - \frac{1}{2} E_k E_k \frac{\partial \varepsilon}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\frac{1}{2} E_k E_k \frac{\partial \varepsilon}{\partial \rho}\rho\right) \tag{b}$$

Because $\nabla \times \bar{E} = 0$, $\partial E_i/\partial x_j = \partial E_j/\partial x_i$, so that the second term can be rewritten and combined with the third. (Note the j is a dummy summation variable.)

$$F_i = \frac{\partial}{\partial x_j}(\varepsilon E_i E_j) - \frac{1}{2}\frac{\partial}{\partial x_i}(\varepsilon E_k E_k) + \frac{\partial}{\partial x_i}\left(\frac{1}{2} E_k E_k \frac{\partial \varepsilon}{\partial \rho}\rho\right) \tag{c}$$

Finally, we introduce δ_{ij} (see Eq. 8.1.7) to write (c) in the required form

$$F_i = \frac{\partial T_{ij}}{\partial x_j} \tag{d}$$

where

$$T_{ij} = \varepsilon E_i E_j - \frac{1}{2}\delta_{ij} E_k E_k \left(\varepsilon - \rho \frac{\partial \varepsilon}{\partial \rho}\right) \tag{e}$$

This is identical to Eq. 8.5.46.

PROBLEM 9.1

The equation of motion for a static rod is

$$0 = E \frac{d^2\delta}{dx^2} + F_x \text{ where } F_x = \rho g \tag{a}$$

We can integrate this equation directly and get

$$\delta(x) = -\frac{\rho g}{E}\left(\frac{x^2}{2}\right) + Cx + D, \tag{b}$$

where C and D are arbitrary constants.

Part a

The stress function is $T(x) = E \frac{d\delta}{dx}$, and therefore

$$T(x) = -\rho g x + CE. \tag{c}$$

We have a free end at $x = \ell$ and this implies $T(x=\ell)=0$. Now we can write the stress as

$$T(x) = -\rho g x + \rho g \ell. \tag{d}$$

The maximum stress occurs at $x = 0$ and is $T_{max} = \rho g \ell$. Equating this to the maximum allowable stress, we have

$$2 \times 10^9 = (7.8 \times 10^3)(9.8)\ell$$

hence

$$\ell = 2.6 \times 10^4 \text{ meters.}$$

Part b

From part (a)

$$T(x) = -\rho g x + \rho g \ell \tag{e}$$

The fixed end at $x = 0$ implies that $D = 0$, so now we can write the displacement

$$\delta(x) = -\frac{\rho g}{E}\left(\frac{x^2}{2}\right) + \frac{\rho g \ell}{E}(x) \tag{f}$$

Part c

$$\delta(\ell) = -\frac{\rho g}{E}\,\frac{\ell^2}{2} + \frac{\rho g \ell}{E}(\ell) = \frac{\rho g \ell^2}{2E} \tag{g}$$

For $\ell = 2.6 \times 10^4$ meters, $\delta(\ell) = 129$ meters. This appears to be a large displacement, but note that the total unstressed length is 26,000 meters.

PROBLEM 9.2

Part a

The equation of motion for a static rod is

$$0 = E\frac{d^2\delta}{dx^2} + \rho g \tag{a}$$

If we define $x' = x - L_1$, we can write the solutions for δ in rod 1 and in rod 2 as

$$\delta_1(x) = -\frac{\rho_1 g}{E_1}\left(\frac{x^2}{2}\right) + C_1 x + D_1 \tag{b}$$

and

$$\delta_2(x') = -\frac{\rho_2 g}{E_2}\left(\frac{x'^2}{2}\right) + C_2 x' + D_2 \tag{c}$$

where C_1, C_2, D_1, and D_2 are arbitrary constants. Since $T = E\frac{d\delta}{dx}$ we can also write the tensions,

$$T_1(x) = -\rho_1 g x + E_1 C_1 \tag{d}$$

and

$$T_2(x') = -\rho_2\, g x' + E_2 C_2 \tag{e}$$

We must have four boundary conditions to evaluate the constants and they are:

$$\delta_1(x{=}0) = 0, \tag{f}$$

$$\delta_2(x'{=}0) = \delta_1(x{=}L_1) \tag{g}$$

$$0 = -A_1 T_1(x{=}L_1) + A_2 T_2(x'{=}0) + mg, \tag{h}$$

and

$$0 = -A_2 T_2(x'{=}L_2) + Mg + f_x^e \tag{i}$$

where f_x^e is found using the Maxwell stress tensor

$$f_x^e = \frac{\varepsilon_o A_M V_o^2}{2d^2} \tag{j}$$

where we assume $d \gg \delta^{(2)}(x'{=}L_2)$.

Equations (f), (g), (h) and (i) serve to define the constants of integration. Substitution of (b)-(e) shows that

$$D_1 = 0 \tag{k}$$

$$-\frac{\rho_1 g}{E_1}\left(\frac{L_1^2}{2}\right) + C_1 L_1 + D_1 - D_2 = 0 \tag{ℓ}$$

PROBLEM 9.2 (Continued)

$$-A_1[-\rho_1 g L_1 + E_1 C_1] + A_2[E_2 C_2] + mg = 0 \tag{m}$$

$$-A_2[-\rho_2 g L_2 + E_2 C_2] + Mg + \frac{\varepsilon_o A_M V_o^2}{2d^2} = 0 \tag{n}$$

Solution of these expressions, beginning with (n), gives

$$C_2 = \left[Mg + \frac{\varepsilon_o A_M V_o^2}{2d^2} + \rho_2 g L_2 A_2\right] \frac{1}{A_2 E_2} \tag{o}$$

and hence

$$C_1 = \left[mg + \rho_1 g L_1 A_1 + A_2 E_2 C_2\right] \frac{1}{A_1 E_1}$$

$$= \left\{[(M+m) + \rho_1 L_1 A_1 + \rho_2 L_2 A_2]g + \frac{\varepsilon_o A_M V_o^2}{2d^2}\right\} \frac{1}{A_1 E_1} \tag{p}$$

$$D_2 = \frac{L_1}{A_1 E_1} \left\{[(M+m) + \frac{\rho_1 L_1 A_1}{2} + \rho_2 L_2 A_2]g + \frac{\varepsilon_o A_M V_o^2}{2d^2}\right\} \tag{q}$$

$$D_1 = 0 \tag{r}$$

Thus, (b) and (c) are determined.

PROBLEM 9.3

Part a

Longitudinal displacements on the rod satisfy the wave equation

$$\rho \frac{\partial^2 \delta}{\partial t^2} = E \frac{\partial^2 \delta}{\partial x^2} \text{ and the stress } T = E \frac{\partial \delta}{\partial x}\,. \tag{a}$$

We can write $\delta(x,t) = \mathrm{Re}[\hat{\delta}(x)e^{j\omega t}]$ for sinusoidal excitations. $\hat{\delta}(x)$ can be written as $\hat{\delta}(x) = C_1 \sin \beta x + C_2 \cos \beta x$ where $\beta = \omega\sqrt{\rho/E}$. The two constants are found from the boundary conditions

$$M \frac{\partial^2 \delta}{\partial t^2}(\ell,t) = - AT(\ell,t) + f(t) \tag{b}$$

$$\delta(0,t) = 0. \tag{c}$$

These conditions become

$$-M\omega^2 \hat{\delta}(\ell) = - AE \frac{d\hat{\delta}}{dx}(\ell) + f_o \tag{d}$$

and

$$\hat{\delta}(0) = 0 \tag{e}$$

PROBLEM 9.3 (Continued)

for sinusoidal excitations.

Now we find $C_2 = 0$ and

$$C_1 = \frac{f_o}{AE\beta\cos\beta\ell - M\omega^2\sin\beta\ell}\;. \tag{f}$$

Hence,

$$\delta(x,t) = \frac{\sin\beta x}{AE\beta\cos\beta\ell - M\omega^2\sin\beta\ell}\,\mathrm{Re}[f_o e^{j\omega t}] \tag{g}$$

and

$$T(x,t) = E\frac{\partial\delta}{\partial x} = \frac{E\beta\cos\beta x}{AE\beta\cos\beta\ell - M\omega^2\sin\beta\ell}\,\mathrm{Re}[f_o e^{j\omega t}] \tag{h}$$

Part b

At $x = \ell$,

$$\delta(\ell,t) = \frac{1}{AE\beta\cot\beta\ell - M\omega^2}\,\mathrm{Re}[f_o e^{j\omega t}] \tag{i}$$

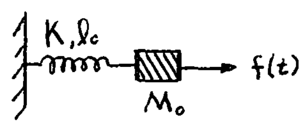
where $\beta\cot\beta\ell = \omega\sqrt{\rho/E}\cot(\omega\ell\sqrt{\rho/E})$.

For small ω, $\cot(\omega\ell\sqrt{\rho/E}) \rightarrow \dfrac{1}{\omega\ell\sqrt{\frac{\rho}{E}}}$ and

$$\delta(\ell,t) \rightarrow \frac{1}{\frac{AE}{\ell} - M\omega^2}\,f(t) \tag{j}$$

This equation is as used to describe a mass on the end of a massless spring:

$$M_o\frac{d^2x}{dt^2} = -Kx + f(t) \tag{k}$$



and

$$x = \mathrm{Re}[\hat{x}e^{j\omega t}],$$

$$-M_o\omega^2\hat{x} = -K\hat{x} + f_o, \tag{ℓ}$$

or

$$x = \frac{1}{K - M_o\omega^2}\,f(t) \tag{m}$$

Comparing (j) and (ℓ) we note that

$$K = \frac{AE}{\ell} \text{ and } \ell_o = \ell. \tag{n}$$

Our comparison is complete and since $M \gg \rho A\ell$ we can use the massless spring model with a mass $M_o = M$ on the end.

PROBLEM 9.4

A response that can be represented purely as a wave traveling in the negative x direction implies that there be no wave reflection at the left-hand boundary. We must have

$$v(0,t) + \frac{1}{\sqrt{\rho E}} T(0,t) = 0 \tag{a}$$

as seen in Sec. 9.1.1b.

This condition can be satisfied by a viscous damper alone:

$$AT(0,t) + Bv(0,t) = 0 \tag{b}$$

Hence, we can write

$$\begin{aligned} B &= A\sqrt{\rho E} \\ M &= 0 \\ K &= 0. \end{aligned} \tag{c}$$

PROBLEM 9.5

Part a

At $x = \ell$ the boundary condition is

$$0 = - AT(\ell,t) - B \frac{\partial \delta}{\partial t} (\ell,t) + f(t) \tag{a}$$

Part b

We can write the solution as

$$\hat{\delta}(x) = C_1 \sin \beta x + C_2 \cos \beta x, \tag{b}$$

where $\beta = \omega\sqrt{\frac{\rho}{E}}$. At $x = 0$ there is a fixed end, hence $\hat{\delta}(x=0) = 0$ and $C_2 = 0$. At $x = \ell$ our boundary condition becomes

$$F_o = j\omega B\hat{\delta}(x=\ell) + AE \frac{d\hat{\delta}}{dx} (x=\ell), \tag{c}$$

or in terms of C_1;

$$F_o = j\omega BC_1 \sin \beta\ell + AE\beta C_1 \cos \beta\ell \tag{d}$$

After solving for C_1, we can write our solution as

$$\hat{\delta}(x) = \frac{F_o \sin\beta x}{AE\beta\cos\beta\ell + j\omega B\sin\beta\ell} \tag{e}$$

Part c

For ω real and $B>0$, δ cannot be infinite with a finite-applied force, because the denominator of $\hat{\delta}(x)$ can never be zero.

Physically, $B>0$ implies that the system is damped and energy would be

PROBLEM 9.5 (Continued)

dissipated for each cycle of operation, hence a perfect resonance cannot occur. However, there will be frequencies which will maximize the amplitude.

PROBLEM 9.6

First, we can calculate the force of magnetic origin, f_x, on the rod. If we define $\delta(\ell,t)$ to be the a.c. deflection of the rod at $x = \ell$, then using Ampere's law and the Maxwell stress tensor (Eq. 8.5.41 with magnetostriction ignored) we find

$$f_x = \frac{\mu_o A N^2 I^2}{2[d-\delta(\ell,t)]^2} \qquad \text{(a)}$$

This result can also be obtained using the energy methods of Chap. 3 (See Appendix E, Table 3.1). Since $d >> \delta(\ell,t)$, we may linearize f_x:

$$f_x \simeq \frac{\mu_o A N^2 I^2}{2d^2} + \frac{\mu_o A N^2 I^2}{d^3}\ \delta(\ell,t) \qquad \text{(b)}$$

The first term represents a constant force which is balanced by a static deflection on the rod. If we assume that this static deflection is included in the equilibrium length ℓ, then we need only use the last term of f_x to compute the dynamic deflection $\delta(\ell,t)$. In the bulk of the rod we have the wave equation; for sinusoidal variations

$$\delta(x,t) = \mathrm{Re}[\hat{\delta}(x)e^{j\omega t}] \qquad \text{(c)}$$

we can write the complex amplitude $\hat{\delta}(x)$ as

$$\hat{\delta}(x) = C_1 \sin \beta x + C_2 \cos \beta x \qquad \text{(d)}$$

where $\beta = \omega\sqrt{\frac{\rho}{E}}$. At $x = 0$ we have a fixed end, so $\hat{\delta}(o) = 0$ and $C_2 = 0$. At $x = \ell$ the boundary condition is

$$0 = f_x - AE\frac{\partial\delta}{\partial x}(\ell,t), \qquad \text{(e)}$$

or

$$0 = \frac{\mu_o A N^2 I^2}{d^3}\ \hat{\delta}(x{=}\ell) - AE\frac{d\hat{\delta}}{dx}(x{=}\ell) \qquad \text{(f)}$$

Substituting we obtain

$$\frac{\mu_o A N^2 I^2}{d^3} C_1 \sin \beta\ell = C_1 AE\beta \cos \beta\ell \qquad \text{(g)}$$

Our solution is $\hat{\delta}(x) = C_1 \sin \beta x$ and for a non-trivial solution we must have $C_1 \neq 0$. So, divide (g) by C_1 and obtain the resonance condition:

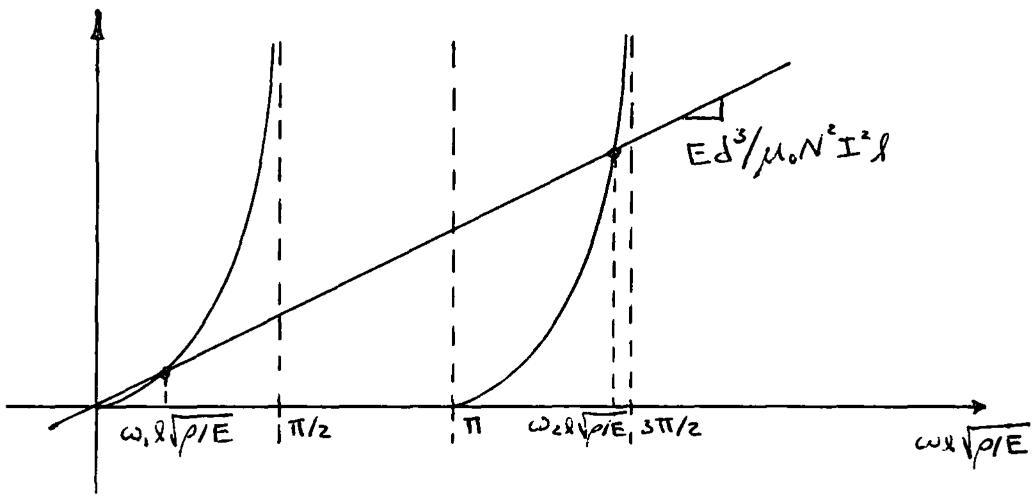
PROBLEM 9.6 (Continued)

$$\left(\frac{\mu_o AN^2I^2}{d^3}\right) \sin \beta\ell = AE\beta \cos \beta\ell \tag{h}$$

Substituting $\beta = \omega\sqrt{\frac{\rho}{E}}$ and rearranging, we have

$$\frac{Ed^3}{\mu_o N^2I^2\ell}\left(\omega\ell\sqrt{\frac{\rho}{E}}\right) = \tan\left(\omega\ell\sqrt{\frac{\rho}{E}}\right) \tag{i}$$

which, when solved for ω, yields the eigenfrequencies. Graphically, the first two eigenfrequencies are found from the sketch.



Notice that as the current I is increased, the slope of the straight line decreases and the first eigenfrequency (denoted by ω_1) goes to zero and then seemingly disappears for still higher currents. Actually ω_1 now becomes <u>imaginary</u> and can be found from the equation

$$\frac{Ed^3}{\mu_o N^2I^2\ell}\left(|\omega_1|\ell\sqrt{\frac{\rho}{E}}\right) = \tanh\left(|\omega_1|\ell\sqrt{\frac{\rho}{E}}\right) \tag{j}$$

Just as there are negative solutions to (i), $-\omega_1$, $-\omega_2$ .. etc., so there are now solutions $\pm j|\omega_1|$. Thus, because ω_1 is imaginary, the system is unstable,

PROBLEM 9.6 (Continued)

(amplitude of one solution growing in time).

Hence when the slope of the straight line becomes less than unity, the system is unstable. This condition can be stated as:

$$\text{STABLE} \longrightarrow \frac{Ed^3}{\mu_o N^2 I^2 \ell} > 1 \tag{k}$$

or

$$\text{UNSTABLE} \longrightarrow \frac{Ed^3}{\mu_o N^2 I^2 \ell} < 1 \tag{ℓ}$$

PROBLEM 9.7

Part a

$\delta(x,t)$ satisfies the wave equation

$$\rho \frac{\partial^2 \delta}{\partial t^2} = E \frac{\partial^2 \delta}{\partial x^2} \tag{a}$$

and the stress is $T = E \frac{\partial \delta}{\partial x}$. We can write

$$\delta(x,t) = \text{Re}[\hat{\delta}(x) e^{j\omega t}] \tag{b}$$

and substitution into the wave equation gives

$$\frac{d^2\hat{\delta}}{dx^2} + \beta^2 \hat{\delta} = 0. \tag{c}$$

$$\beta = \omega \sqrt{\frac{\rho}{E}}$$

For $x > 0$ we have,

$$\hat{\delta}_a(x) = \hat{C}_1 \sin \beta x + \hat{C}_2 \cos \beta x \tag{d}$$

and

$$\hat{T}_a(x) = \hat{C}_1 E\beta \cos \beta x - \hat{C}_2 E\beta \sin \beta x \tag{e}$$

and for $x < 0$ we have,

$$\hat{\delta}_b(x) = \hat{C}_3 \sin \beta x + \hat{C}_4 \cos \beta x \tag{f}$$

and

$$\hat{T}_b(x) = \hat{C}_3 E\beta \cos \beta x - \hat{C}_4 E\beta \sin \beta x \tag{g}$$

Part b

There are four constants to be determined; thus we need four boundary conditions. At the right end (x=L), we have

$$\hat{\delta}_a(x=L) = 0 \tag{h}$$

PROBLEM 9.7 (Continued)

and the left end,

$$\hat{\delta}_b(x=-L) = \delta_o e^{-j(\frac{\pi}{2})} \tag{i}$$

There are two conditions at the middle (x=0),

$$\hat{\delta}_a(x=0^+) = \hat{\delta}_o(x=0^-) \tag{j}$$

and

$$-M\omega^2\,\hat{\delta}_a(x=0) = A\hat{T}_a(x=0^+) - A\hat{T}_b(x=0^-) - 4K\hat{\delta}_a(x=0) \tag{k}$$

Part c

Solving for $\hat{C}_1, \hat{C}_2, \hat{C}_3$, and $\hat{C}_4$ we obtain

$$\hat{C}_1 = \frac{-\delta_o AE\beta e^{-j\frac{\pi}{2}}\cot\beta L}{\sin\beta L(4K+2AE\beta\cot\beta L-M\omega^2)} \tag{ℓ}$$

$$\hat{C}_2 = \frac{\delta_o AE\beta e^{-j\frac{\pi}{2}}}{\sin\beta L(4K+2AE\beta\cot\beta L-M\omega^2)} \tag{m}$$

$$\hat{C}_3 = \frac{\delta_o AE\beta e^{-j\frac{\pi}{2}}\cot\beta L}{\sin\beta L(4K+2AE\beta\cot\beta L-M\omega^2)} - \frac{\delta_o e^{-j\frac{\pi}{2}}}{\sin\beta L} \tag{n}$$

$$\hat{C}_4 = \hat{C}_2 \tag{o}$$

Thus, (b), (e), and (g) with these constants give the desired stress distribution.

PROBLEM 9.8

In terms of the complex amplitudes, (ℓ) and (r) become

$$\hat{T}'(0) = \frac{L_o I}{aA}\hat{i}_i \qquad (\ell) - \text{text} \tag{a}$$

and

$$\hat{T}'(\ell) = \frac{L_o I}{aA}\hat{i}' \qquad (r) - \text{text} \tag{b}$$

where $\hat{i}' = -G\hat{v}_o$.

Equation (t) without the approximation becomes

$$\hat{v}_o = -j\omega\frac{GL_o(\mu+\mu_o)}{\mu-\mu_o}\hat{v}_o + j\omega\frac{L_o I}{a}\hat{\delta}_o \tag{c}$$

Using the steady-state solutions for the rod, we can solve for T(x) in terms of the boundary values $\hat{T}(0)$ and $\hat{T}(\ell)$:

PROBLEM 9.8 (Continued)

$$\hat{T}(x) = \hat{T}(0)\,\frac{\sin[k(\ell-x)]}{\sin[k\ell]} + \hat{T}(\ell)\,\frac{\sin[kx]}{\sin[k\ell]} \tag{d}$$

then

$$\hat{\delta} = \frac{1}{\omega\sqrt{\rho E}}\left[\hat{T}(0)\,\frac{\cos[k(\ell-x)]}{\sin[k\ell]} - \hat{T}(\ell)\,\frac{\cos[kx]}{\sin[k\ell]}\right] \tag{e}$$

From (a) and (b), this becomes

$$\hat{\delta}(\ell) = \hat{\delta}_o = \frac{1}{\omega\sqrt{\rho E}}\left[\frac{L_o I}{aA}\,\hat{i}_i\,\frac{1}{\sin[k\ell]} + \frac{GL_o I}{aA}\,\hat{v}_o\,\frac{\cos[k\ell]}{\sin[k\ell]}\right] \tag{f}$$

Thus, in view of (c) solved for $\hat{\delta}_o$, we obtain the system function

$$H(\omega) = \frac{\hat{v}_o}{\hat{i}_i} = \frac{-\frac{1}{G}}{\cos[k\ell]+j\sqrt{\rho E}\,(\frac{A}{G})(\frac{a}{L_o I})^2\sin[k\ell] - \left[\frac{\omega GL_o(\mu+\mu_o)}{(\mu-\mu_o)}\right]\sqrt{\rho E}\,(\frac{A}{G})(\frac{a}{L_o I})^2\sin[k\ell]} \tag{g}$$

PROBLEM 9.9

Part a

First of all, $y(t) = \delta(-L,t)$ where $\delta(x,t) = Re[\hat{\delta}(x)e^{j\omega t}]$. We can write the solution for $\hat{\delta}$ as $\hat{\delta}(x) = C_1\sin\beta x + C_2\cos\beta x$, where $\beta = \omega\sqrt{\rho/E}$. The C_2 is zero because of the fixed end at $x = 0(\hat{\delta}(0) = 0)$. At the other end we have

$$M\,\frac{\partial^2\delta}{\partial t^2}(-L,t) = A_2E\,\frac{\partial\delta}{\partial x}(-L,t) + f^e(t) \tag{a}$$

Using the Maxwell stress tensor, (or the energy method of Chap. 3) we find

$$f^e(t) = \frac{A\mu_o N^2}{2}\left\{\frac{[I_o - I(t)]^2}{[d-D+\delta(-L,t)]^2} - \frac{[I_o + I(t)]^2}{[d-D-\delta(-L,t)]^2}\right\} \tag{b}$$

which when linearized becomes,

$$f^e(t) \simeq -C_I I(t) - C_y\,\delta(-L,t), \tag{c}$$

where

$$C_I = \frac{2N^2\mu_o AI_o}{(d-D)^2};\quad C_y = \frac{2N^2\mu_o AI_o^2}{(d-D)^3}$$

Our boundary condition (a) becomes

$$-M\omega^2\,\hat{\delta}(-L) = A_2E\,\frac{d\hat{\delta}}{dx}(-L) - C_I\hat{I} - C_y\,\hat{\delta}(-L) \tag{d}$$

Solving for C_1 we obtain

PROBLEM 9.9 (Continued)

$$C_1 = \frac{C_I \hat{I}}{A_2 E\beta\cos\beta L - (M\omega^2 - C_y)\sin\beta L}, \tag{e}$$

and we can write our solution as

$$y(t) = \mathrm{Re}[-C_1 \sin\beta L \, e^{j\omega t}]. \tag{f}$$

Part b

The transducer is itself made from solid materials having characteristics that do not differ greatly from those of the rod. Thus, there is the question of whether the elastic response of the transducer materials is of importance. Under the assumption that the rod and transducer are constructed from materials having essentially the same elastic properties, the assumption that the yoke and plunger are rigid, but that the rod supports acoustic waves, is justified provided the rod is long compared to the largest dimension of the transducer, and that an acoustic wavelength is long compared to the largest transducer dimension. (See Sec. 9.1.3).

PROBLEM 9.10

Part a

At the outset, we can write the equation of motion for the massless plate:

$$-aT(\ell,t) + f^e(t) = M\frac{\partial^2\delta}{\partial t^2}(\ell,t) \cong 0 \tag{a}$$

Using the Maxwell stress tensor we find the force of electrical origin $f^e(t)$ to be

$$f^e(t) = \frac{\varepsilon_o A}{2}\left[\frac{(V_o + v(t))^2}{(d-\delta(\ell,t))^2} - \frac{(V_o - v(t))^2}{(d+\delta(\ell,t))^2}\right] \tag{b}$$

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Since $v(t) << V_o$ and $\delta(\ell,t) << d$, we can linearize $f^e(t)$:

$$f^e(t) \simeq \left[\frac{2\varepsilon_o A V_o^2}{d^3}\right]\delta(\ell,t) + \left[\frac{2\varepsilon_o A V_o}{d^2}\right]v(t) \tag{c}$$

Recognizing that $T(\ell,t) = E\frac{\partial\delta}{\partial x}(\ell,t)$ we can write our boundary condition at $x = \ell$ in the desired form:

$$aE\frac{\partial\delta}{\partial x}(\ell,t) = \frac{2\varepsilon_o A V_o^2}{d^3}\delta(\ell,t) + \frac{2\varepsilon_o A V_o}{d^2}v(t) \tag{d}$$

PROBLEM 9.10 (Continued)

Longitudinal displacements in the rod obey the wave equation and for an assumed form of $\delta(x,t) = \mathrm{Re}[\hat{\delta}(x)e^{j\omega t}]$ we can write $\hat{\delta}(x) = C_1\sin\beta x + C_2\cos\beta x$, where $\beta = \omega\sqrt{\rho/E}$. At $x = 0$ we have a fixed end, thus $\hat{\delta}(x{=}0) = 0$ and $C_2 = 0$. From part (a) and assuming sinusoidal time dependence, we can write our boundary condition at $x = \ell$ as

$$aE\,\frac{d\hat{\delta}}{dx}(\ell) = \frac{2\varepsilon_o A V_o^2}{d^3}\,\hat{\delta}(\ell) + \frac{2\varepsilon_o A V_o}{d^2}\,\hat{V} \qquad \text{(e)}$$

Solving

$$C_1 = \frac{2\varepsilon_o A V_o\,\hat{V}}{aEd^2\beta\cos\beta\ell - \dfrac{2\varepsilon_o A V_o^2}{d}\sin\beta\ell} \qquad \text{(f)}$$

Finally, we can write our solution as

$$\delta(x,t) = \left[\frac{2\varepsilon_o A V_o\ \sin\beta x}{aEd^2\beta\cos\beta\ell - \dfrac{2\varepsilon_o A V_o^2}{d}\sin\beta\ell}\right]\mathrm{Re}[\hat{V}e^{j\omega t}] \qquad \text{(g)}$$

*

PROBLEM 9.11

Part a

For no elastic wave reflection at the right-hand boundary we must have a boundary condition of the form

$$v(0,t) + \frac{1}{\sqrt{\rho E}}\,T(0,t) = 0 \qquad \text{(a)}$$

(from Sec. 9.1.1b). Since $v(0,t) = \frac{\partial\delta}{\partial t}(0,t)$, we can write

$$-\sqrt{\rho E}\,\frac{\partial\delta}{\partial t}(0,t) = T(0,t) \qquad \text{(b)}$$

If we write the boundary condition at $x = 0$ for our example we obtain

$$0 = -\,ST(0,t) + f_x^e(t), \qquad \text{(c)}$$

or for perturbations

$$0 = -\,ST(0,t) + f_{a.c.}^e(t) \qquad \text{(d)}$$

Combining (b) and (d)

$$f_{a.c.}^e(t) = -\,S\sqrt{\rho E}\,\frac{\partial\delta}{\partial t}(0,t) \qquad \text{(e)}$$

PROBLEM 9.11 (Continued)

and since $\partial\delta/\partial t\ (0,t) = dy_s/dt$,

$$f^e_{a.c.}(t) = -\ S\sqrt{\rho E}\ \frac{dy_s}{dt} \tag{f}$$

The perturbation electric force can be found using the Maxwell stress tensor (using a surface of integration similar to that illustrated by Prob. 8.10):

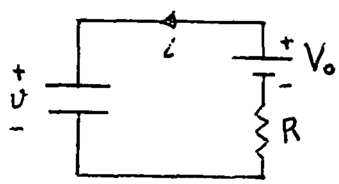
$$f^e_x(t) = \frac{\varepsilon_o v^2 D}{a} \simeq \frac{\varepsilon_o V_o^2 D}{a} + \frac{2\varepsilon_o V_o D v_s}{a} \tag{g}$$

where we associate $f^e_{a.c.}(t) = \frac{2\varepsilon_o V_o D v_s}{a}$.

Equation (f) now becomes

$$\frac{2\varepsilon_o V_o D v_s}{a} = -\ S\sqrt{\rho E}\ \frac{dy_s}{dt} \tag{h}$$

Now that we have dealt with the force balance we can write the circuit equations.



The capacitance of the device is found to be

$$C = \frac{2\varepsilon_o y D}{a}$$

Note that $q = Cv$ and $i = \frac{dq}{dt}$. The basic circuit equation is

$$v + iR = V_o = v + R\,\frac{dq}{dt} = v + R\left[C\,\frac{dv}{dt} + v\,\frac{dC}{dt}\right] \tag{i}$$

Substituting, we obtain

$$V_o = v + RC\,\frac{dv}{dt} + \frac{2\varepsilon_o DR}{a}\,\frac{dy}{dt} \tag{j}$$

and for perturbation quantities,

$$0 = v_s + RC_o\,\frac{dv_s}{dt} + \frac{2\varepsilon_o D V_o R}{a}\ \frac{dy_s}{dt}\ . \tag{k}$$

Since $\omega \ll \frac{1}{RC_o}$, $v_s \gg RC_o\ dv_s/dt$ and now we have

$$0 = v_s + \frac{2\varepsilon_o D V_o R}{a}\,\frac{dy_s}{dt} \tag{\ell}$$

PROBLEM 9.11 (Continued)

Equations (h) and (ℓ) must be satisfied simultaneously and this can occur only if

$$\frac{2\varepsilon_o D V_o R}{a} = \frac{aS\sqrt{\rho E}}{2\varepsilon_o V_o D} \tag{m}$$

Finally from (m) we have the condition on the d.c. voltage,

$$V_o = \frac{a}{2\varepsilon_o D}\left[\frac{S\sqrt{\rho E}}{R}\right]^{1/2} \tag{n}$$

PROBLEM 9.12

Part a

Note that there is no mutual capacitance between the two pairs. We can find the capacitance of the left-hand pair of plates to be

$$C_2 = \frac{\varepsilon d(\frac{w}{2} - y_2)}{h} + \frac{\varepsilon_o d(\frac{w}{2} + y_2)}{h} \tag{a}$$

The current i_2 can be found from $i_2 = dq_2/dt = d(V_oC_2)/dt = V_o\ dC_2/dt$, and upon substitution of C_2 we obtain

$$i_2 = \left[\frac{-(\varepsilon-\varepsilon_o)V_o d}{h}\right]\frac{dy_2}{dt} \tag{b}$$

If we solve for y_2 in terms of v_s our job will be done.

Define the y-axis from left to right with y = 0 at $y_1 = 0$. Assume all constant forces (with $v_s = 0$) to be balanced and consider only the perturbations. If we assume for the rod $\delta(y,t) = \text{Re}[\hat{\delta}(y)e^{j\omega t}]$ then we can write

$$\hat{\delta}(y) = C_1 \sin \beta y + C_2 \cos \beta y \tag{c}$$

where $\beta = \omega\sqrt{\rho/E}$. (We have assumed that the electrical forces act only on the surfaces of the rod. This is evident from the form of the force density, Eq. 8.5.45, if the effect of electrostriction can be ignored.) At y = 0 there is no perturbation force and for a.c. deflections we have a free end condition:

$$0 = T(0,t) \Rightarrow E\frac{d\hat{\delta}}{dy}(y = 0) = 0 \tag{d}$$

This forces C_1 to be zero. At y = ℓ we can write the boundary condition as

$$0 = -\,hdT(\ell,t) + f^e_{a.c.}(t)$$

PROBLEM 9.12 (Continued)

Using the Maxwell stress tensor (or energy methods, as in Sec. 8.54)

$$f_1^e(t) = \frac{(\varepsilon-\varepsilon_o)d}{2h}(V_o + v_s)^2 \tag{e}$$

Linearizing and ignoring the d.c. term we have

$$f_{1a.c.}^e(t) = \frac{(\varepsilon-\varepsilon_o)V_o d}{h} v_s.$$

From the boundary condition for complex amplitudes we obtain

$$0 = -\,hdE\frac{d\hat{\delta}}{dy}(\ell) + \frac{(\varepsilon-\varepsilon_o)V_o d}{h}\hat{v}_s \tag{f}$$

Substituting and solving for C_2;

$$C_2 = \frac{-(\varepsilon-\varepsilon_o)V_o}{h^2 E\beta\sin\beta\ell}\hat{v}_s. \tag{g}$$

Recognizing that $y_2(t) = \delta(0,t)$, we can now write

$$y_2(t) = \text{Re}\left[\frac{-(\varepsilon-\varepsilon_o)V_o\hat{v}_s}{h^2 E\beta\sin\beta\ell}e^{j\omega t}\right] \tag{h}$$

Since $i_2 = \frac{-(\varepsilon-\varepsilon_o)V_o d}{h}\frac{dy_2}{dt}$, we have

$$i_2 = \text{Re}\left[\frac{j\omega(\varepsilon-\varepsilon_o)^2 V_o^2 d}{h^3 E\beta\sin\beta\ell}\hat{v}_s e^{j\omega t}\right] \tag{i}$$

Finally, we can write

$$Y(j\omega) = \frac{\hat{i}_2}{\hat{v}_s} = \frac{j\omega(\varepsilon-\varepsilon_o)^2 V_o^2 d}{h^3 E\beta\sin\beta\ell} \tag{j}$$

Part b

The poles can be found from

$$h^3 E\beta\sin\beta\ell = 0 \tag{k}$$

where $\beta = \omega\sqrt{\rho/E}$. The lowest nonzero frequency can be found from

$\sin(\omega\ell\sqrt{\rho/E} = 0$ to be

$$\omega = \frac{\pi}{\ell\sqrt{\rho/E}} \tag{ℓ}$$

Note that the $\omega = 0$ is a pole because the rod is free to translate slowly between the plates.

PROBLEM 9.13

Part a

The flux for the left-hand transducer is

$$\lambda_\ell = \frac{\mu_o N^2}{g} 2\pi R(a-\delta(0,t))i_\ell \tag{a}$$

and for the right-hand one,

$$\lambda_r = \frac{\mu_o N^2}{g} 2\pi R(a+\delta(L,t))i_r \tag{b}$$

For this electrically linear situation we have $W_m' = \frac{1}{2} Li^2 = \frac{1}{2} \lambda i$ and $f = \frac{\partial W_m'}{\partial \delta}$. Hence we find, to linear terms

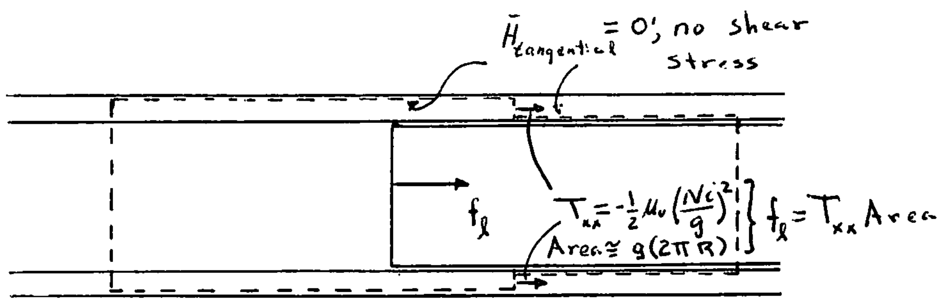
$$f_\ell = -\frac{\mu_o N^2}{g} \pi R(I_o^2 + 2I_o i) \tag{c}$$

and, because $i_r = I_o - Gv_{out}$

$$f_r = \frac{\mu_o N^2}{g} \pi R(I_o^2 - 2I_o\, Gv_{out}) \tag{d}$$

Part b

For the left-hand transducer, an acceptable stress-tensor surface is shown below,



and the mirror-image is acceptable for the right-hand transducer. Application of $f_x = \oint_S T_{xj} n_j da$ to the two surfaces yields the same result as in part (a).

PROBLEM 9.13 (Continued)

Part c

The wave equation holds in the rod for $\delta(x,t)$. Assuming $\delta = \mathrm{Re}[\hat{\delta}(x)e^{j\omega t}]$, we have $\hat{\delta}(x) = C_1 \sin\beta x + C_2 \cos\beta x$, where $\beta = \omega\sqrt{\rho/E}$. At $x = 0$, $f_\ell = -T(0,t)(\pi R^2)$ which yields

$$\hat{T}(0) = \frac{2\mu_o N^2 I_o \hat{I}}{Rg} = C_I \hat{I},$$

which in turn implies $C_1 = \frac{C_I \hat{I}}{E\beta}$. At $x = \ell$, $f_r = T(L,t)(\pi R^2)$, which will yield C_2.

The only other relation we need is the electrical circuit equation, which we can find from

$$\hat{v}_{out} = \frac{d\lambda_r}{dt} \qquad\qquad C_2 = \frac{C_I}{E\beta\sin\beta\ell}\left(\hat{I}\cos\beta\ell + G\hat{v}_{out}\right)$$

to be

$$\hat{v}_{out} = \frac{I_o L_1\, j\omega\, \hat{\delta}(L)}{a(1+j\, GL_1\omega)} \tag{e}$$

where $L_1 = \mu_o N^2(2\pi Ra)/g$.

Finally we can write $G(\omega)$ as

$$G(\omega) = \frac{\hat{v}_{out}}{\hat{I}} = \frac{j\omega I_o L_1 C_I}{aE\beta\sin\beta L(1+jGL_1\omega) - j\omega GC_I I_o L_1\cos\beta\ell} \tag{f}$$

Part d

If $G \ll \frac{1}{L_1\omega}$ so that the self inductance of the output transducer is negligible and the system is matched so that $a\sqrt{E\rho} = G\, C_I\, I_o L_1$ we have

$$\frac{\hat{v}_{out}}{\hat{I}} = \frac{jI_o L_1 C_I}{a\sqrt{E\rho}\,[\sin\beta L - j\cos\beta L]} \tag{g}$$

and

$$\left|\frac{\hat{v}_{out}}{\hat{I}}\right| = \frac{I_o\, L_1\, C_I}{a\sqrt{E\rho}} \tag{h}$$

PROBLEM 9.14

Part a

With no perturbations and no volume force in the rod we know that the stress, $T(x_1)$, will be constant. At $x_1 = 0$,

$$0 = -AT(x_1 = 0) + f^e \tag{a}$$

where, using the Maxwell stress tensor, $f^e = \frac{\varepsilon_o V_o^2 A_1}{2d^2}$. Hence,

PROBLEM 9.14 (Continued)

$$T(x_1) = \frac{\varepsilon_o V_o^2 A_1}{2Ad^2} . \qquad \text{(b)}$$

Part b

The velocity of the wave will be $v_p = \sqrt{E/\rho}$ and the transit time will be $t_d = L/v_p$.

Using Table 9.1 we have

$$t_d = \frac{1}{5100} = 1.96 \times 10^{-4} \text{ sec.}$$

Part c

This part is similar to Prob. 9.11, where our condition for no reflection is

$$f^e_{a.c.}(t) = - A\sqrt{\rho E}\, \frac{\partial \delta}{\partial t}\,(0,t) \qquad \text{(c)}$$

Using the Maxwell stress tensor

$$f^e = \frac{\varepsilon_o A_1 v^2}{2d^2} \cong \frac{\varepsilon_o A_1 V_o^2}{2d^2} + \frac{\varepsilon_o A_1 V_o}{d^2}\, v'$$

where $v = v' + V_o$. Here, we ignore the effect on f^e of the change in d resulting from the motion of the plate.

Writing the circuit equation we have

$$iR + v = V_o = R\,\frac{dq}{dt} + v = R\left(C\,\frac{dv}{dt} + v\,\frac{dC}{dt}\right) \qquad \text{(d)}$$

The capacitance C is

$$\frac{\varepsilon_o A_1}{d-\delta(0,t)} \simeq \frac{\varepsilon_o A_1}{d} + \frac{\varepsilon_o A_1}{d^2}\,\delta(0,t) \qquad \text{(e)}$$

Our equation becomes

$$0 = v' + R\,\frac{\varepsilon_o A_1}{d}\,\frac{dv'}{dt} + RV_o\,\frac{\varepsilon_o A_1}{d^2}\,\frac{\partial \delta}{\partial t}\,(0,t) \qquad \text{(f)}$$

and since

$$v' \gg \frac{\varepsilon_o A_1 R}{d}\,\frac{dv'}{dt},$$

we have

PROBLEM 9.14 (continued)

$$v' = -\frac{RV_o\varepsilon_o A_1}{d^2}\frac{\partial\delta}{\partial t}(0,t) \tag{g}$$

Now we can use this result to write $f^e_{a.c.} = \varepsilon_o A_1 V_o\, v'/d^2$, and the condition that this force take the form of (c) requires

$$A\sqrt{\rho E}\, d^4 = RV_o^2\, \varepsilon_o^2 A_1^2 , \tag{h}$$

or equivalently

$$R = \frac{A\sqrt{\rho E}\, d^4}{\varepsilon_o^2 A_1^2 V_o^2} \tag{i}$$

PROBLEM 9.15

Part a

We have from the problem statement

$$\psi(z+\Delta z) - \psi(z) = \beta\tau\Delta z .$$

If we take the limit $\Delta z \to 0$, then we obtain

$$\tau = \frac{1}{\beta}\frac{\partial\psi}{\partial z} .$$

Part b

We can write the equation of motion directly as

$$(J\Delta z)\frac{\partial^2\psi}{\partial t^2} = \tau(z+\Delta z,t) - \tau(z,t).$$

Dividing by Δz we have

$$J\frac{\partial^2\psi}{\partial t^2} = \frac{\tau(z+\Delta z,t)-\tau(z,t)}{\Delta z}$$

Taking the limit $\Delta z \to 0$ we obtain

$$J\frac{\partial^2\psi}{\partial t^2} = \frac{\partial\tau}{\partial z}$$

Part c

Substituting the result of part (a) into the result of part (b) we get

$$J\beta\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial z^2} .$$

PROBLEM 9.16

Part a

We seek to write Newton's law for motions in the z direction of a slice of the material having x thickness dx. In our situation the mass is $\rho adzdx$, where the acceleration is $\partial^2\delta_z/\partial t^2$. The net force due to the stress is

$$F_z = [T_{zx}(x+dx) - T_{zx}(x)]a\ dz \tag{a}$$

and

$$\rho \frac{\partial^2\delta_z}{\partial t^2} dx\ adz = [T_{zx}\ (x+dx) - T_{zx}(x)]a\ dz \tag{b}$$

Finally, in the limit $dx \to 0$ we have

$$\rho \frac{\partial^2\delta_z}{\partial t^2} = \frac{\partial T_{zx}}{\partial x} \tag{c}$$

Part b

The shear strain, e_{zx}, is defined so that it is proportional to $\delta_z(x+dx) - \delta_z(x)$ normalized to the distance between points dx. If $T_{zx}=2G\ e_{zx}$ then in the limit $dx \to 0$ $T_{zx} = G\ \partial\delta_z/\partial x$ if we define

$$e_{zx} = \frac{1}{2}\frac{\partial\delta_z}{\partial x}\ . \tag{d}$$

The 1/2 is included to subtract out rigid body rotation, a point that is important in dealing with three-dimensional motions (see Chap. 11, Sec. 11.2.1a).

Part c

From part (a),

$$\rho \frac{\partial^2\delta_z}{\partial t^2} = \frac{\partial T_{zx}}{\partial x} \tag{e}$$

Using the result of part (b) we have

$$\rho \frac{\partial^2\delta_z}{\partial t^2} = G\frac{\partial^2\delta_z}{\partial x^2}\ ; \tag{f}$$

the wave equation for shear waves with the propagational velocity

$$v_p = \sqrt{\frac{G}{\rho}}\ .$$

PROBLEM 9.17

Part a

Conservation of mass implies: net mass out per unit time = time rate of decrease of stored mass

$$[\rho v + \frac{\partial(\rho v)}{\partial x} \Delta x]A - (\rho v)A = -\frac{\partial}{\partial t}[\rho(\Delta x)A] \quad \text{(a)}$$

As $\Delta x \to 0$, we have

$$\frac{\partial}{\partial x}(\rho v) + \frac{\partial \rho}{\partial t} = 0 \quad \text{(b)}$$

If we write $\rho = \rho_o + \rho'(x,t)$ and $v = v(x,t)$ then we obtain by substitution

$$\rho_o \frac{\partial v}{\partial x} + \frac{\partial(\rho' v)}{\partial x} = -\frac{\partial \rho'}{\partial t} \quad \text{(c)}$$

Retaining only first-order terms we have

$$\rho_o \frac{\partial v}{\partial x} = -\frac{\partial \rho'}{\partial t} \quad \text{(d)}$$

as desired.

Part b

Conservation of momentum implies:

time rate of increase of stored momentum = net momentum in per unit time + externally applied force

$$\frac{\partial}{\partial t}(\rho v \Delta x A) = -[\rho v^2 + \frac{\partial(\rho v^2)}{\partial x} \Delta x]A + (\rho v^2)A + pA - (p + \frac{\partial p}{\partial x} \Delta x)A \quad \text{(e)}$$

as $\Delta x \to 0$, we have

$$\frac{\partial(\rho v)}{\partial t} = -\frac{\partial(\rho v^2)}{\partial x} - \frac{\partial p}{\partial x} \quad \text{(f)}$$

Expanding we have

$$\rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) + \underbrace{v\left(\frac{\partial(\rho v)}{\partial x} + \frac{\partial \rho}{\partial t}\right)}_{\text{this term is zero by conservation of mass}} = -\frac{\partial p}{\partial x} \quad \text{(g)}$$

PROBLEM 9.17 (continued)

Finally we have

$$\rho\left(\frac{\partial v}{\partial t} + v\,\frac{\partial v}{\partial t}\right) = -\frac{\partial p}{\partial x} \tag{h}$$

Substituting the perturbation quantities and retaining only the first order terms we obtain

$$\rho_o\,\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial x} \tag{i}$$

Part c

In terms of perturbation quantities we can write

$$p' = a^2\,\rho' \tag{j}$$

where

$$a^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_{\rho_o}.$$

Substitution for p' yields the two equations

$$\rho_o\,\frac{\partial v}{\partial x} = -\frac{\partial \rho'}{\partial t} \tag{k}$$

and

$$-a^2\,\frac{\partial \rho'}{\partial x} = \rho_o\,\frac{\partial v}{\partial t}. \tag{ℓ}$$

Combining we obtain

$$\frac{\partial^2 v}{\partial t^2} = a^2\,\frac{\partial^2 v}{\partial x^2} \quad \text{(scalar wave equation)} \tag{m}$$

Part d

If we substitute $v = \mathrm{Re}[\hat{v}(x)e^{j\omega t}]$ in the above equation we obtain

$$\frac{d^2\hat{v}(x)}{dx^2} + \frac{\omega^2}{a^2}\,\hat{v}(x) = 0 \tag{n}$$

which has solutions of the form

$$\hat{v}(x) = C_1\,\sin\left(\frac{\omega}{a}\,x\right) + C_2\,\cos\left(\frac{\omega}{a}\,x\right). \tag{o}$$

PROBLEM 9.17 (continued)

A rigid wall at $x = 0$ imples that $\hat{v}(x=0) = 0$. The drive at $x = \ell$ and the equations of part (c) imply that

$$\frac{d\hat{v}}{dx} = -\frac{j\omega\hat{p}_o}{a^2\rho_o} \tag{p}$$

at $x = \ell$.

The solution for $\hat{v}$ is

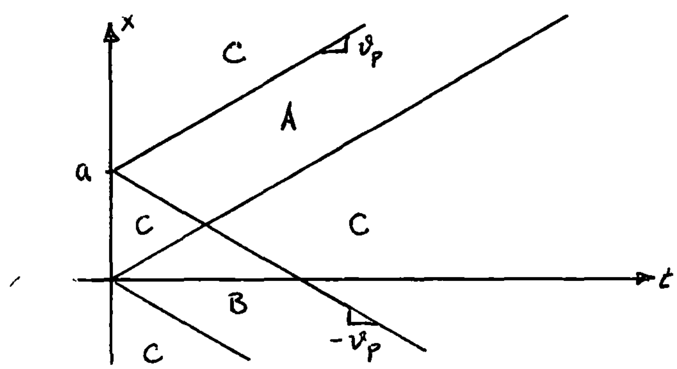
$$\hat{v}(x) = -\frac{j\hat{p}_o}{a\rho_o}\,\frac{\sin(\frac{\omega}{a}x)}{\cos(\frac{\omega}{a}\ell)} \tag{q}$$

and we can now obtain $v(x,t)$: for $\hat{p}_o$ real

$$v(x,t) = \frac{\hat{p}_o}{a\rho_o}\,\frac{\sin(\frac{\omega}{a}x)}{\cos(\frac{\omega}{a}\ell)}\sin\omega t. \tag{r}$$

PROBLEM 9.18

We can calculate the values of $d\delta+/d\alpha$ and $d\delta-/d\beta$ for three regions of the x-t plane as defined below.



Referring to equations from text, 9.1.23 and 9.1.24, 9.1.27 and 9.1.28:

Region A:

$$\frac{d\delta+}{d\alpha} = -\frac{1}{2}\frac{v_m}{v_p},\quad \frac{d\delta-}{d\beta} = 0 \tag{a}$$

and

$$T = -\frac{E}{2}\frac{v_m}{v_p} \tag{b}$$

PROBLEM 9.18 (continued)

Region B:

$$\frac{d\delta+}{d\alpha} = 0, \; \frac{d\delta-}{d\beta} = \frac{1}{2}\frac{v_m}{v_p} \tag{c}$$

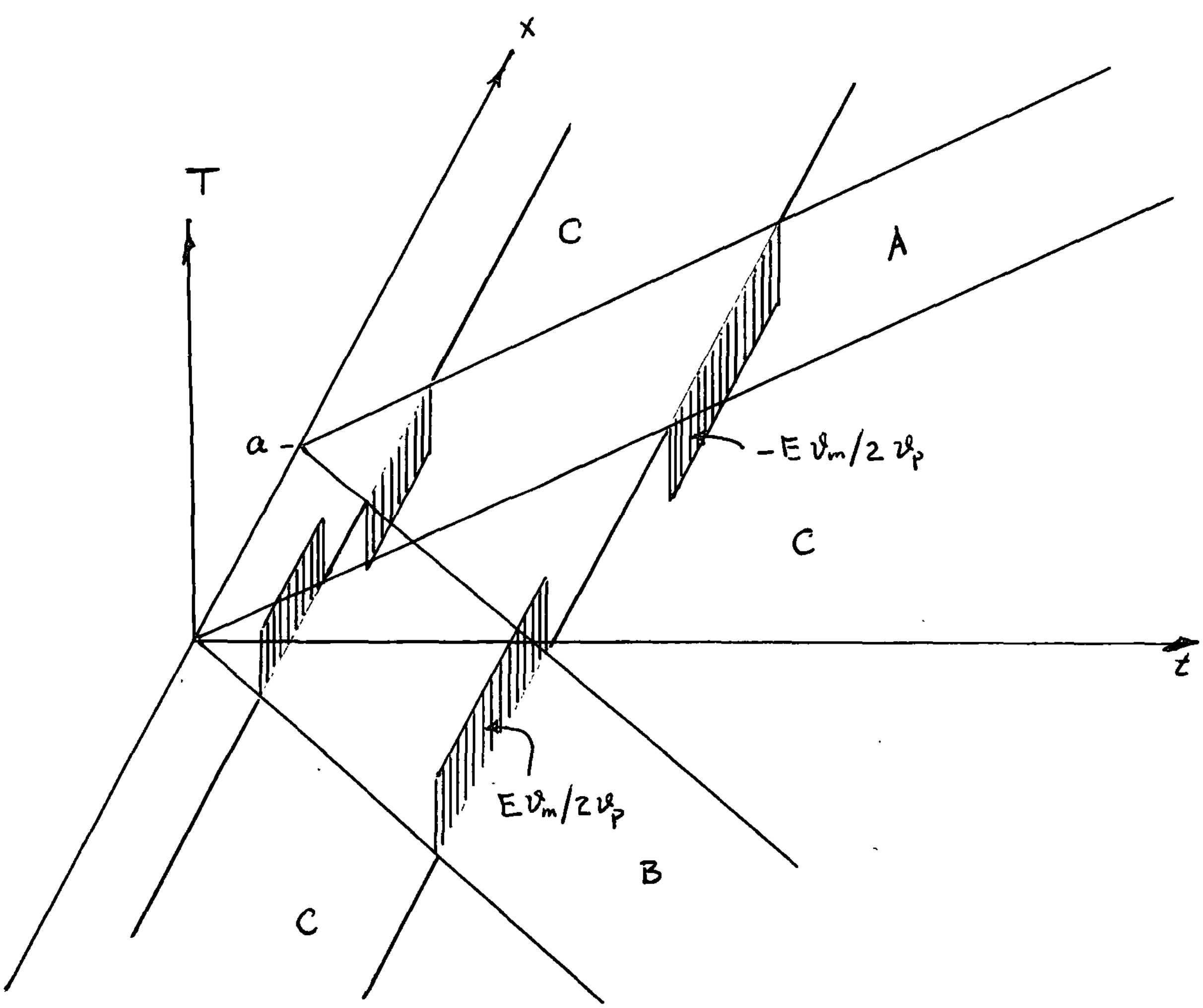
and

$$T = \frac{E}{2}\frac{v_m}{v_p} \tag{d}$$

Region C:

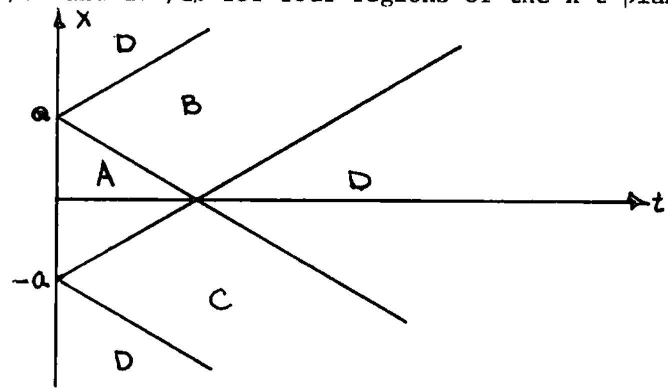
$$\frac{d\delta+}{d\alpha} = \frac{d\delta-}{d\beta} = 0 \text{ and } T = 0. \tag{e}$$

Plotting T(x,t) in the x-t plane we have



PROBLEM 9.19

We can find $d\delta+/d\alpha$ and $d\delta-/d\beta$ for four regions of the x-t plane:



Referring to equations from the text 9.1.23, 9.1.24 and 9.1.27, 9.1.28 we have,

Region A:

$$\frac{d\delta+}{d\alpha} = \frac{1}{2}\frac{T(\alpha)}{E}, \quad \frac{d\delta-}{d\beta} = \frac{1}{2}\frac{T(\beta)}{E} \tag{a}$$

Region B:

$$\frac{d\delta+}{d\alpha} = \frac{1}{2}\frac{T(\alpha)}{E}, \quad \frac{d\delta-}{d\beta} = 0 \tag{b}$$

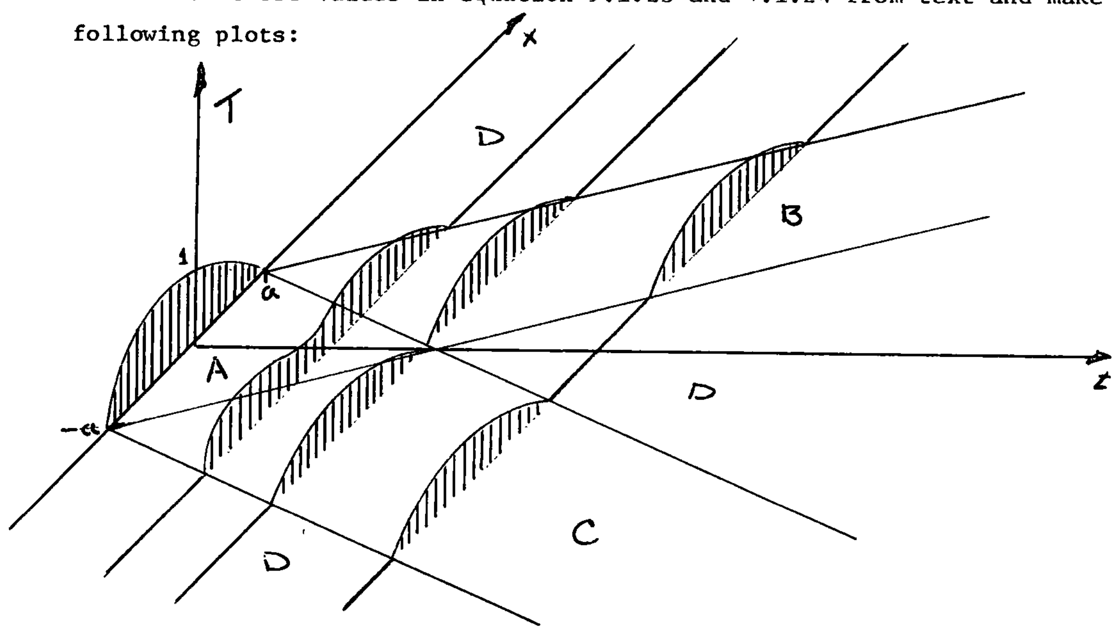
Region C:

$$\frac{d\delta+}{d\alpha} = 0, \quad \frac{d\delta-}{d\beta} = \frac{1}{2}\frac{T(\alpha)}{E} \tag{c}$$

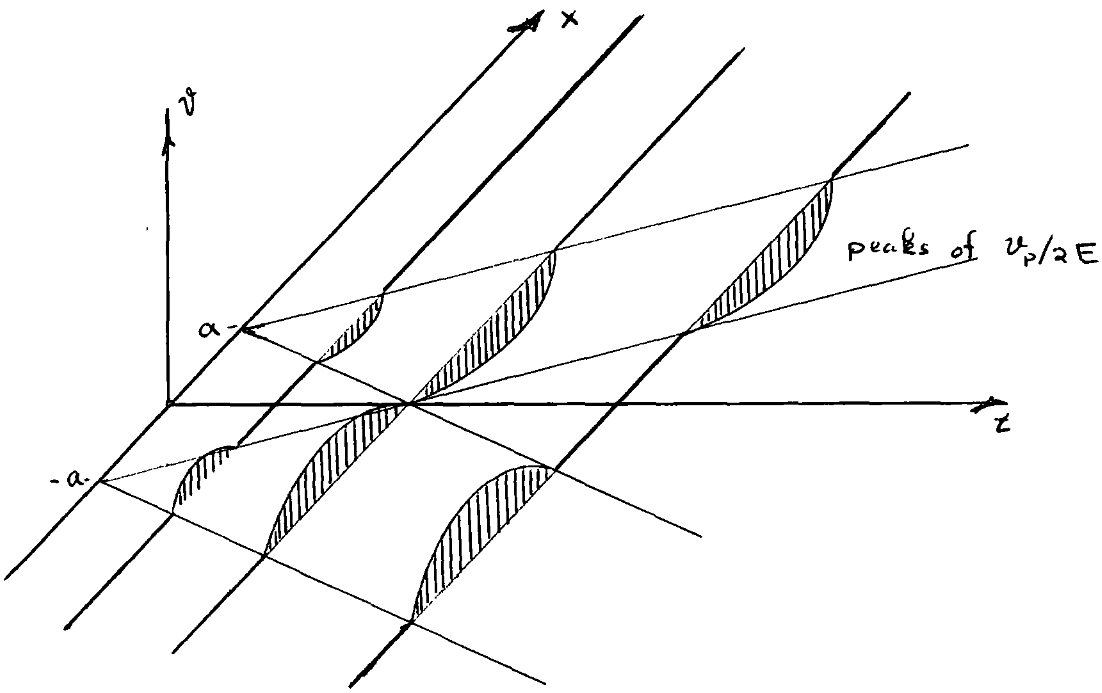
Region D:

$$\frac{d\delta+}{d\alpha} = \frac{d\delta-}{d\beta} = 0 \tag{d}$$

We can use these values in equation 9.1.23 and 9.1.24 from text and make the following plots:



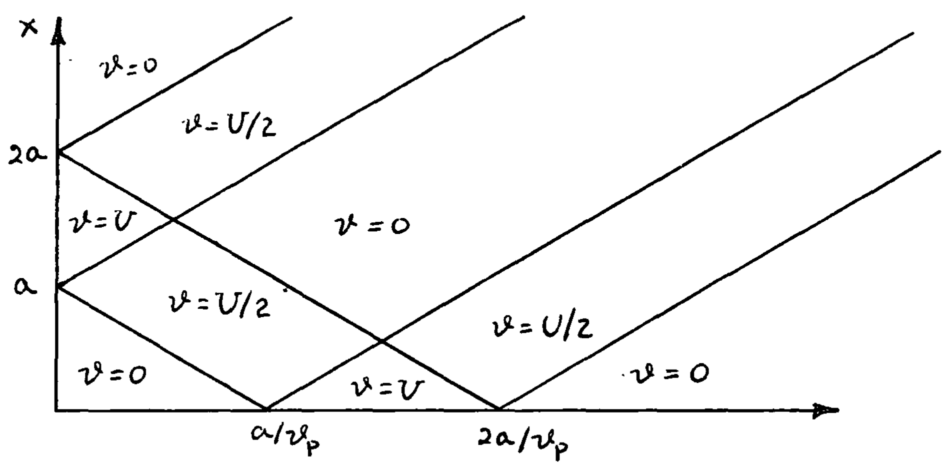
PROBLEM 9.19 (continued)



PROBLEM 9.20

Part a

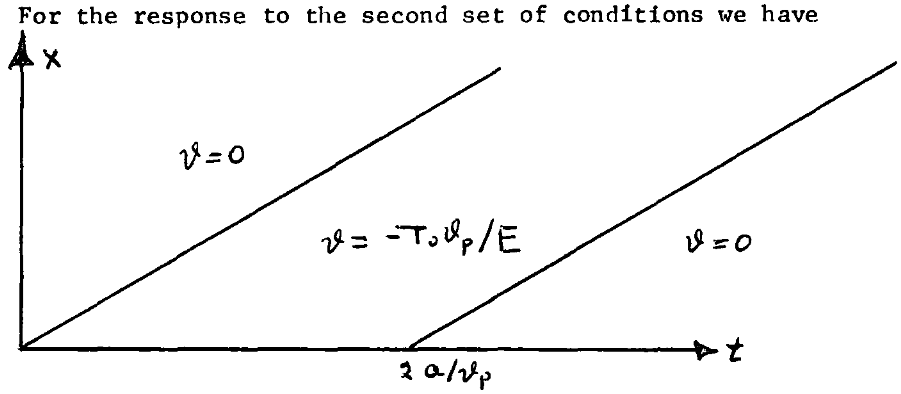
The free end at x = 0 implies that T(0,t) = 0 and using equations 9.1.23 through 9.1.26 we can easily find that velocity pulses "bounce off" x = 0 boundary with the same sign and magnitude. For the x-t plane we can indicate the values for v(x,t):



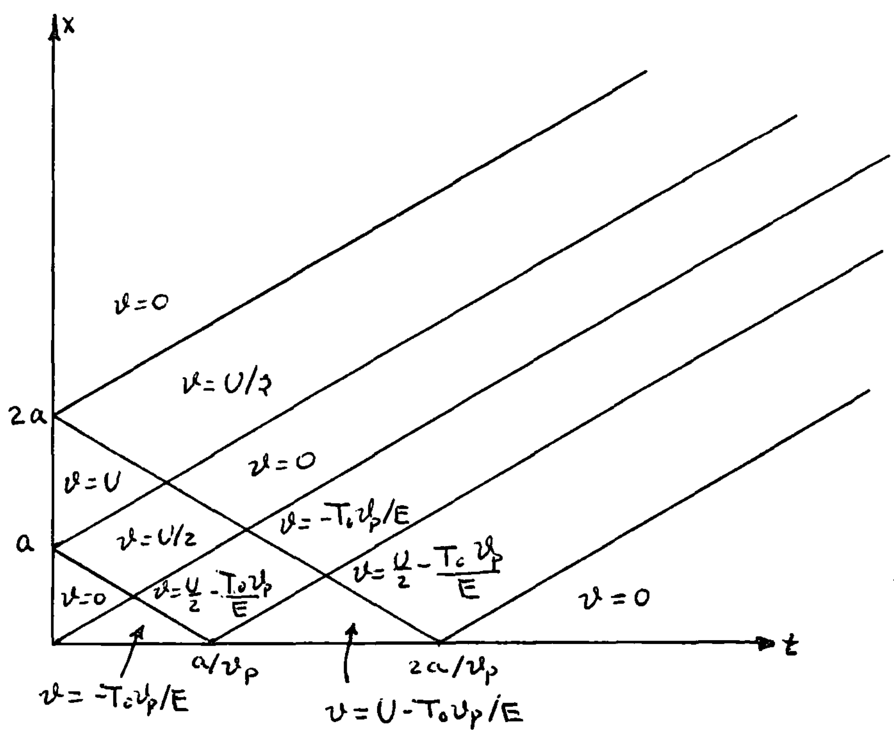
PROBLEM 9.20 (continued)

Part b

We can make use of part (a) if we use superposition. Consider the superposition of boundary and initial conditions; a free end, $T(0,t) = 0$ with the initial conditions in part (a) and the $T(0,t)$ as shown in Fig. 9.P20b with initial conditions on T and v zero. Since the system is linear, we can add the velocities that result from the two situations and thus have the net velocity. For the response to the second set of conditions we have



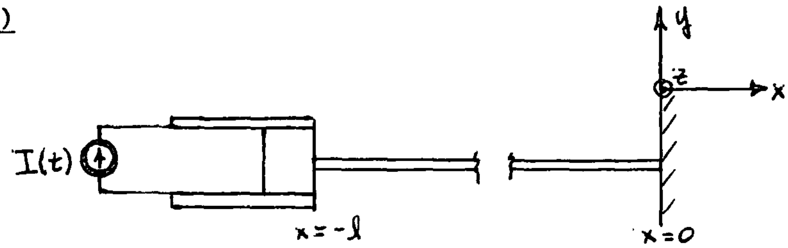
Add this velocity set to the set in part (a) and we obtain:



PROBLEM 9.21

Part a

With the current returned on the inside surface the field in the air gap is $H_z = -\frac{I(t)}{D}$



and the force per unit area acting on the inside surface is

$$T_x = -\frac{1}{2}\mu_o\frac{I^2}{D^2} \tag{a}$$

The force is $f_x = -T_x\, aD = \frac{1}{2}\frac{\mu_o a}{D} I^2(t)$ and the boundary condition at $x = -\ell$ is

$$M\frac{\partial^2\delta}{\partial t^2}(-\ell,t) = \frac{1}{2}\frac{\mu_o a}{D} I^2(t) + AT(-\ell,t) \tag{b}$$

Part b

The current will flow on the surface when the time τ is much shorter than the characteristic diffusion time τ_d over the length b:

$$\tau_d \gg \tau \quad \text{or } \mu_o\sigma b^2 \gg \tau \tag{c}$$

Part c

In order to ignore the mass M, the inertial term must be small compared to $AT(-\ell,t)$. For $t < \tau$, $\delta_- = 0$ on the rod, and from Eqs. 9.1.23 and 9.1.24,

$$T(-\ell,t) = -\frac{E}{v_p}\frac{\partial\delta}{\partial t}(-\ell,t) \tag{d}$$

Thus

$$\left|M\frac{\partial^2\delta}{\partial t^2}(-\ell,t)\right| \ll \left|\frac{AE}{v_p}\frac{\partial\delta}{\partial t}\right| \tag{e}$$

or

$$M \ll AE\,\tau/v_p \tag{f}$$

Our boundary condition in part (a) now becomes:

$$0 = \frac{1}{2}\frac{\mu_o a}{D} I^2(t) + AT(-\ell,t) \tag{g}$$

Since there is a fixed end at $x = 0$ we know that a stress wave traveling in the +x direction will reflect at $x = 0$ with the same wave returning in the -x direction. To satisfy the condition $v(0,t) = 0$, Eq. 9.1.23 shows that $d\delta_+/d\alpha = d\delta_-/d\beta$ at $x = 0$. Thus, from Eq. 9.1.24, the stress is twice that

PROBLEM 9.21 (continued)

initiated at the left end

$$T_r = \frac{\mu_o a}{DA} I_o^2 \quad \text{(h)}$$

PROBLEM 9.22

Part a

We have $W = W'$ and $U = C + U'$ where W' and U' are perturbations from equilibrium. Rewriting the equations we have

$$\frac{\partial W'}{\partial t} + (W')\frac{\partial W'}{\partial x} + \frac{\partial U'}{\partial x} + \frac{K}{(C+U')^3}\frac{\partial U'}{\partial x} = 0 \quad \text{(a)}$$

and

$$\frac{\partial U'}{\partial t} + (C+U')\frac{\partial W'}{\partial x} + (W')\frac{\partial U'}{\partial x} = 0 \quad \text{(b)}$$

Neglecting all second-order perturbation terms we have

$$\frac{\partial W'}{\partial t} + (1 + \frac{K}{C^3})\frac{\partial U'}{\partial x'} = 0 \quad \text{(c)}$$

$$\frac{\partial U'}{\partial t} + (C)\frac{\partial W'}{\partial x} = 0 \quad \text{(d)}$$

Part b

Multiplying the above two equations by $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$, respectively, we have

$$\frac{\partial^2 W'}{\partial t^2} + (1 + \frac{K}{C^3})\frac{\partial^2 U'}{\partial x \partial t} = 0 \quad \text{(e)}$$

and

$$\frac{\partial^2 U'}{\partial x \partial t} + (C)\frac{\partial^2 W'}{\partial x^2} = 0 \quad \text{(f)}$$

Eliminating U' we obtain

$$\frac{\partial^2 W'}{\partial t^2} = C(1 + \frac{K}{C^3})\frac{\partial^2 W'}{\partial x^2} \quad \text{(g)}$$

which is the familiar wave equation with wave velocity $v_p = \sqrt{C(1 + \frac{K}{C^3})}$.
We can write the solution as $W' = \mathrm{Re}[\hat{W}(x)e^{j\omega t}]$ where

$$\hat{W}(x) = C_1 \sin \beta x + C_2 \cos \beta x \quad \text{(h)}$$

with $\beta = \omega/v_p$.
At $x = 0$, $W = W' = 0$ and hence $C_2 = 0$. At $x = -L$, $W = W' = W_o \cos \omega t$, or equivalently $\hat{W}(-L) = W_o$, hence $C_1 = -W_o/\sin\beta L$. Upon substitution we find that

PROBLEM 9.22 (continued)

the solution is

$$W = W' = -\frac{W_o \sin \beta x}{\sin \beta L} \cos \omega t . \qquad (i)$$

PROBLEM 9.23

Part a

This part is similar to Prob. 9.24 with two simplifications:

$V_o = 0$ and

the mass is M/unit width (M_w) instead of 2M. The two separate relations yielding the natural frequencies are

$$\sin\left(\omega L \sqrt{\frac{\sigma_m}{S}}\right) = 0 \qquad (a)$$

and

$$\frac{2\sigma_m}{M\omega\sqrt{\frac{\sigma_m}{S}}} = \tan\left(\omega L\sqrt{\frac{\sigma_m}{S}}\right) \qquad (b)$$

(a) yields $\omega L\sqrt{\sigma_m/S} = n\pi$ where $n = 1, 2, \ldots$ and corresponds to solutions which are "odd", or $\xi(x) = -\xi(-x)$. (b) can be solved graphically and corresponds to solutions which are "even", or $\xi(x) = \xi(-x)$.

Part b

The effect of raising M is to reduce the eigenfrequencies of the "even" modes. The "odd" solutions predicted by (a) are independent of the mass M. This is physically reasonable since there is a node at the mass, and since the mass doesn't move there is no inertial force. For the "even" solutions predicted by (b), we notice that if M = 0 we have essentially the natural frequencies of a membrane of length 2L. As $M \to \infty$, the system responds like two different membranes of length L. The infinite mass acts like a rigid boundary.

PROBLEM 9.24

Part a

We can use the Maxwell Stress Tensor to find the forces of electric origin. If f_u^e corresponds to the force due to the upper electrode and f_ℓ^e corresponds to the force due to the lower electrode, then we have:

$$\vec{f}_u^e(t) = \frac{\varepsilon_o V_o^2 A}{2[d-\xi(0,t)]^2}\,\vec{i}_y \qquad (a)$$

PROBLEM 9.24 (Continued)

$$\vec{f}^{\,e}_{\ell}\ (t) = -\frac{\varepsilon_o V_o^2 A}{2[d+\xi(0,t)]^2}\vec{i}_y \tag{b}$$

Our equation for the membranes is $\sigma_m\ \partial^2\xi/\partial t^2 = S\ \frac{\partial^2\xi}{\partial x^2}$ and if we assume $\xi=\mathrm{Re}[\hat{\xi}(x)e^{j\omega t}]$, then we can write

$$\hat{\xi}(x) = C_1 \sin\beta x + C_2 \cos\beta x \tag{c}$$

for $x > 0$ and

$$\hat{\xi}(x) = C_3 \sin\beta x + C_4 \cos\beta x \tag{d}$$

for $x < 0$ where $\beta = \omega\sqrt{\sigma_m/S}$.

Our boundary condition will yield the four constants. We have

$$\begin{aligned}&\hat{\xi}(x = -L) = 0\\ &\hat{\xi}(x = L) = 0\\ &\hat{\xi}(x = 0^+) = \hat{\xi}(x = 0^-)\end{aligned} \tag{e}$$

and

$$2M\frac{\partial^2\xi}{\partial t^2}(0,t) = Sw\left[\frac{\partial\xi}{\partial x}(0^+) - \frac{\partial\xi}{\partial x}(0^-)\right] + f^e_u(t) + f^e_{\ell}(t) \tag{f}$$

which reduces to

$$-2M\omega^2\ \hat{\xi}(0) = Sw\left[\frac{d\hat{\xi}}{dx}(0^+) - \frac{d\hat{\xi}}{dx}(0^-)\right] + \frac{2\varepsilon_o V_o^2 A}{d^3}\hat{\xi}(0) \tag{g}$$

after we linearize $[f^e_u(t) + f^e_{\ell}(t)]$. Substituting, we immediately find $C_2 = C_4$. Writing the remaining equations we have

$$0 = -C_3 \sin\beta L + C_2 \cos\beta L \tag{h}$$

$$0 = C_1 \sin\beta L + C_2 \cos\beta L \tag{i}$$

$$0 = Sw\beta\ C_1 + \left[\frac{2\varepsilon_o V_o^2 A}{d^3} + 2M\omega^2\right] C_2 - Sw\beta\ C_3 \tag{j}$$

If we eliminate the constants by setting the determinant of the coefficients C_1, C_2, and C_3 equal to zero, we obtain two separate relations:

$$\sin\beta L = 0 \text{ and } \frac{Sw\beta}{\frac{\varepsilon_o V_o^2 A}{d^3} + M\omega^2} = \tan\beta L. \tag{k}$$

PROBLEM 9.24 (continued)

Substituting for β we have

$$\sin\left(\omega L\sqrt{\frac{\sigma_m}{S}}\right) = 0 \text{ and } \frac{Sw\omega\sqrt{\frac{\sigma_m}{S}}}{\frac{\varepsilon_o V_o^2 A}{d^3} + M\omega^2} = \tan \omega L\sqrt{\frac{\sigma_m}{S}} \qquad (\ell)$$

The first relation implies that $\omega L\sqrt{\sigma_m/S} = n\pi$ where $n = 1, 2, \ldots.$ The second relation can be solved graphically.

Part b

As V_o is increased from $V_o = 0$, the lowest natural frequency decreases. When V_o approaches the value

$$\sqrt{\frac{Swd^3}{\varepsilon_o AL}},$$

the lowest natural frequency approaches zero; as V_o is further increased, there will be an imaginary solution for ω and the system will be unstable.

PROBLEM 9.25

Part a

The force of the lower plunger is $f_\ell^m = \frac{\partial W'}{\partial \xi} = \frac{L_o}{2a^2} i_\ell^2$. By symmetry the upper plunger has a force $f_u^m = -(L_o/a^2)\, i_u^2$. From the circuit $i_\ell^2 = (I_o + i_1)^2 \simeq I_o^2 + 2I_o i_1$ and $i_u^2 = (I_o - i_1)^2 \simeq I_o^2 - 2I_o i_1$. Hence the total magnetic force is

$$f^m = \frac{2L_o I_o i_1}{a} = \frac{2L_o I_o G}{a}\frac{\partial \xi}{\partial x}(0,t) \qquad (a)$$

Writing the force balance on the tip of the wire at $x = -\ell$ we have

$$f\frac{\partial \xi}{\partial x}(-\ell,t) + \frac{2L_o I_o G}{a}\frac{\partial \xi}{\partial x}(0,t) = 0 \qquad (b)$$

Part b

Away from the ends

$$M\frac{\partial^2 \xi}{\partial t^2} = f\frac{\partial^2 \xi}{\partial x^2} \qquad (c)$$

and if $\xi = \mathrm{Re}[\hat{\xi}(x)e^{j\omega t}]$ then

$$\hat{\xi}(x) = C_1 \sin \beta x + C_2 \cos \beta x \qquad (d)$$

where $\beta \simeq \omega\sqrt{m/f}$. $\xi(0,t) = 0$ implies that $C_2 = 0$. From part (a) we have

PROBLEM 9.25 (continued)

$$f \frac{d\hat{\xi}}{dx}(-\ell) + \frac{2L_oI_oG}{a}\frac{d\hat{\xi}}{dx}(0) = 0. \tag{e}$$

Upon substitution we obtain

$$f\beta\, C_1 \cos \beta\ell + \frac{2L_oI_o}{a} G\beta C_1 = 0 \tag{f}$$

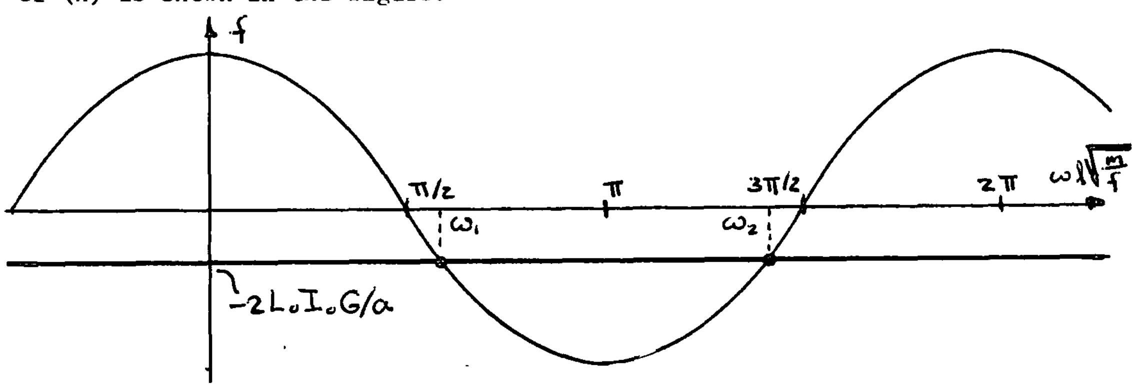
Since C_1 must be finite for a finite response, we have

$$f\beta \cos \beta\ell + \frac{2L_oI_o}{a} G\beta = 0, \tag{g}$$

or

$$f \cos \omega\ell\sqrt{\frac{m}{f}} + \frac{2L_oI_oG}{a} = 0 \tag{h}$$

(We have ruled out one solution, because it is trivial.) A graphical solution of (h) is shown in the figure.



Part c

If G = 0, then

$$\omega\ell \sqrt{\frac{m}{f}} = (\frac{2n+1}{2})\pi \tag{i}$$

with n = 0, 1, 2,...

Part d

From the figure, ω_1 increases toward $\omega_1\ell\sqrt{m/f} = \pi$ and ω_2 decreases toward the same value. They come together at $G = af/2L_oI_o$ and seemingly disappear if $G > \frac{af}{2L_oI_o}$.

PROBLEM 9.25 (continued)

Part e

If $|G| > \frac{af}{2L_o I_o}$, then (h) has imaginary solutions for ω, hence the system will be unstable.

PROBLEM 9.26

Part a

First of all we notice that $y(t) = \xi(-L,t)$. For the membrane $\sigma_m \frac{\partial^2\xi}{\partial t^2} = S\frac{\partial^2\xi}{\partial x^2}$ and if $\xi = \mathrm{Re}[\hat{\xi}(x)e^{j\omega t}]$ then $\hat{\xi}(x) = C_1 \sin\beta x + C_2 \cos\beta x$ where $\beta = \omega\sqrt{\sigma_m/S}$. At $x = 0$, $\hat{\xi}(x=0) = 0$ and therefore $C_2 = 0$. At $x = -L$, we can write the boundary condition

$$M\frac{\partial^2\xi}{\partial t^2}(-L,t) = SD\frac{\partial\xi}{\partial x}(-L,t) + f_y^m(t) \qquad \text{(a)}$$

We can find $f_y^e(t)$ using Ampere's Law and the Maxwell stress tensor

$$f_y^e(t) = \frac{\mu_o AN^2}{2}\left[\frac{(I_o + I(t))^2}{(d-D-\xi(-L,t))^2} - \frac{(I_o - I(t))^2}{(d-D+\xi(-L,t))^2}\right] \qquad \text{(b)}$$

Since $I_o \gg I(t)$ and $(d-D) \gg \xi(-L,t)$ then we can linearize:

$$f_y^e(t) \simeq 2N^2 A\mu_o\left[\frac{I_o}{(d-D)^2}I(t) + \frac{I_o^2}{(d-D)^3}\xi(-L,t)\right] \qquad \text{(c)}$$

Substitution of (c) into (a) and definition of $C_I \equiv \frac{2N^2 A\mu_o I_o}{(d-D)^2}$ and $C_y \equiv \frac{2N^2 A\mu_o I_o^2}{(d-D)^3}$ gives

$$M\frac{\partial^2\xi}{\partial t^2}(-L,t) = SD\frac{\partial\xi}{\partial x}(-L,t) + C_I I(t) + C_y\,\xi(-L,t) \qquad \text{(d)}$$

or in complex form,

$$-M\omega^2\,\hat{\xi}(-L) = SD\frac{\partial\hat{\xi}}{\partial x}(-L) + C_I\hat{I} + C_y\,\hat{\xi}(-L) \qquad \text{(e)}$$

After solving for C_1, we can write

$$\hat{\xi}(x) = \frac{C_I \sin\beta x\,\hat{I}}{(M\omega^2+C_y)\sin\beta L - SD\beta\cos\beta L} \qquad \text{(f)}$$

or finally

PROBLEM 9.26 (continued)

$$\hat{y} = \frac{C_I \sin \beta L \, \hat{I}}{SD\beta \cos \beta L - (M\omega^2 + C_y)\sin \beta L} \tag{g}$$

where $y(t) = \mathrm{Re}[\hat{y}\, e^{j\omega t}]$.

Part b

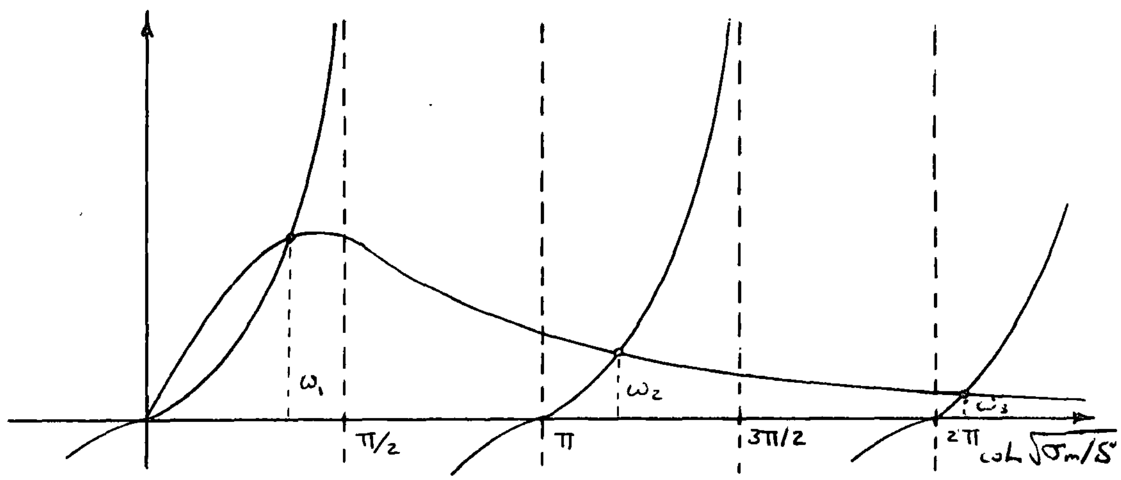
To find the resonance frequencies we look at the poles of $\hat{y}/\hat{I}$. This amounts to finding the zeros of the denominator of $\hat{y}/\hat{I}$. We have

$$SD\omega\sqrt{\frac{\sigma_m}{S}} \cos \omega L\sqrt{\frac{\sigma_m}{S}} = [M\omega^2 + C_y]\sin \omega L\sqrt{\frac{\sigma_m}{S}} \tag{h}$$

or

$$\frac{SD\omega\sqrt{\frac{\sigma_m}{S}}}{M\omega^2 + C_y} = \tan(\omega L\sqrt{\frac{\sigma_m}{S}}) \tag{i}$$

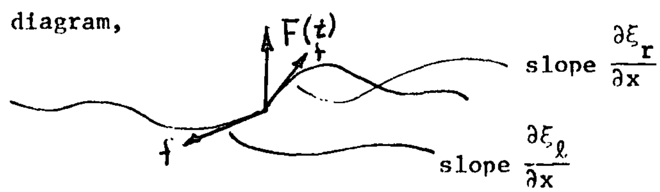
We can represent the solution graphically:



PROBLEM 9.27

Part a

The boundary condition may be obtained by applying force equilibrium using the following diagram,



PROBLEM 9.27 (continued)

thus

$$F(t) = f\left[\frac{\partial\xi_\ell}{\partial x}(0^-) - \frac{\partial\xi_r}{\partial x}(0^+)\right] \tag{a}$$

Part b

For the odd solution, $\xi_\ell(x,t) = -\,\xi_r(-x,t)$ and it follows that $\frac{\partial\xi_\ell}{\partial x} - \frac{\partial\xi_r}{\partial x} = 0$. This implies that the odd solution is not excited by the force F(t).

Part c

For the even solution, $\xi_\ell(x,t) = \xi_r(-x,t)$, we have $\frac{\partial\xi_\ell}{\partial x} = -\frac{\partial\xi_r}{\partial x}$ and the boundary condition (a) from part (a) becomes

$$F(t) = -\,2f\,\frac{\partial\xi_r}{\partial x} \quad \text{at } x = 0 \tag{b}$$

For $0<x<\ell$ we have

$$m\,\frac{\partial^2\xi}{\partial t^2} = f\,\frac{\partial^2\xi}{\partial x^2} \tag{c}$$

with $\xi(x,t) = 0$ at $x = \ell$.

For $t < 0$, this reducss to

$$\frac{\partial^2\xi}{\partial x^2} = 0 \tag{d}$$

and we obtain

$$\xi(x) = \frac{F_o\ell}{2f}\left(1 - \frac{x}{\ell}\right) \quad \text{for } 0<x<\ell \tag{e}$$

Part d

We now have a combined transient and driven response, as discussed in Sec. 9.2.1. By contrast with the developments of that section, we now have a boundary condition at $x = 0$ on the slope $\partial\xi_r/\partial x$ (see (b) of part (c)). Our program is: ($\xi\equiv\xi_r$ in the following)

i. Find the driven sinusoidal steady-state response. This satisfies the boundary conditions:

$$F_o \cos\omega t = -\,2f\,\frac{\partial\xi}{\partial x}(0,t) \tag{f}$$

$$\xi(\ell,t) = 0 \tag{g}$$

ii. Find normal modes, which satisfy homogeneous boundary conditions;

$$\frac{\partial\xi}{\partial x}(0,t) = 0 \tag{h}$$

$$\xi(\ell,t) = 0 \tag{i}$$

The sum of these modes takes the form of a Fourier series.

PROBLEM 9.27 (continued)

iii. Superimpose (i) and (ii) and use the initial conditions found in parts (a)-(c) to evaluate the arbitrary coefficients.

The driven response is of the form

$$\xi = \mathrm{Re}(C_1 \sin \beta x + C_2 \cos \beta x)e^{j\omega t};\ \beta = \omega\sqrt{\frac{m}{f}} \tag{j}$$

a linear combination which satisfies (g)

$$\xi = \mathrm{Re}C_3 \sin \beta(x-\ell)e^{j\omega t} \tag{k}$$

while (f) evaluates C_3 and the driven response is

$$\xi = -\mathrm{Re}\frac{F_o \sin \beta(x-\ell)e^{j\omega t}}{2f\beta \cos \beta\ell} \tag{ℓ}$$

The normal modes are in this lossless case the resonances of the driven response and occur as $\cos \beta\ell = 0$. Thus

$$\omega_n \ell\sqrt{\frac{m}{f}} = (\frac{2n+1}{2})\pi,\ n = 0, 1, 2, 3... \tag{m}$$

and the total solution for $0<x<\ell$ is

$$\xi = -\frac{F_o \sin\beta(x-\ell)}{2f\beta \cos \beta\ell}\cos \omega t + \sum_{n=0}^{\infty}[A_n^+ e^{j\omega_n t} + A_n^- e^{-j\omega_n t}]\sin[(\frac{2n+1}{2})\frac{\pi}{\ell}(x-\ell)] \tag{n}$$

The coefficients A_n^+ and A_n^- are evaluated by requiring that

$$\xi(x,0) = \frac{F_o \ell}{2f}(1-\frac{x}{\ell}) = -\frac{F_o \sin\beta(x-\ell)}{2f\beta \cos \beta\ell} + \sum_{n=0}^{\infty}(A_n^+ + A_n^-)\sin[(\frac{2n+1}{2})\frac{\pi}{\ell}(x-\ell)] \tag{o}$$

and

$$\frac{\partial\xi}{\partial t}(x,0) = 0 = \sum_{n=0}^{\infty}[j\omega_n A_n^+ - j\omega_n A_n^-]\sin[(\frac{2n+1}{2})\frac{\pi}{\ell}(x-\ell)] \tag{p}$$

This last condition is satisfied if $A_n^+ = A_n^-$. The A_n^+'s follow from (o) by using the orthogonality of the functions $\sin[(2n+1/2)\frac{\pi}{\ell}(x-\ell)]$ and $\sin[(2m+1\ /2)\frac{\pi}{\ell}(x-\ell)$, $m \neq n$, over the interval ℓ.

PROBLEM 10.1

Part a

At $x = 0$, the net force on an incremental length of the string has to be zero.

$$-2B\,\frac{\partial\xi}{\partial t} - f\,\frac{\partial\xi}{\partial x} = 0$$

This is the required boundary condition at $x = 0$.

Part b

The power absorbed by the dashpots is the product of force $2B\,\partial\xi/\partial t$ and the velocity $\partial\xi/\partial t$. Thus

$$P = 2B\left(\frac{\partial\xi}{\partial t}\right)^2$$

If we solve Eq. 10.1.6 for

$$\xi(x,t) = \mathrm{Re}[\hat{\xi}\,e^{j(\omega t-kx)}]$$

and assume that $\omega < \omega_c$ we get

$$\xi(x,t) = \mathrm{Re}\,\{[A_1\,\sinh|k|x + A_2\,\cosh|k|x]e^{j\omega t}\}$$

where

$$k = \left[\frac{\omega^2 - \omega_c^2}{v_s^2}\right]^{1/2}$$

We can calculate A_1 and A_2 using the boundary condition of part (a) and the boundary condition at $x = \ell$

$$\xi(-\ell,t) = \mathrm{Re}\,\xi_o\,e^{j\omega t}$$

We then get

$$A_1 = -\frac{j\xi_o 2B\omega}{[f|k|\cosh|k|\ell + j\omega 2B\,\sinh|k|\ell]}$$

$$A_2 = \frac{\xi_o f|k|}{[f|k|\cosh|k|\ell + j\omega 2B\,\sinh|k|\ell]}$$

If we plug these values into the expression for power, and then time average, we have

$$\langle P\rangle = \frac{B(f|k|\xi_o\omega)^2}{[(f|k|\cosh|k|\ell)^2 + (2B\omega\,\sinh|k|\ell)^2]}$$

where it is convenient to use the identity

$$\langle \mathrm{Re}\,Ae^{j\omega t}\,\mathrm{Re}Be^{j\omega t}\rangle = \frac{1}{2}\,AB^*$$

PROBLEM 10.2

Part a

We use Eq. (10.1.6)

$$\frac{\partial^2\xi}{\partial t^2} = v_s^2 \frac{\partial^2\xi}{\partial x^2} - \omega_c^2\xi \ , \ \omega_c^2 = \frac{Ib}{m}$$

Assume solutions $\xi = \mathrm{Re}[(Ae^{-jkx} + Be^{jkx})e^{j\omega t}]$. The dispersion equation is:

$$k^2 = \frac{\omega_d^2 - \omega_c^2}{v_s^2}$$

Now use the boundary conditions, which require

$$A\ e^{jk\ell} + Be^{-jk\ell} = \xi_d$$

$$-jk[A - B] = 0$$

(i) $\omega_d < \omega_c$ (below cutoff)

$$\xi = \frac{\xi_d \cosh \alpha x}{\cosh \alpha\ell} \cos \omega_d t$$

$$\alpha = \sqrt{(\omega_c^2 - \omega_d^2)/v_s^2}$$

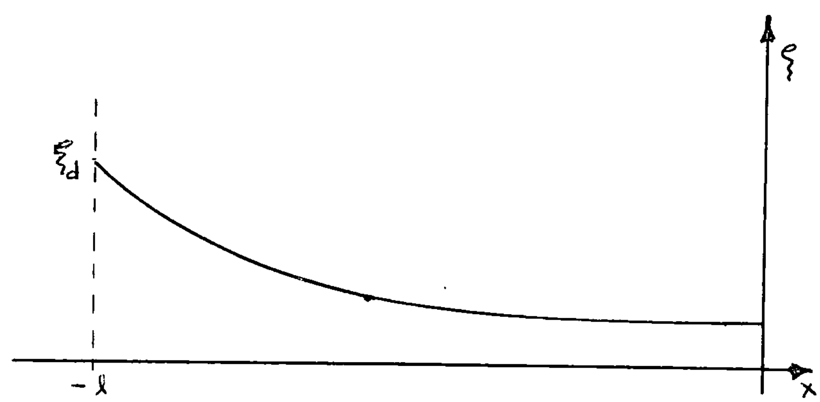
(ii) $\omega_d > \omega_c$ (above cutoff)

$$\xi(x,t) = \frac{\xi_d \cos \beta x}{\cos \beta\ell} \cos \omega_d t$$

$$\beta = \sqrt{(\omega_d^2 - \omega_c^2)/v_s^2}$$

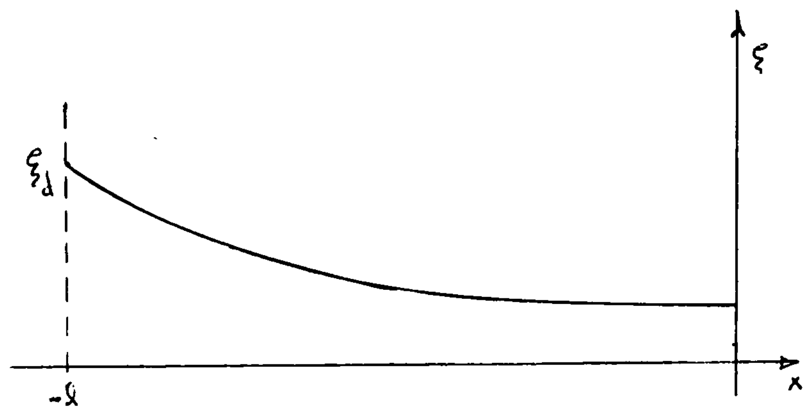
Part b

$\omega_d = 0$

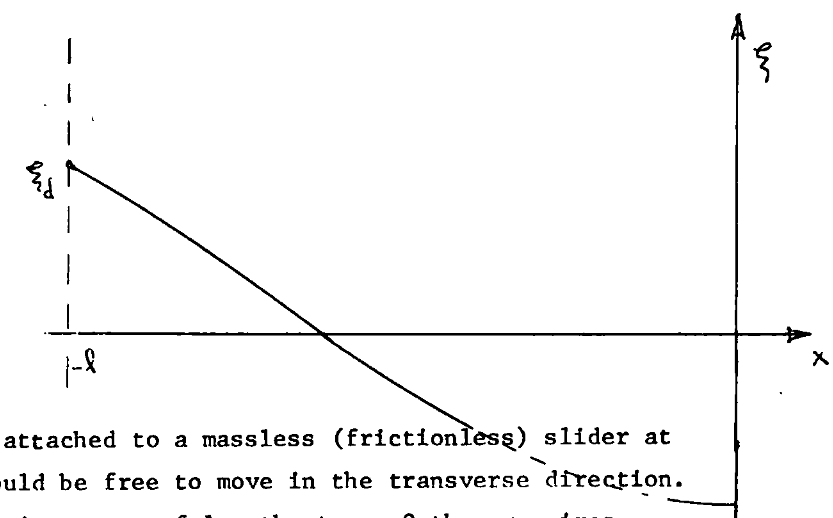


PROBLEM 10.2 (continued)

$\omega_d < \omega_c$



$\omega_d > \omega_c$



Part c

The string might be attached to a massless (frictionless) slider at $x = 0$, so that the end would be free to move in the transverse direction. Force equilibrium for the increment of length at $x = 0$ then requires $\partial\xi/\partial x = 0$ at $x = 0$.

PROBLEM 10.3

Part a

From Eq. 10.1.10 we have

$$k = \left[\frac{\omega^2 - \omega_c^2}{v_s^2}\right]^{1/2}$$

with our solution of the form

$$\xi(x,t) = \mathrm{Re}\left(A_1\, e^{j(\omega t-kx)} + A_2\, e^{j(\omega t+kx)}\right)$$

We have the boundary conditions

PROBLEM 10.3(continued)

$$\frac{\partial \xi}{\partial x}(0,t) = 0$$

and

$$\frac{\partial \xi}{\partial x}(-\ell,t) = 0.$$

From the first boundary condition, we obtain

$$A_1 = A_2 \; ; \; \xi(x,t) = \text{Re } A_3 \cos kx \, e^{j\omega t}$$

From the second boundary condition, we obtain

$$\sin k\ell = 0$$

This implies that

$$k = \frac{n\pi}{\ell} \; ; \; n = 0,1,2,3\ldots$$

Note that by contrast with the case where the ends are fixed, $n = 0$ is a valid (nontrivial) and crucial solution. It corresponds to an eigenmode which is simply a rigid body translation.

From Eq. 10.1.7

$$\omega^2 = k^2 v_s^2 + \omega_c^2$$

Therefore, the eigenfrequencies are

$$\omega = \pm \left[\omega_c^2 + \left(\frac{n\pi}{\ell} v_s\right)^2\right]^{1/2}$$

For the $n = 0$ mode, $\omega = \pm \omega_c$.

Part b

With I as in Fig. 10.1.9, we have the same equations as in part (a) if we replace ω_c^2 by $-\omega_c^2$. Therefore, for this case, the eigenfrequencies are

$$\omega = \left[\left(\frac{n\pi}{\ell} v_s\right)^2 - \omega_c^2\right]^{1/2}$$

Part c

With I as in Fig. 10.1.9, the $\bar{I}\times\bar{B}$ force is destabilizing, as a small perturbation from $x = 0$ tends to increase this force. If ω in part (b) became imaginary, the equilibrium $\xi = 0$ would become unstable as the solutions are unbounded in time. This will happen as

$$\left(\frac{n\pi}{\ell} v_s\right)^2 - \omega_c^2 < 0$$

PROBLEM 10.3 (continued)

or in terms of the current

$$I > \frac{n}{b}\left(\frac{\pi}{\ell} v_s\right)^2$$

Note that any finite current makes the n = 0 mode unstable, since for this mode there is no elastic restoring force.

PROBLEM 10.4

Multiply the system equation by $\frac{\partial \xi}{\partial t}$,

$$m \frac{\partial \xi}{\partial t} \cdot \frac{\partial^2 \xi}{\partial t^2} = f \frac{\partial \xi}{\partial t} \cdot \frac{\partial^2 \xi}{\partial x^2} - Ib \frac{\partial \xi}{\partial t} \cdot \xi + \frac{\partial \xi}{\partial t} \cdot F(x,t)$$

Proper substitution of partial differential identities yields:

$$\frac{\partial}{\partial t}\left[\frac{m}{2}\left(\frac{\partial \xi}{\partial t}\right)^2 + \frac{f}{2}\left(\frac{\partial \xi}{\partial x}\right)^2 + \frac{Ib}{2}\xi^2\right] - \frac{\partial}{\partial x}\left[f \frac{\partial \xi}{\partial x} \cdot \frac{\partial \xi}{\partial t}\right] = \frac{\partial \xi}{\partial t} \cdot F(x,t)$$

PROBLEM 10.5

We have that

$$\xi(x,t) = \mathrm{Re}\left[\hat{\xi}_+ e^{j(\omega t - kx)} + \hat{\xi}_- e^{j(\omega t + kx)}\right]$$

Part a

For k real, we might write this in the form

$$\xi(x,t) = \frac{1}{2}\left[\hat{\xi}_+ e^{j(\omega t - kx)} + \hat{\xi}_+^* e^{-j(\omega t - kx)} + \hat{\xi}_- e^{j(\omega t + kx)} + \hat{\xi}_-^* e^{-j(\omega t + kx)}\right]$$

From Prob. 10.4 we have that the power carried by the string is

$$P = - f \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial t}.$$

If we do the indicated differentiations, then substitute into this expression, and then time average, we will obtain

$$\langle P \rangle = \frac{f \omega k}{2}\left[\hat{\xi}_+ \hat{\xi}_+^* - \hat{\xi}_- \hat{\xi}_-^*\right]$$

Part b

For k purely imaginary

$$k = j\beta$$

with β real, we can write $\xi(x,t)$ in the form

PROBLEM 10.5 (continued)

$$\xi(x,t) = \frac{1}{2}\left[\hat{\xi}_+ e^{j\omega t+\beta x} + \hat{\xi}_+^* e^{-j\omega t+\beta x} + \hat{\xi}_- e^{j\omega t-\beta x} + \hat{\xi}_-^* e^{-j\omega t-\beta x}\right]$$

If we again substitute into our expression for power and average over time we obtain

$$\langle P\rangle = -\frac{j\omega\beta f}{2}[\hat{\xi}_+^* \hat{\xi}_- - \hat{\xi}_+ \hat{\xi}_-^*]$$

From (b), we see that it is possible to have a net power flow from two evanescent waves, but not from a single evanescent wave. Suppose that a single evanescent wave did carry power away from the driving source. This would correspond physically to a string driven at the left and infinite to the right. With $\omega_d<\omega_c$, the response as $x \to \infty$ becomes vanishingly small; clearly there can be no power flow at $x \to \infty$. Yet, there is no mechanism for power absorption by the string and so there can be no power flow into the string from the drive. With a dissipative load, a second evanescent wave is established, decaying to the left, and the conditions for power flow are met.

PROBLEM 10.6

From the dispersion relation, we calculate:

$$v_g \equiv \frac{\partial\omega}{\partial k} = v_s\left[1 - \frac{\omega_c^2}{\omega^2}\right]^{1/2}$$

Now, assuming a single forward traveling wave:

$$\xi = \xi_+ \cos[\omega t - k(\omega)x]$$

Then:

$$\langle W\rangle = \left(\frac{m\omega^2}{4} + \frac{fk^2}{4} + \frac{Ib}{4}\right)\xi_+^2$$

$$\langle P\rangle = \left(\frac{fk\omega}{2}\right)\xi_+^2$$

Thus, substitution gives

$$\frac{\langle P\rangle}{\langle W\rangle} = \frac{fk\omega/2}{\left(\frac{m\omega^2}{4} + \frac{fk^2}{4} + \frac{Ib}{4}\right)}$$

$$= v_s\left[1 - \frac{\omega_c^2}{\omega^2}\right]^{1/2} = v_g$$

which is the desired relation. This result is of some general significance, but has been shown here for a particular case.

PROBLEM 10.7

Part a

The equations of motion for the membranes are

$$\sigma_m \frac{\partial^2 \xi_1}{\partial t^2} = S \frac{\partial^2 \xi_1}{\partial x^2} + T_1$$

$$\sigma_m \frac{\partial^2 \xi_2}{\partial t^2} = S \frac{\partial^2 \xi_2}{\partial x^2} + T_2$$

where T_1 and T_2 are the transverse magnetic forces/area. If the membranes extend a distance w into the paper, and if we define regions 1, 2, and 3 as the top, middle, and bottom regions respectively in Fig. 10P.7, the flux in each region is

$$\lambda_1 = \mu_o H_1 \, w(d-\xi_1)$$

$$\lambda_2 = \mu_o H_2 \, w(d+\xi_1-\xi_2)$$

$$\lambda_3 = \mu_o H_3 \, w(d+\xi_2)$$

where H_1, H_2, and H_3 are the magnetic field intensities within each region. Since the flux is conserved, when $\xi_1 = \xi_2 = 0$ we have

$$\lambda_1 = \lambda_3 = - \mu_o H_o \, wd \qquad \lambda_2 = + \mu_o H_o \, wd$$

Therefore,

$$\bar{H}_1 = - \frac{H_o d}{d-\xi_1} \bar{i}_x$$

and

$$\bar{H}_2 = + \frac{H_o d}{d+\xi_1-\xi_2} \bar{i}_x$$

and

$$\bar{H}_3 = - \frac{H_o d}{d+\xi_2} \bar{i}_x.$$

We will use the Maxwell stress tensor to calculate T_1 and T_2, using a pill-box volume enclosing a section of surface on each membrane.

We then obtain

$$T_1 \cong - \frac{\mu_o}{2} [H_1^2 - H_2^2]$$

and

$$T_2 \cong - \frac{\mu_o}{2} [H_2^2 - H_3^2]$$

Substituting the expression for the $\bar{H}$ fields, and realizing that $\xi_1 \ll d$ and $\xi_2 \ll d$, we finally obtain for the forces

PROBLEM 10.7 (continued)

$$T_1 = -\frac{\mu_o H_o^2 (2\xi_1 - \xi_2)}{d}$$

and

$$T_2 = -\frac{\mu_o H_o^2 (2\xi_2 - \xi_1)}{d}$$

Our equations of motion are then

$$\sigma_m \frac{\partial^2 \xi_1}{\partial t^2} = S \frac{\partial^2 \xi_1}{\partial x^2} - \frac{\mu_o H_o^2}{d} (2\xi_1 - \xi_2)$$

and

$$\sigma_m \frac{\partial^2 \xi_2}{\partial t^2} = S \frac{\partial^2 \xi_2}{\partial x^2} - \frac{\mu_o H_o^2}{d} (2\xi_2 - \xi_1)$$

Part b

We assume that

$$\xi_1 = \text{Re}\, \hat{\xi}_1 e^{j(\omega t - kx)}$$

and

$$\xi_2 = \text{Re}\, \hat{\xi}_2 e^{j(\omega t - kx)}$$

We can substitute these functions into the equations of motion from part (a), and solve for the relation between ω and k such that the 2 equations of motion are consistent. This dispersion relation is

$$-\sigma_m \omega^2 + Sk^2 + \frac{2\mu_o H_o^2}{d} = \pm \frac{\mu_o H_o^2}{d}$$

We see that the dispersion equation factors into two dispersion relations. If we substitute this relation back into the equations of motion from part (a), we see that we obtain even and odd solutions.

The dispersion relation

$$\omega^2 = \frac{Sk^2}{\sigma_m} + \frac{\mu_o H_o^2}{\sigma_m d}$$

yields

$$\xi_1 = \xi_2.$$

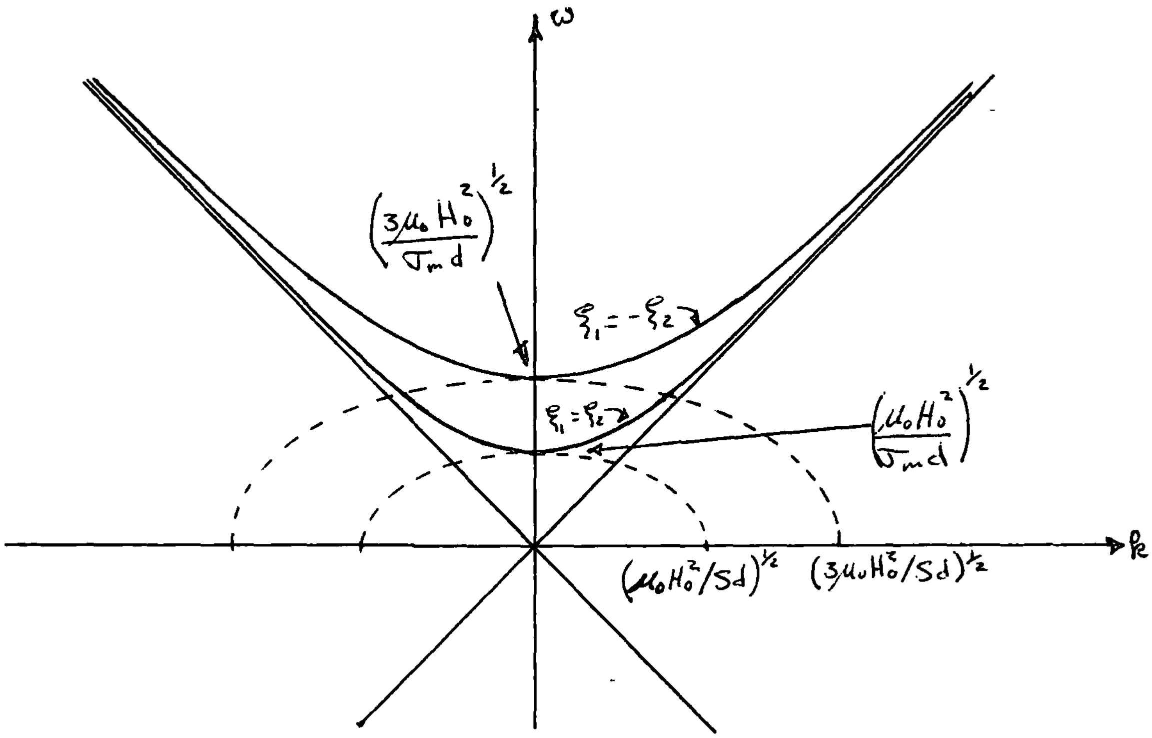
The dispersion relation

$$\omega^2 = \frac{Sk^2}{\sigma_m} + \frac{3\mu_o H_o^2}{\sigma_m d}$$

yields $\xi_1 = -\xi_2$.

PROBLEM 10.7 (continued)

Plotting ω versus k, we obtain



_______ k real

- - - - k imaginery

From the plot we see that the lowest frequency for which we have propagation (k real) for the even mode is

$$\omega_{ce} = \left(\frac{\mu_o H_o^2}{\sigma_m d}\right)^{1/2}$$

For the odd mode, the cut off frequency is

$$\omega_{co} = \left(\frac{3\mu_o H_o^2}{\sigma_m d}\right)^{1/2}$$

Part d

We are given the boundary conditions that at x = 0

$$\xi_1 = 0 \quad \xi_2 = 0$$

and at $x = -\ell$

$$\xi_1 = -\xi_2 = \text{Re}\ \xi_o\, e^{j\omega t}.$$

PROBLEM 10.7 (continued)

From the boundary condition, we see that our solution is purely odd. Therefore

$$k = \left[\frac{\omega^2\sigma_m}{S} - \frac{3\mu_o H_o^2}{Sd}\right]^{1/2}$$

We assume a solution of the form

$$\xi_1(x,t) = -\,\xi_2(x,t) = \mathrm{Re}\{A_1\, e^{j(\omega t-kx)} + A_2\, e^{j(\omega t+kx)}\}$$

Evaluating A_1 and A_2 through the boundary conditions, we obtain

$$A_1 = -\,A_2 = \frac{\xi_o}{e^{jk\ell} - e^{-jk\ell}}$$

Therefore

$$\xi_1(x,t) = -\xi_2(x,t) = \mathrm{Re}\,\frac{\xi_o[e^{-jkx} - e^{+jkx}]e^{j\omega t}}{[e^{jk\ell} - e^{-jk\ell}]}$$

For $\omega = 0$, k is pure imaginary. We define $k = j\beta$, with β real with value

$$\beta = \left(\frac{3\mu_o H_o^2}{Sd}\right)^{1/2}$$

Therefore

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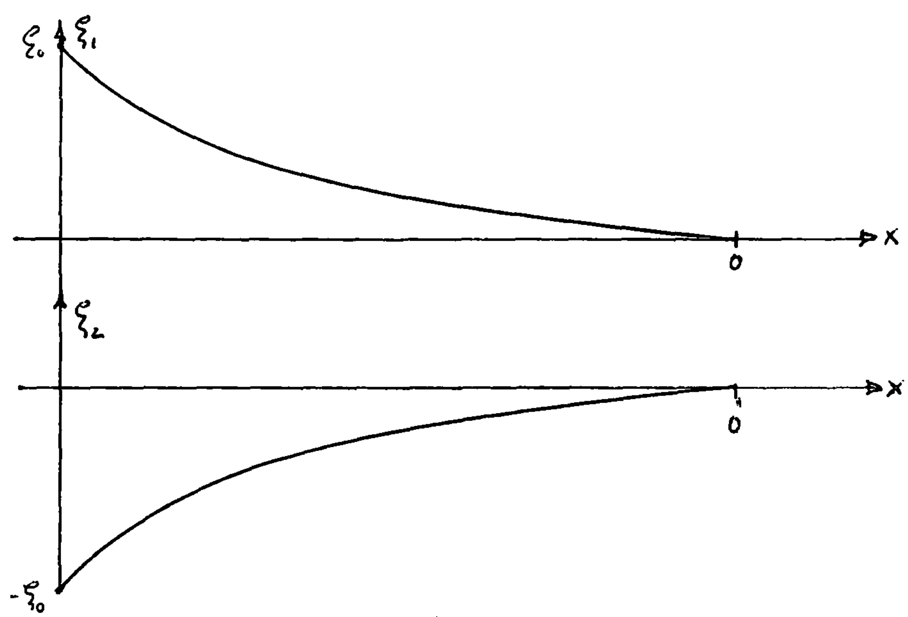
$$\xi_1(x,t) = -\,\frac{\xi_o \sinh \beta x}{\sinh \beta\ell} \cos\omega t$$

A sketch appears below.



PROBLEM 10.8

Part a

The given equations follow by writing out Maxwell's equations and assuming $\bar{E}$ and $\bar{H}$ have the given directions and dependences.

Part b

The force equation for an incremental volume element is

$$\bar{F} = \bar{i}_x \, mn_e \frac{\partial v_x}{\partial t} \tag{a}$$

where $\bar{F}$ is the force density due to electrical forces on the electrons

$$\bar{F} = - \bar{i}_x \, en_e E_x \tag{b}$$

Thus,

$$-en_e E_x = mn_e \frac{\partial v_x}{\partial t} \tag{c}$$

Part c

As the electrons move, they give rise to the current density

$$J_x \cong - en_e \, v_x \text{ (linearized)} \tag{d}$$

Part d

Assume $e^{j(\omega t-kx)}$ dependence and (c) and (d) require

$$\hat{J}_x = - j \frac{e^2 n_e}{\omega m} \hat{E}_x \tag{e}$$

$$= - j\omega\varepsilon_o \left[\frac{\omega_\rho^2}{\omega^2}\right]\hat{E}_x \tag{f}$$

where $\omega_\rho = \sqrt{e^2 n_e / m\varepsilon_o}$ is called the plasma frequency. (See page 600)
Combining this with Maxwell's equations:

$$k^2 = \frac{\omega^2}{c^2}\left[1 - \frac{\omega_\rho^2}{\omega^2}\right] \; ; \; c = \frac{1}{\sqrt{\varepsilon_o \mu_o}} \tag{g}$$

Part e

We have a dispersion which yields evanescent waves below the plasma (cutoff) frequency. Below this frequency, the electrons respond to the electric field associated with the wave in such a way as to reflect rather than transmit an incident electromagnetic wave.

Part f

Waves impinging upon a boundary between free space and plasma will be totally reflected if the wave frequency $\omega < \omega_p$. The plasma frequency for the ionosphere

PROBLEM 10.8 (continued)

is typically

$$f_p \sim 10 \text{ MH}_z$$

This result explains why AM broadcasts ($500 \text{ KH}_z < f < 1500 \text{ KH}_z$) can commonly be monitored all over the world, whereas FM ($88 \text{ MH}_z < f < 108 \text{ MH}_z$) has a range limited to "line-of-sight".

PROBLEM 10.9

In the regions

$$x < -\ell \text{ and } x > 0$$

the equation of motion for the string is

$$\frac{\partial^2 \xi}{\partial t^2} = v_s^2 \frac{\partial^2 \xi}{\partial x^2}$$

In the region $-\ell \leq x \leq 0$, this equation is modified due to the magnetic force to

$$\frac{\partial^2 \xi}{\partial t^2} = v_s^2 \frac{\partial^2 \xi}{\partial x^2} - \omega_c^2 \xi$$

If we assume

$$\xi(x,t) = \text{Re}\{\hat{\xi}\, e^{j(\omega t - kx)}\}$$

and substitute back into the equations of motion we obtain the dispersion relations

$$k = \pm \left[\frac{\omega^2 - \omega_c^2}{v_s^2}\right]^{1/2} \qquad -\ell \leq x \leq 0$$

$$k = \pm \frac{\omega}{v_s} \qquad x < -\ell, x > 0$$

The boundary conditions are

$$\text{at } x = -\ell \qquad \hat{\xi} = \hat{\xi}_o$$

$$\text{at } x = 0 \qquad \xi \text{ and } \frac{\partial \xi}{\partial x} \text{ must be continuous.}$$

We assume that

$$\xi(x,t) = \text{Re}\left\{\left[A\, e^{-\beta x} + B\, e^{+\beta x}\right] e^{j\omega t}\right\} \qquad \text{for } -\ell < x < 0$$

PROBLEM 10.9 (continued)

where

$$\beta = \left[\frac{\omega_c^2 - \omega^2}{v_s^2}\right]^{1/2} \qquad \text{for } \omega < \omega_c$$

and

$$\xi(x,t) = \mathrm{Re}\,\{\hat{\xi}_b\, e^{-jk_b x}\, e^{j\omega t}\} \quad \text{for } x > 0$$

where

$$k_b = \frac{\omega}{v_s}$$

Using the above boundary conditions, we obtain

$$A = \frac{\xi_o(\beta + jk_b)}{2(\beta \cosh \beta\ell + jk_b \sinh \beta\ell)}$$

$$B = \frac{\xi_o(\beta - jk_b)}{2(\beta \cosh \beta\ell + jk_b \sinh \beta\ell)}$$

But $\hat{\xi}_b = A + B$

Therefore

$$\frac{\hat{\xi}_b}{\xi_o} = \frac{1}{[\cosh \beta\ell + \frac{jk_b}{\beta} \sinh \beta\ell]}$$

Part b

As $\ell \to 0$

$$\frac{\hat{\xi}_b}{\hat{\xi}_o} \to 1$$

As $\ell \to \infty$

$$\frac{\hat{\xi}_b}{\hat{\xi}_o} \to 0$$

PROBLEM 10.10

Part a

The equation of motion for the string is

$$m\,\frac{\partial^2 \xi}{\partial t^2} = f\,\frac{\partial^2 \xi}{\partial x^2} + S - mg \tag{a}$$

PROBLEM 10.10 (continued)

where, for small deflections ξ in the "1/r" field from Q,

$$S \cong \frac{qQ}{2\pi\varepsilon_o d}\left[1 + \frac{\xi}{d}\right]$$

In static equilibrium, $\xi = 0$ and from (a)

$$qQ = 2\pi d\varepsilon_o \cdot mg \tag{b}$$

Part b

The perturbation equation of motion remains;

$$m\frac{\partial^2\xi}{\partial t^2} = f\frac{\partial^2\xi}{\partial x^2} + \left(\frac{qQ}{2\pi d^2\varepsilon_o}\right)\xi \tag{c}$$

Assume $e^{j(\omega t-kx)}$ dependence and (c) requires ($v_s = \sqrt{f/m}$)

$$\omega^2 = v_s^2k^2 - \frac{qQ}{2\pi d^2\varepsilon_o m} \tag{d}$$

or from (b),

x3k1k9

$$\omega^2 = v_s^2k^2 - \frac{g}{d}$$

The boundary conditions require $k = n\pi/\ell$, and for stability the most critical mode is $n = 1$; thus

$$v_s^2\left(\frac{\pi}{\ell}\right)^2 > \frac{g}{d} \tag{e}$$

$$m < \frac{fd}{g}\left(\frac{\pi}{\ell}\right)^2 \tag{f}$$

Part c

Increase f, d, or decrease ℓ.

PROBLEM 10.11

$$m\frac{\partial^2\xi}{\partial t^2} = f\frac{\partial^2\xi}{\partial x^2} + S - mg$$

where $S = (\bar{I}\times\bar{B})_{r=\xi_o}$ and $|B| = \frac{\mu_o I_o}{2\ r}$, r the radial distance from the fixed wire.

Therefore $S = \frac{\mu_o I_o I^o}{2\pi r}$.

For static equilibrium

PROBLEM 10.11 (continued)

$$S = mg = \frac{\mu_o I_o I}{2\pi\xi_o}$$

Therefore

$$I = \frac{2\pi mg\xi_o}{\mu_o I_o}$$

Note that $I_o I > 0$ for the required equilibrium.

Part b

The force per unit length is linearized to obtain the perturbation equation.

$$S = \frac{\mu_o I_o I}{2\pi(\xi_o+\xi)} \approx \frac{\mu_o I_o I}{2\pi}\left[\frac{1}{\xi_o} - \frac{\xi}{\xi_o^2}\right]$$

Therefore

$$m\frac{\partial^2\xi}{\partial t^2} = f\frac{\partial^2\xi}{\partial x^2} - \frac{\mu_o I_o I}{2\pi\xi_o^2}\xi$$

Part c

Assuming $e^{j(\omega t-kx)}$ solutions, the dispersion relation is

$$-m\omega^2 = -fk^2 - \frac{\mu_o I_o I}{2\pi\xi_o^2}$$

Solving for ω, we obtain

$$\omega = \left[k^2\frac{f}{m} + \frac{\mu_o I_o I}{2\pi m\xi_o^2}\right]^{1/2}$$

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As long as $I_o I > 0$ the equilibrium will always be stable as ω will always be real. Note that this condition is required for the desired static equilibrium to exist.

PROBLEM 10.12

The equation of motion is given as

$$m\frac{\partial^2\xi}{\partial t^2} = f\frac{\partial^2\xi}{\partial x^2} + P\xi \qquad \text{(a)}$$

Part a

Boundary conditions follow from force equilibrium for the ends of the wire

PROBLEM 10.12 (continued)

$$\text{(i)}\quad -2K\xi(0,t) + f\,\frac{\partial\xi(0,t)}{\partial x} = 0 \tag{b}$$

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$$\text{(ii)}\quad 2K\xi(\ell,t) + f\,\frac{\partial\xi(\ell,t)}{\partial x} = 0 \tag{c}$$

Part b

The dispersion relation follows from (a) as

$$\omega^2 = v_s^2\,k^2 - \frac{P}{m}\;;\quad v_s = \sqrt{f/m} \tag{d}$$

where solutions have been assumed of the following form:

$$\xi = \mathrm{Re}[(A\sin kx + B\cos kx)e^{j\omega t}] \tag{e}$$

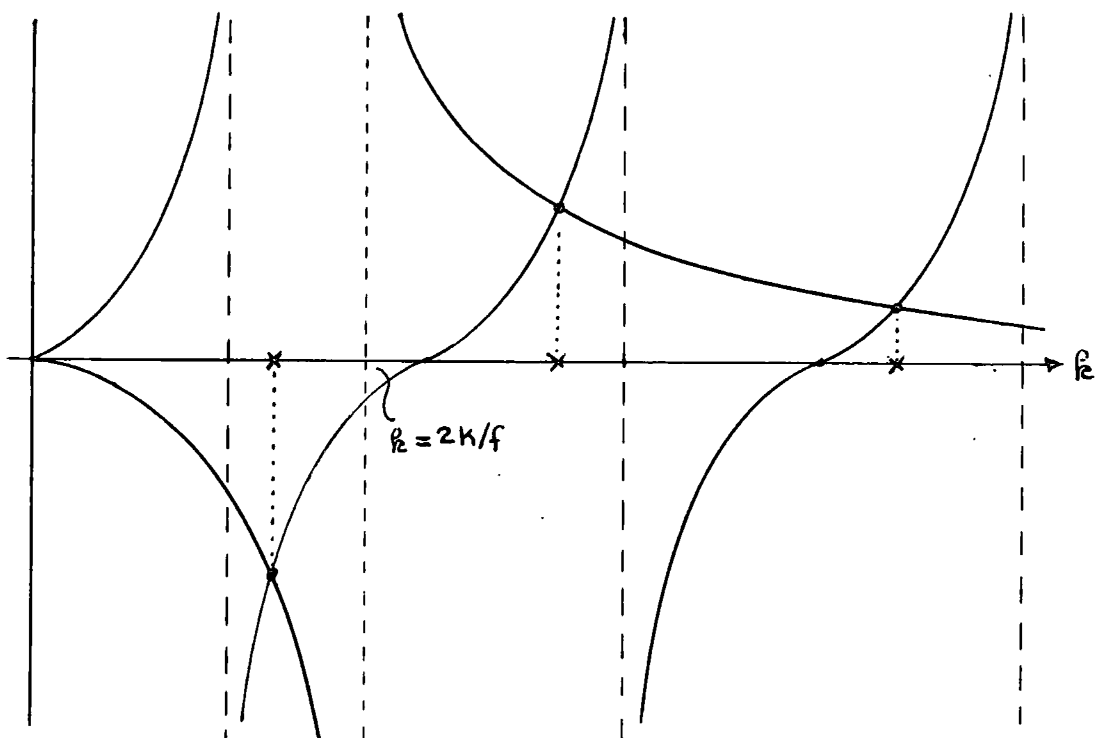
Application of the boundary conditions yields a transcendental equation for k:

$$\tan k\ell = \frac{4Kf\,k}{f^2k^2-4K^2} \tag{f}$$

where, from (d),

$$k = \frac{1}{v_s}\sqrt{\omega^2 + P/m} \tag{g}$$

Thus, (f) is the desired equation for the natural frequencies.



PROBLEM 10.12 (continued)

Part c

As $K \to 0$, the lowest root of graphic solution goes to $k \to 0$, for which stability criterion is:

$$0 > \frac{P}{m}$$

PROBLEM 10.13

Part a

This problem is very similar to that of problem 10.7. Using the same reasoning as in that problem, we obtain

$$\sigma_m \frac{\partial^2 \xi_1}{\partial t^2} = S \frac{\partial^2 \xi_1}{\partial x^2} + \frac{\varepsilon_o V_o^2}{d^3} (2\xi_1 - \xi_2)$$

$$\sigma_m \frac{\partial^2 \xi_2}{\partial t^2} = S \frac{\partial^2 \xi_2}{\partial x^2} + \frac{\varepsilon_o V_o^2}{d^3} (2\xi_2 - \xi_1)$$

Part b

Assuming sinusoidal solutions in time and space, the dispersion relation is

$$-\sigma_m \omega^2 + Sk^2 - \frac{2\varepsilon_o V_o^2}{d^3} = \pm \frac{\varepsilon_o V_o^2}{d^3}$$

We have a dispersion relation that factors into two parts. The odd mode, $\xi_1 = -\xi_2$ has the dispersion relation

$$\omega = \left[\frac{Sk^2}{\sigma_m} - \frac{3\varepsilon_o V_o^2}{\sigma_m d^3}\right]^{1/2}$$

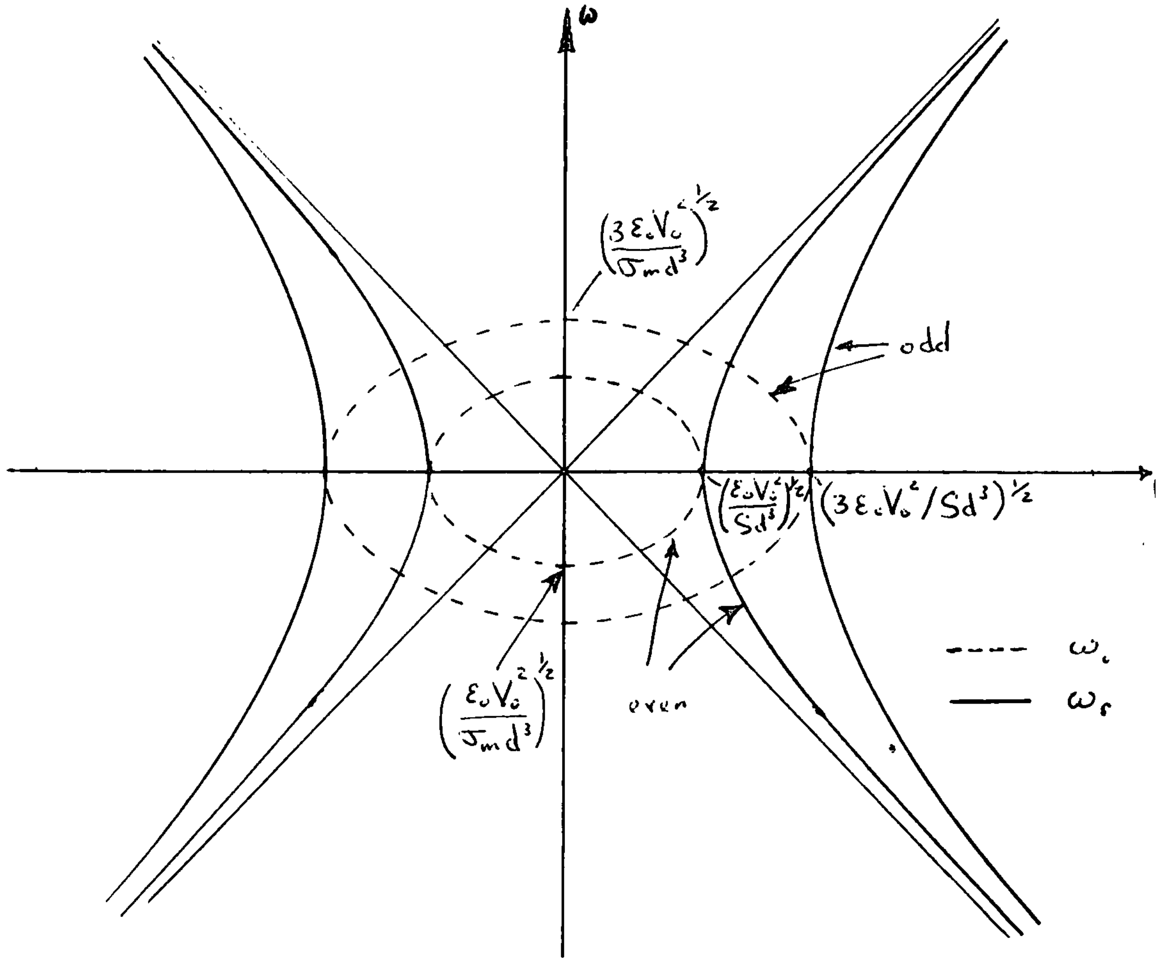
The even mode, $\xi_1 = \xi_2$ has the dispersion relation

$$\omega = \left[\frac{Sk^2}{\sigma_m} - \frac{\varepsilon_o V_o^2}{\sigma_m d^3}\right]^{1/2}$$

Part c

A plot of the dispersion relation appears below.

PROBLEM 10.13 (continued)



Part d

The lowest allowed value of k is $k = \frac{\pi}{L}$ since the membranes are fixed at $x = 0$ and $x = L$. Therefore the first mode to go unstable is the even mode. ODD

This happens as

$$\left(\frac{3\varepsilon_o V_o^2}{Sd^3}\right) = \frac{\pi^2}{L^2}$$

or

$$V_o = \left|\frac{\pi^2}{L^2}\ \frac{Sd^3}{\varepsilon_o 3}\right|^{1/2}$$

PROBLEM 10.14

The equation of motion is

$$\frac{\partial^2 \xi}{\partial t^2} = v_s^2 \frac{\partial^2 \xi}{\partial x^2} - \nu \frac{\partial \xi}{\partial t} \tag{a}$$

Part a

The dispersion for this system is:

$$\omega^2 - j\nu\omega - v_s^2 k^2 = 0 \tag{b}$$

We may solve for ω,

$$\omega = j\left[\frac{\nu}{2} \pm \sqrt{\left(\frac{\nu}{2}\right)^2 - v_s^2 k^2}\right] \equiv j[\alpha \pm \gamma] \tag{c}$$

We assume solutions of the form:

$$\xi(x,t)\ \mathrm{Re}\left\{\sum_{n\ \text{odd}} [A_n e^{-(\alpha+\gamma_n)t} + B_n e^{-(\alpha-\gamma_n)t}] \sin \frac{n\pi x}{\ell}\right\} \tag{d}$$

Now, we may use the initial condition on $\frac{\partial \xi}{\partial t}$ to relate A_n and B_n. Thus we obtain:

$$\xi(x,t) = \mathrm{Re}\left\{\sum_{n\ \text{odd}} A_n [e^{-\gamma_n t} - \left(\frac{\alpha+\gamma_n}{\alpha-\gamma_n}\right) e^{\gamma_n t}] e^{-\alpha t} \sin \frac{n\pi x}{\ell}\right\} \tag{e}$$

Now, we apply the initial condition on $\xi(x,t = 0)$ to determine A_n.

$$\xi(x,0) = \sum_{n\ \text{odd}} A_n \left[\frac{2\gamma_n}{\gamma_n - \alpha}\right] \sin \frac{n\pi x}{\ell} \equiv \sum_{n\ \text{odd}} A_n' \sin \frac{n\pi x}{\ell} \tag{f}$$

The coefficient A_n' is determined from a Fourier analysis of the displacement:

$$A_n' = \frac{4\xi_o}{n\pi}, \tag{g}$$

So that:

$$A_n = \left(\frac{\gamma_n - \alpha}{2\gamma_n}\right)\left(\frac{4\xi_o}{n\pi}\right) \tag{h}$$

Part b

There is one important difference between this problem and the magnetic diffusion problems of Chap. VII. While magnetic diffusion is "true diffusion" and satisfies the normal diffusion equation, the string equation is basically a wave equation modified by viscosity. Hence, we note (c) that especially the

PROBLEM 10.14 (continued)

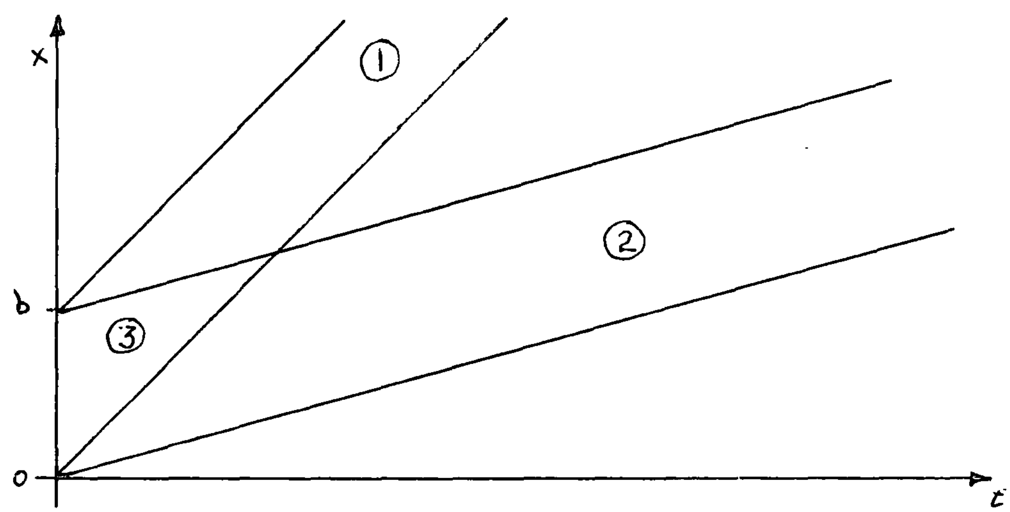
higher modes in the solution to this problem have sinusoidal time dependence as well as decay. Magnetic diffusion as discussed in Chap. 7 exhibits no such oscillation, because there is no mathematical analog to the inertia of the string. If we had included the effects of electromagnetic wave propagation (displacement current) the analogy would be more complete.

PROBLEM 10.15

From Chap. 10, page 588, Eqs. (e) and (f) we have

$$\frac{d\xi_+}{d\alpha} = \frac{(v_s - U)}{2v_s}\frac{\partial\xi}{\partial x} - \frac{1}{2v_s}\frac{\partial\xi}{\partial t}$$

$$\frac{d\xi_-}{d\beta} = \frac{(v_s + U)}{2v_s}\frac{\partial\xi}{\partial x} + \frac{1}{2v_s}\frac{\partial\xi}{\partial t}$$



Since $\frac{\partial\xi}{\partial x}$ $(x = 0) = 0$, we have the following relations in the three regions.

Region 1

$$\frac{d\xi_+}{d\alpha} = -\frac{V_o}{2v_s}\ ;\ \frac{d\xi_-}{d\beta} = 0$$

Region 2

$$\frac{d\xi_+}{d\alpha} = 0;\ \frac{d\xi_-}{d\beta} = \frac{V_o}{2v_s}$$

Region 3

$$\frac{d\xi_+}{d\alpha} = -\frac{V_o}{2v_s}\ ;\ \frac{d\xi_-}{d\beta} = \frac{V_o}{2v_s}$$

PROBLEM 10.15 (continued)

In the other regions, the derivatives are zero. From Eq. 10.2.10 on page 586,

$$\frac{\partial \xi}{\partial x} = \frac{d\xi_+}{d\alpha} + \frac{d\xi_-}{d\beta}$$

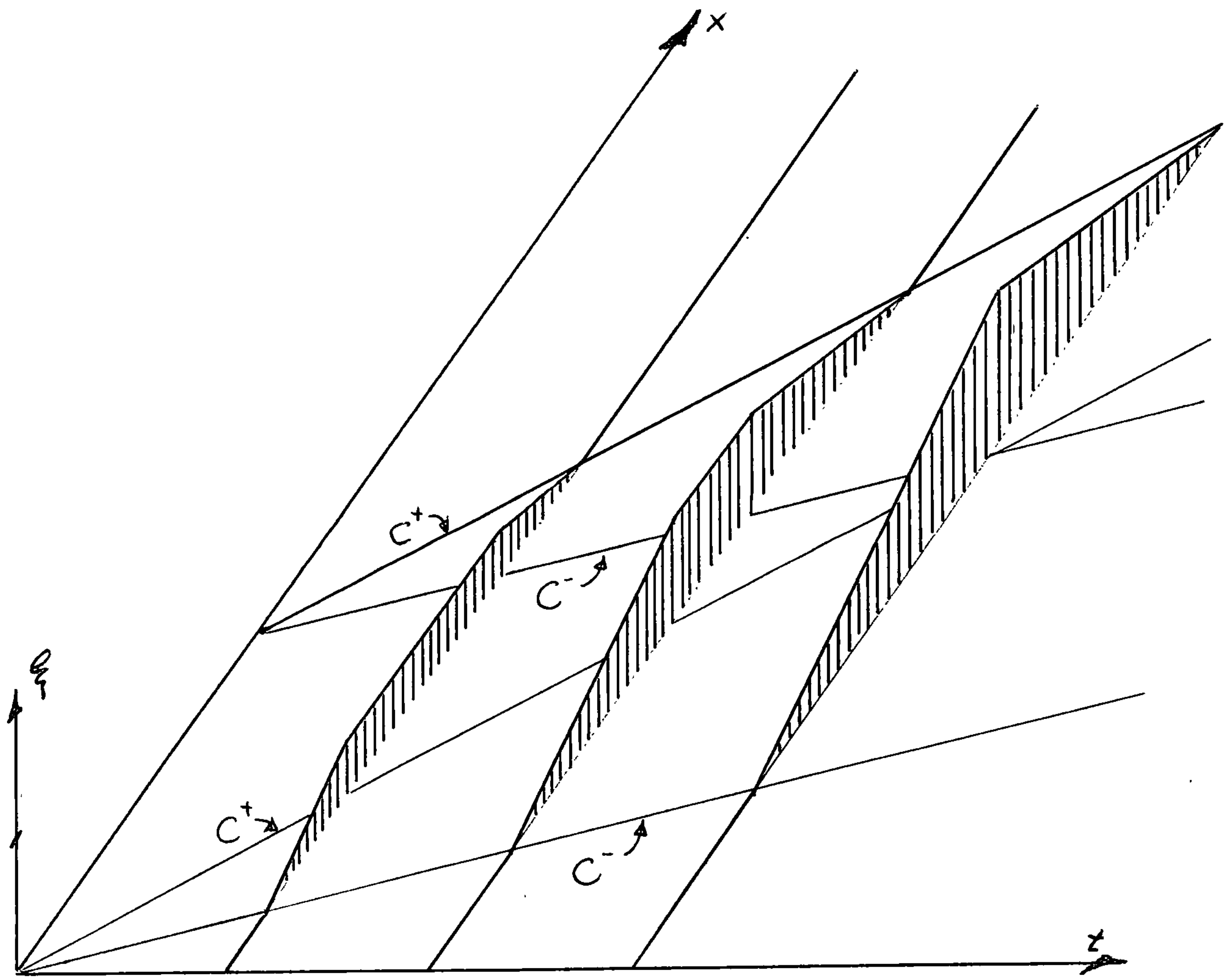
we have

$$\frac{\partial \xi}{\partial x} = \frac{V_o}{2v_s} [u_{-1}(\beta)-u_{-1}(\beta-b) - u_{-1}(\alpha) + u_{-1}(\alpha-b)]$$

Integrating with respect to x, we obtain

$$\xi(x,t) = \frac{V_o}{2v_s} [u_{-2}(\beta) - u_{-2}(\beta-b) - u_{-2}(\alpha) + u_{-2}(\alpha-b)]$$

A sketch of this deflection is shown in the figure.



PROBLEM 10.16

Part a

The equation of motion is simply

$$m\frac{\partial^2\xi}{\partial t^2} = f\frac{\partial^2\xi}{\partial x^2}$$

The dispersion equation follows as:

$$(\omega-kU)^2 = v_s^2k^2$$

or

$$k_{\pm} = \frac{\omega}{U\pm v_s} = \frac{\omega(U\mp v_s)}{U^2-v_s^2} \equiv \alpha \mp \beta$$

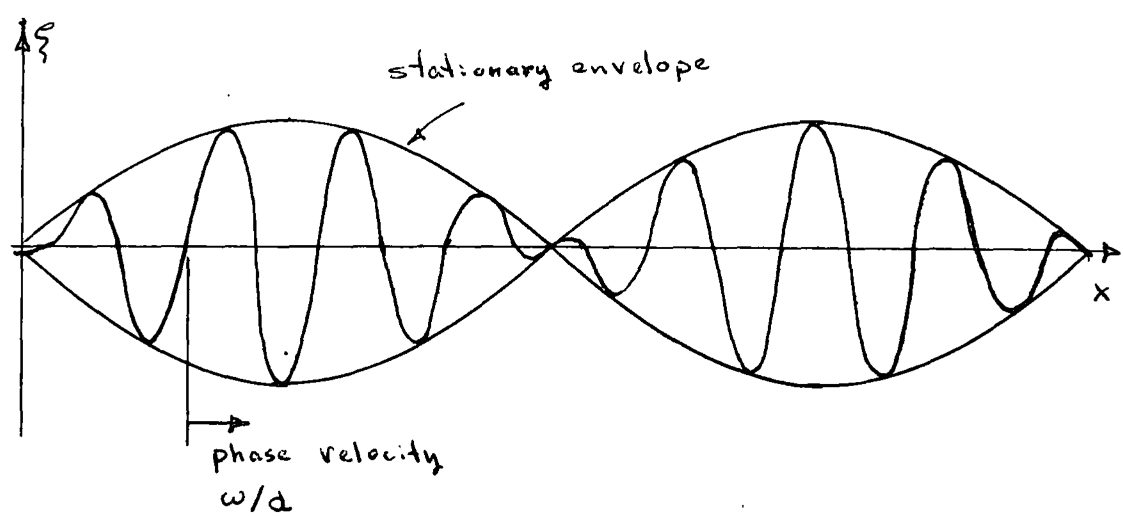
Where solutions are assumed of the form:

$$\xi(x,t) = \mathrm{Re}\{(\xi_+e^{j\beta x} + \xi_-e^{-j\beta x})e^{j(\omega t-\alpha x)}\}$$

The boundary conditions are both applied at x = 0, because string is moving at a "supersonic" velocity.

$$\xi(x,t) = \xi_o\{\cos\beta x\cos[\omega t-\alpha x] - \frac{U}{v_s}\sin\beta x\sin[\omega t-\alpha x]\}$$

Part b



PROBLEM 10.17

We use Eq. 10.2.9

$$\left(\frac{\partial}{\partial t} + U\,\frac{\partial}{\partial x}\right)^2 \xi = v_s^2\,\frac{\partial^2\xi}{\partial x^2}$$

Assuming sinusoidal solutions in time and space we obtain the dispersion relation

$$(\omega - kU)^2 = k^2 v_s^2$$

Thus

$$k = \frac{\omega}{U \pm v_s} = \frac{\omega(U \mp v_s)}{U^2 - v_s^2}$$

We let

$$\alpha = \frac{\omega U}{U^2 - v_s^2}$$

$$\beta = \frac{\omega v_s}{U^2 - v_s^2}$$

Therefore, $k = \alpha \mp \beta$ and

$$\xi(x,t) = \text{Re}[A\,e^{-j(\alpha-\beta)x} + B\,e^{-j(\alpha+\beta)x}]e^{j\omega t}$$

The boundary conditions are

$\xi(x = 0) = 0$ which implies $A = -B$

$\xi(x = -\ell) = \xi_o$

Therefore

$$\xi(x,t) = \text{Re}\,A[e^{-j(\alpha-\beta)x} - e^{-j(\alpha+\beta)x}]e^{j\omega t}$$

$$= \text{Re}\,A2j\,\sin\beta x\;e^{j(\omega t-\alpha x)}$$

However,

$$\xi(-\ell,t) = \text{Re}\,\xi_o\,e^{j\omega t}$$

Therefore

$$\xi(x,t) = -\frac{\xi_o}{\sin\beta\ell}\,\sin\beta x\,\cos[\omega t - \alpha(x+\ell)]$$

Part b

For $\xi = 0$ at $x = 0$ and at $x = -\ell$ we must have $\beta = n\pi/\ell$

$$\frac{\omega v_s}{U^2 - v_s^2} = \frac{n\pi}{\ell}$$

or

PROBLEM 10.17 (continued)

$$\omega = \frac{n\pi}{\ell}\,\frac{(U^2 - v_s^2)}{v_s}$$

These are the natural frequencies of the wire.

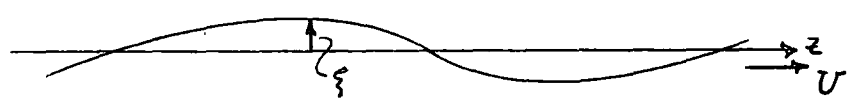
Part c

The results are meaningful only for $|U| < |v_s|$. If this inequality were not true, we would not be able to use a downstream boundary condition to determine upstream behavior and arrive at a result that would be obtained by "turning the driv on". That is, if $U > v_s$ the predictions are not consistent with causality.

PROBLEM 10.18

Part a

In the limit of wavelength short compared to the radius, we may "unwrap" the system:



$$m\left(\frac{\partial}{\partial t} + U\,\frac{\partial}{\partial z}\right)^2 \xi = f\,\frac{\partial^2\xi}{\partial z^2} \tag{a}$$

Now let $z \to R\theta$, $U \to R\Omega$. Then, it follows that

$$m\left(\frac{\partial}{\partial t} + \Omega\,\frac{\partial}{\partial \theta}\right)^2 \xi = \frac{f}{R^2}\,\frac{\partial^2\xi}{\partial \theta^2} \tag{b}$$

where $\Omega_s = \sqrt{f/(m\,R^2)}$

Part b

The initial conditions are

$$\partial\xi/\partial t(\theta, t = 0) = 0 \tag{c}$$

$$\xi(\theta, t = 0) = \begin{cases} \xi_o & 0 \le \theta \le \pi/4 \\ 0, & \text{elsewhere} \end{cases} \tag{d}$$

Solutions take the form

$$\xi = \xi_+(\alpha) + \xi_-(\beta) \tag{e}$$

where

PROBLEM 10.18 (continued)

$$\alpha = \theta - \Omega_s t$$
$$\beta = \theta - 3\Omega_s t$$

Because $\partial\xi/\partial t\ (t = 0) = 0$,

$$= -\Omega_s \frac{d\xi_+}{d\alpha} - 3\Omega_s \frac{d\xi_-}{d\beta} \tag{f}$$

Also,

$$\frac{\partial\xi}{\partial\theta}\ (t=0) = \frac{d\xi_+}{d\alpha} + \frac{d\xi_-}{d\beta} = \xi_o[u_o(0) - u_o(\pi/4)] \tag{g}$$

Thus, from (f) and (g)

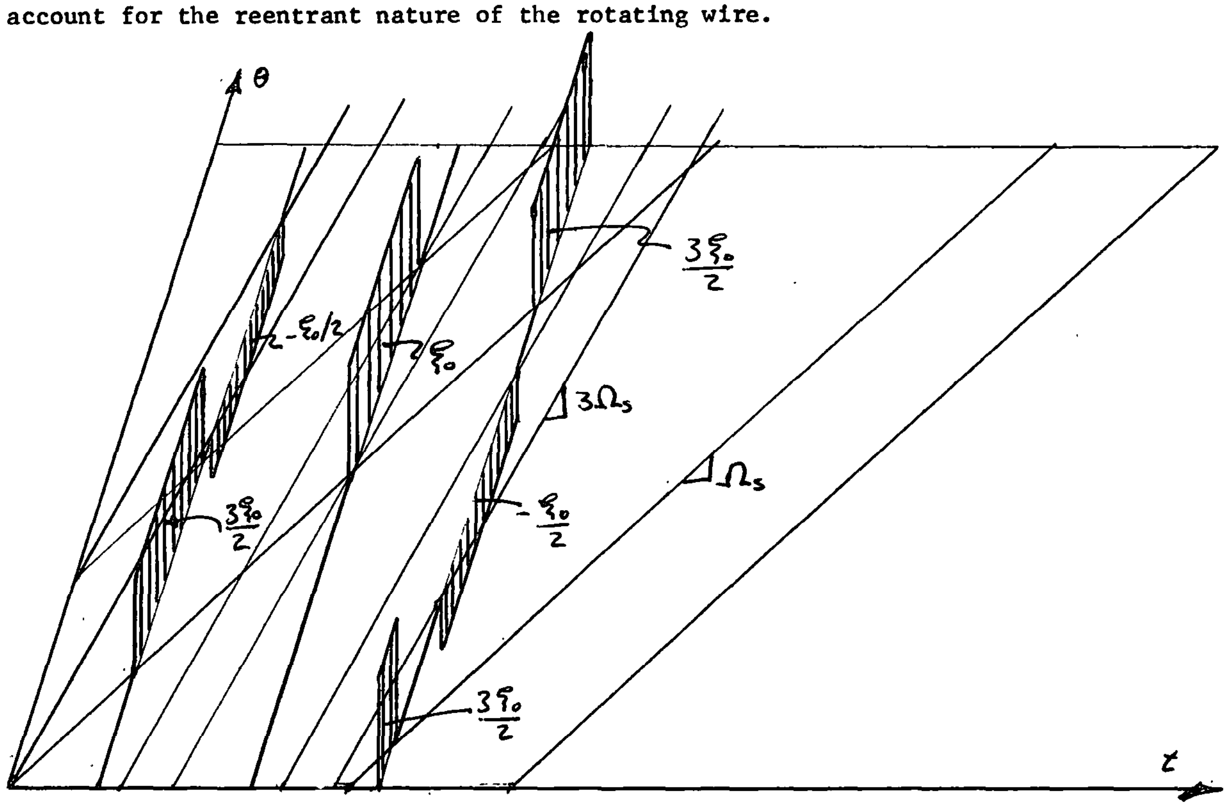
$$\frac{d\xi_+}{d\alpha} = \frac{3}{2}\,\xi_o[u_o(0) - u_o(\pi/4)];\ \text{on } \alpha \tag{h}$$

$$\frac{d\xi_-}{d\beta} = -\frac{1}{2}\,\xi_o[u_o(0) - u_o(\pi/4)];\ \text{on } \beta$$

The solution in the θ-t plane follows from

$$\frac{\partial\xi}{\partial\theta} = \frac{d\xi_+}{d\alpha} + \frac{d\xi_-}{d\beta} \tag{j}$$

and an integration at constant t on θ. The result is shown in the figure. Note that the characteristics that leave the interval $0 < \theta < 2\pi$ at $\theta = 2\pi$ reappear at $\theta = 0$ to account for the reentrant nature of the rotating wire.



PROBLEM 10.19

In the moving frame we can write

$$m \frac{\partial^2 \xi}{\partial t'^2} = f \frac{\partial^2 \xi}{\partial x'^2} + F(x',t') \tag{a}$$

and so from Prob. 10.4, we can write

$$P'_{in} = \frac{\partial W'}{\partial t'} + \frac{\partial P'}{\partial x'} \tag{b}$$

where

$$P'_{in} = F \frac{\partial \xi}{\partial t'} \tag{c}$$

$$W' = \frac{1}{2} m \left(\frac{\partial \xi}{\partial t'}\right)^2 + \frac{1}{2} f \left(\frac{\partial \xi}{\partial x'}\right)^2 \tag{d}$$

$$P' = - f \frac{\partial \xi}{\partial x'} \frac{\partial \xi}{\partial t'} \tag{e}$$

But $\frac{\partial}{\partial x'} = \frac{\partial}{\partial x}$ and $\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + U \frac{\partial}{\partial x}$

Therefore (c)-(e) become

$$P'_{in} = F\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right)\xi \tag{f}$$

$$W' = \frac{1}{2} m\left(\frac{\partial \xi}{\partial t} + U \frac{\partial \xi}{\partial x}\right)^2 + \frac{1}{2} f\left(\frac{\partial \xi}{\partial x}\right)^2 \tag{g}$$

$$P' = - f \frac{\partial \xi}{\partial x} \left(\frac{\partial \xi}{\partial t} + U \frac{\partial \xi}{\partial x}\right) \tag{h}$$

The conservation of energy equation, in terms of fixed frame coordinates, becomes

$$P'_{in} = \frac{\partial W'}{\partial t} + U \frac{\partial W'}{\partial x} + \frac{\partial P'}{\partial x} \tag{i}$$

$$= \frac{\partial W'}{\partial t} + \frac{\partial}{\partial x} (P' + W'U) \tag{j}$$

If we let

$$P_{in} = P'_{in}$$

$$W = W' \tag{k}$$

$$P = P' + W'U$$

we can write

$$P_{in} = \frac{\partial W}{\partial t} + \frac{\partial P}{\partial x} \tag{ℓ}$$

which is the required form.

PROBLEM 10.20

The equation of motion is given by Eq. 10.2.33, and hence the dispersion equation is 10.2.36;

$$k = \eta \pm j\gamma \tag{a}$$

where

$$\eta = \omega_d U/(U^2-v_s^2)$$

$$\gamma = v_s \sqrt{(U^2-v_s^2)k_c^2-\omega_d^2}\ /(U^2-v_s^2)$$

Solutions are assumed of the form

$$\xi = \text{Re}[A \sinh \gamma x + B \cosh \gamma x]e^{j(\omega t-\eta x)} \tag{b}$$

Boundary conditions require;

$$B = \xi_o \tag{c}$$

$$A = j\eta\xi_o/\gamma \tag{d}$$

Thus

$$\xi = \text{Re}\ \xi_o[\frac{j\eta}{\gamma} \sinh \gamma x + \cosh \gamma x]e^{j(\omega t-\eta x)} \tag{e}$$

The deflection has an envelope with an essentially exponentially increasing dependence on x, with the instantaneous deflection traveling in the + x direction.

PROBLEM 10.21 "m" SHOULD BE "σ_m" THROUGHOUT PROBLEM

Part a

The equation of motion is

$$\sigma_m \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right)^2 \xi = S \frac{\partial^2\xi}{\partial x^2} - mg + T \tag{a}$$

with $T = \dfrac{\varepsilon_o}{2} \dfrac{V_o^2}{(d-\xi)^2} \simeq \dfrac{\varepsilon_o}{2} V_o^2 \left[\dfrac{1}{d^2} + \dfrac{2\xi}{d^3}\right]$

For equilibrium, $\xi = 0$ and from (a)

$$\frac{\varepsilon_o V_o^2}{2d^2} = mg \tag{b}$$

or

$$V_o = \left[\frac{2mg\ d^2}{\varepsilon_o}\right]^{1/2} \tag{c}$$

PROBLEM 10.21 (continued)

Part b

With solutions of the form $e^{j(\omega t-kx)}$ the dispersion relation is

$$(\omega-kU)^2 = \frac{S}{\sigma_m} k^2 - \frac{\varepsilon_o V_o^2}{\sigma_m d^3} \tag{d}$$

Solving for k, we obtain

$$k = \frac{\omega U \pm \sqrt{\frac{S}{\sigma_m}\omega^2 - \left(U^2 - \frac{S}{\sigma_m}\right)\left(\frac{\varepsilon_o V_o^2}{\sigma_m d^3}\right)}}{(U^2 - \frac{S}{\sigma_m})} \tag{e}$$

For $U > \sqrt{S/\sigma_m}$, and not to have spatially growing waves

$$\frac{S}{\sigma_m}\omega^2 - (U^2 - \frac{S}{\sigma_m})\left(\frac{\varepsilon_o V_o^2}{\sigma_m d^3}\right) > 0 \tag{f}$$

or

$$\omega^2 > \left[(U^2 - \frac{S}{\sigma_m})\frac{\varepsilon_o V_o^2}{Sd^3}\right] \tag{g}$$

POOBLEM 10.22

Part a

Neglecting the curvature of the system, as in Prob. 10.18, we write:

$$\sigma_m \left(\frac{\partial}{\partial t} + \frac{\Omega R}{R}\frac{\partial}{\partial\theta}\right)^2 \xi = \frac{S}{R^2}\cdot\frac{\partial^2\xi}{\partial\theta^2} + T_r \tag{a}$$

where the linearized perturbation force/unit area is

$$T_r = \left(\frac{2\varepsilon_o V_o^2}{a^3}\right)\xi \tag{b}$$

Therefore, the equation of motion is

$$\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\theta}\right)^2 \xi = \Omega_s^2\left(\frac{\partial^2\xi}{\partial\theta^2} + m_c^2\,\xi\right) \tag{c}$$

$$\Omega_s^2 = \frac{S}{\sigma_m R^2}$$

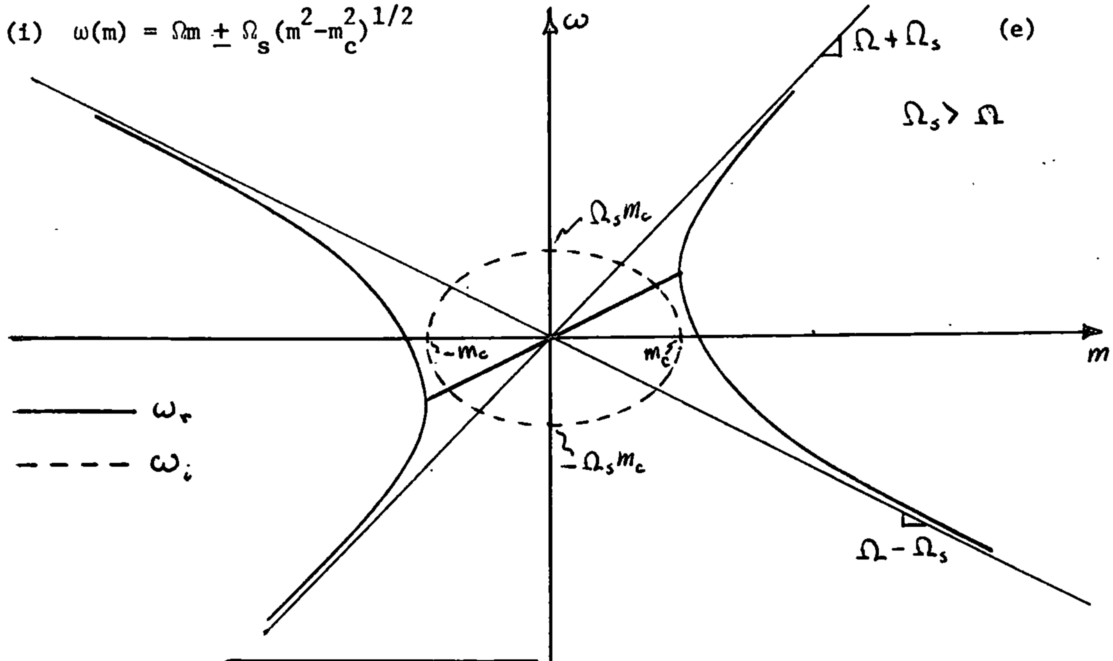
$$m_c^2 = \frac{2\varepsilon_o V_o^2}{a^3}\;\frac{R^2}{S}$$

PROBLEM 10.22

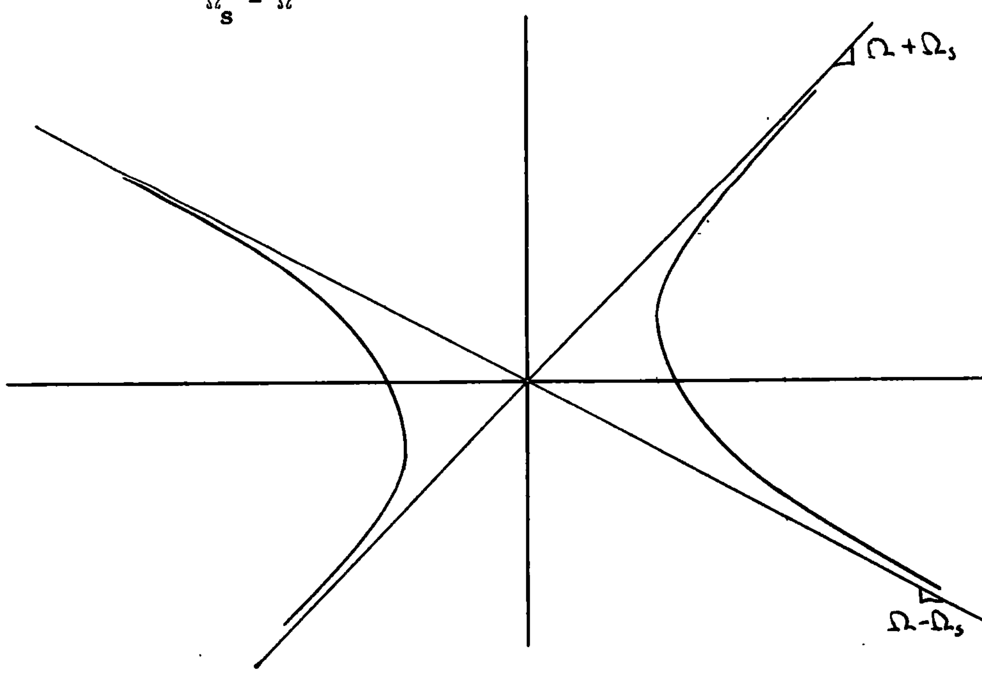
Part b

$$(\omega - m\Omega)^2 = \Omega_s^2(m^2 - m_c^2) \qquad \text{(d)}$$

(i) $\omega(m) = \Omega m \pm \Omega_s(m^2 - m_c^2)^{1/2}$ (e)



(ii) $$m = \frac{\Omega\omega \pm \sqrt{\Omega_s^2\omega^2 + (\Omega_s^2 - \Omega^2)(\Omega_s^2 m_c^2)}}{\Omega_s^2 - \Omega^2} \qquad \text{(f)}$$



PROBLEM 10.22

Part c

Because the membrane closes on itself it can be absolutely unstable regardless of Ω relative to Ω_s. Allowed values of m are determined by the requirement that the deflections be periodic in θ; m = 0, 1,2,3,... Thus, from (e) any finite m_c will lead to instability in the m = 0 mode. Note however that this mode does not meet the requirement that wavelengths be short compared to R.

PROBLEM 10.23

We may take the results of Prob. 10.13, replacing $\frac{\partial}{\partial t}$ by $\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}$ and replacing ω by ω-kU.

Part a

The equations of motion are

$$\sigma_m\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \xi_1 = S\frac{\partial^2 \xi_1}{\partial x^2} + \frac{\varepsilon_o V_o^2}{d^3}(2\xi_1 - \xi_2) \tag{a}$$

and

$$\sigma_m\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\xi_2 = S\frac{\partial^2 \xi_2}{\partial x^2} + \frac{\varepsilon_o V_o^2}{d^3}(2\xi_2 - \xi_1) \tag{b}$$

Part b

The dispersion relation is biquadratic, and factors into

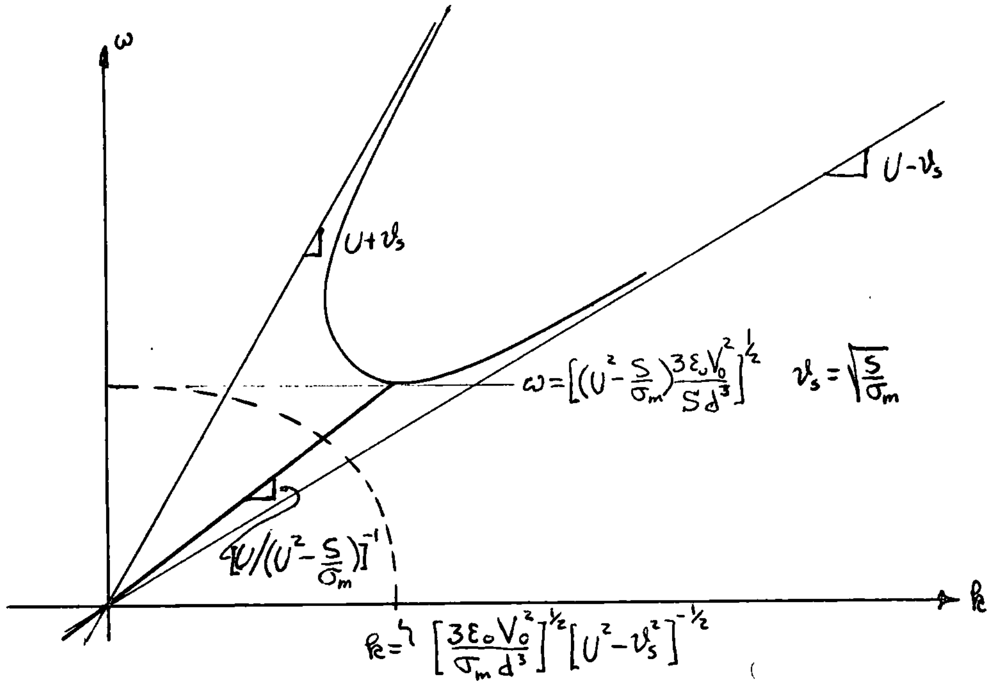
$$-\sigma_m(\omega-kU)^2 + Sk^2 - \frac{2\varepsilon_o V_o^2}{d^3} = \pm\frac{\varepsilon_o V_o^2}{d^3} \tag{c}$$

The ($\pm$) signs correspond to the cases $\xi_1 = -\xi_2$ and $\xi_1 = \xi_2$ respectively, as will be seen in part (d).

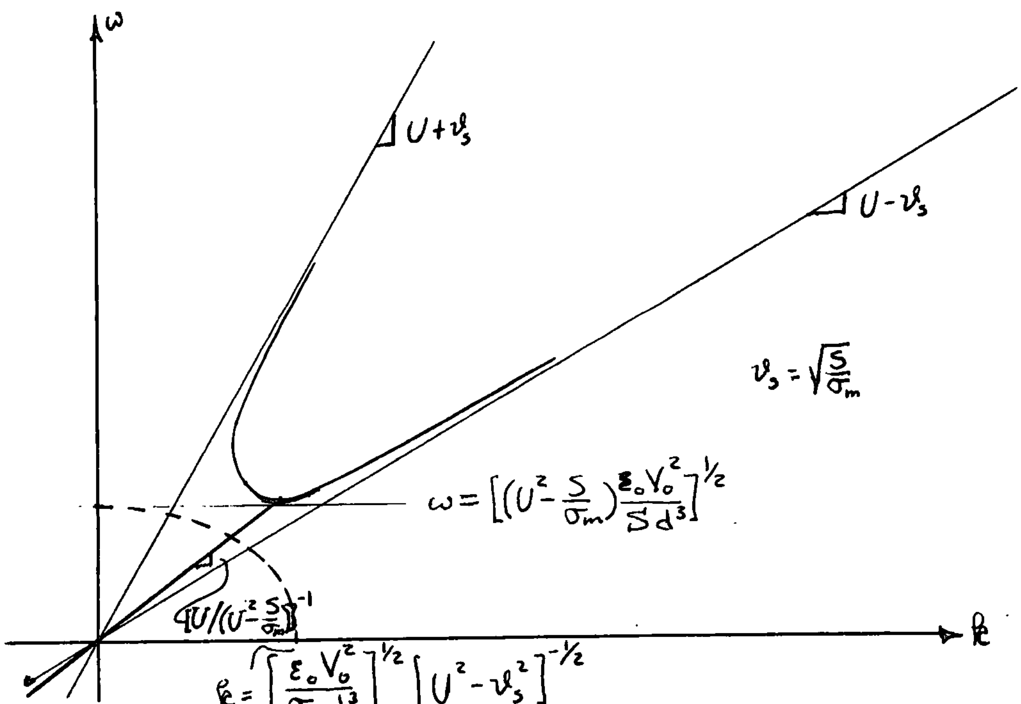
Part c

The dispersion relations are plotted in the figure for $U > \sqrt{S/\sigma_m}$.

PROBLEM 10.23 (continued)



and



PROBLEM 10.23 (continued)

Part d

Let $\xi_1 = \xi_2$. Then (a) and (b) become

$$\sigma_m\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \xi_1 = S\frac{\partial^2 \xi_1}{\partial x^2} + \frac{\varepsilon_o V_o^2}{d^3}\xi_1 \tag{d}$$

and

$$\sigma_m\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \xi_2 = S\frac{\partial^2 \xi_2}{\partial x^2} + \frac{\varepsilon_o V_o^2}{d^3}\xi_2 \tag{e}$$

These equations are identical for $\xi_1 = \xi_2$; the dispersion equation is (c) with the minus sign. Now let $\xi_1 = -\xi_2$ and (a) and (b) require

$$\sigma_m\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \xi_1 = S\frac{\partial^2 \xi_1}{\partial x^2} + \frac{3\varepsilon_o V_o^2}{d^3}\xi_1 \tag{f}$$

and

$$\sigma_m\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \xi_2 = S\frac{\partial^2 \xi_2}{\partial x^2} + \frac{3\varepsilon_o V_o^2}{d^3}\xi_2 \tag{g}$$

These equations are identical for $\xi_1 = -\xi_2$; the dispersion equation is (c) with the + sign.

Part e

$$\xi_1(0,t) = \mathrm{Re}\,\hat{\xi}\,e^{j\omega t} = -\xi_2(0,t) \tag{h}$$

$$\frac{\partial \xi_1}{\partial x} = \frac{\partial \xi_2}{\partial x} = 0 \text{ at } x = 0 \tag{i}$$

The odd mode is excited. Hence, we use the + sign in (c)

$$-\sigma_m(\omega - kU)^2 + Sk^2 - \frac{3\varepsilon_o V_o^2}{d^3} = 0 \tag{j}$$

$$k^2(S - \sigma_m U^2) + 2\sigma_m \omega k U - \sigma_m \omega^2 - \frac{3\varepsilon_o V_o^2}{d^3} = 0 \tag{k}$$

Solving for k, we obtain

$$k_{\pm} = \alpha \pm \beta \tag{ℓ}$$

where $\alpha = \dfrac{\omega U}{U^2 - v_s^2}$ (m)

$$\beta = \frac{\left[\omega^2 v_s^2 - \dfrac{3\varepsilon_o V_o^2 (U^2 - v_s^2)}{\sigma_m d^3}\right]^{1/2}}{U^2 - v_s^2} \tag{n}$$

PROBLEM 10.23 (continued)

with $v_s^2 = S/\sigma_m$.

Therefore

$$\xi_1 = \text{Re}\left\{[A\ e^{-j(\alpha+\beta)x} + B\ e^{-j(\alpha-\beta)x}]e^{j\omega t}\right\} \qquad (o)$$

Applying the boundary conditions, we obtain

$$A = \hat{\xi}\ \frac{(\beta-\alpha)}{2\beta} \qquad (p)$$

$$B = \frac{(\alpha+\beta)\hat{\xi}}{2\beta} \qquad (q)$$

Therefore, if $\hat{\xi}$ is real

$$\xi_1(x,t) = -\ \xi_2(x,t) = \hat{\xi}\ \cos\ \beta x\ \cos(\omega t-\alpha x) - \frac{\alpha}{\beta}\ \hat{\xi}\ \sin\ \beta x\ \sin(\omega t-\alpha x) \qquad (r)$$

Part f

We can see that β can be imaginary, for which we will have spatially growing curves. This can happen when

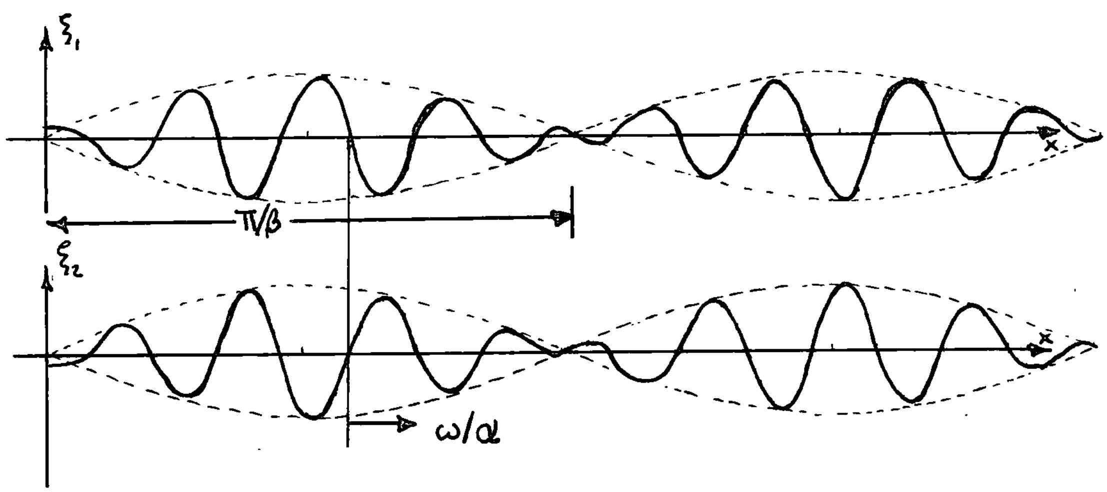
$$\omega^2 v_s^2 - \frac{3\varepsilon_o V_o^2}{\sigma_m d^3}(U^2 - v_s^2) < 0 \qquad (s)$$

or

$$V_o^2 > \frac{\sigma_m d^3\ \omega^2\ v_s^2}{3\varepsilon_o(U^2 - v_s^2)} \qquad (t)$$

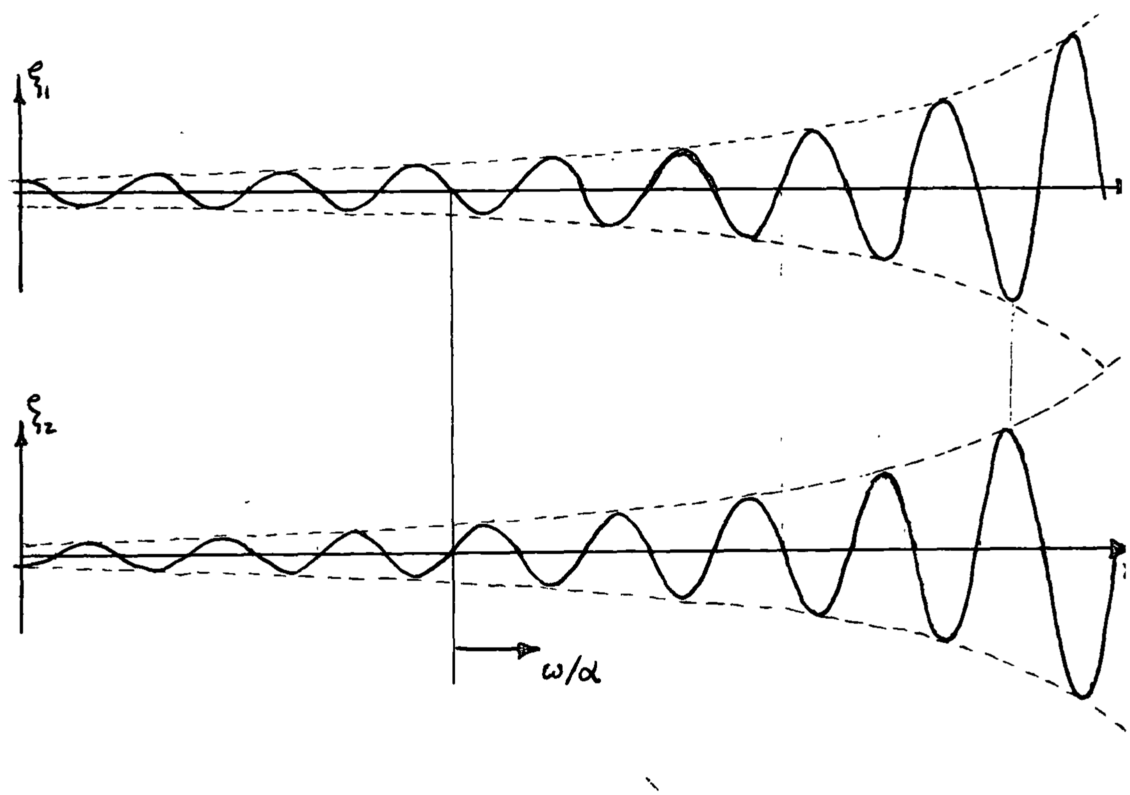
Part g

With $V_o = 0$ and $v > v_s$;



PROBLEM 10.23 (continued)

Amplifying waves are obtained as (t) is satisfied;



PROBLEM 10.24

Part a

The equation of motion for the membrane is:

$$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S \left[\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2}\right] + T_z \tag{a}$$

where

$$T_z = T_{zz} = 2\varepsilon_o V_o^2 \, \xi/s^3 \tag{b}$$

The equation may be rewritten as follows:

PROBLEM 10.24 (continued)

$$\frac{\partial^2\xi}{\partial t^2} = v_s^2\left[\frac{\partial^2\xi}{\partial x^2} + \frac{\partial^2\xi}{\partial y^2} + k_c^2\ \xi\right] \tag{c}$$

where

$$k_c^2 \equiv \frac{2\varepsilon_o V_o^2}{S\ s^3}$$

Assume solutions of the following form:

$$\xi(x,y,t) = \text{Re}[\hat{\xi}\ e^{j(\omega t - k_x x - k_y y)}] \tag{d}$$

The dispersion is:

$$\omega^2 = v_s^2[k_x^2 + k_y^2 - k_c^2] \tag{e}$$

The mode which goes unstable first is the lowest spatial mode:

$$k_x = \frac{\pi}{a}, \quad k_y = \frac{\pi}{b} \tag{f}$$

Instability occurs at

$$k_c^2 = (\frac{\pi}{a})^2 + (\frac{\pi}{b})^2 \tag{g}$$

or,

$$V_o = \{\frac{S\ s^3}{2\varepsilon_o}\ [(\frac{\pi}{a})^2 + (\frac{\pi}{b})^2]\}^{1/2} \tag{h}$$

Part b

The natural frequencies follow from Eq. (e) as

$$\omega_{mn} = v_s[(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2 - k_c^2]^{1/2} \tag{i}$$

Part c

We superimpose eigensolutions to obtain the membrane motion for $t > 0$. The solution that already satisfies the initial condition on velocity is

$$\xi(x,y,t) = \sum_m \sum_n \xi_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \cos\omega_{mn}t \tag{j}$$

where m and n are odd only, since the initial condition on $\xi(x,y,t=0)$ requires no even modes. Now use the principle of orthogonality of modes. Multiply (j) by $\sin(p\pi x/a)\ \sin(q\pi y/b)$ and integrate over the area of the membrane. The left hand side becomes

PROBLEM 10.24 (continued

$$\int_o^b \int_o^a \left[\xi(x,y,t=0)\sin\frac{p\pi x}{a}\sin\frac{q\pi y}{b}\right] dx\,dy \tag{k}$$

$$= \int_o^b \int_o^a J_o u_o\,(x-\frac{a}{2})u_o(y-\frac{b}{2})\sin\frac{p\pi x}{a}\sin\frac{q\pi y}{b}\,dx\,dy$$

Thus, (j) reducesto

$$J_o = \xi_{pq}\,(\frac{a}{2})(\frac{b}{2}) \tag{ℓ}$$

which makes it possible to evaluate the Fourier amplitudes

$$\xi_{mn} = \frac{4J_o}{ab} \tag{m}$$

The desired response is (j) with ξ_{mn} given by (m).

$$\xi(x,y,t) = \sum_{\substack{m\ n\\(odd)}}\sum (\frac{4J_o}{ab})\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\omega_{mn}t$$

Note that the analysis is valid even if the lowest mode(s) is (are) unstable, for which case:

$$\cos\omega_{pq}t \rightarrow \cosh\alpha_{pq}t$$

PROBLEM 10.25

The equation of motion is (see Table 9.2, page 535):

$$\sigma_m\frac{\partial^2\xi}{\partial t^2} = S\left(\frac{\partial^2\xi}{\partial x^2}+\frac{\partial^2\xi}{\partial y^2}\right) \tag{a}$$

With solutions of the form $\xi = \text{Re}\,\hat{\xi}\,e^{j(\omega t-k_x x-k_y y)}$, the dispersion equation is

$$\omega = \pm v_s\sqrt{k_x^2+k_y^2} \tag{b}$$

A particular superposition of these solutions that satisfies the boundary conditions along three of the four edges is

$$\xi = A\sin\frac{n\pi y}{a}\sin k_x(x-b)\cos\omega_o t \tag{c}$$

where in view of (b),

$$\omega_o^2 = v_s^2\,[k_x^2 + (\frac{n\pi}{a})^2] \tag{d}$$

Thus, there is a solution for each value of n, and

PROBLEM 10.25 (continued)

$$\xi = \sum_{n=1}^{\infty} A_n \sin k_n(x-b)\cos \omega_o t \sin \frac{n\pi y}{a} \tag{e}$$

where, from (d)

$$k_n = \left[\left(\frac{\omega_o}{v_s}\right)^2 - \left(\frac{n\pi}{a}\right)^2\right]^{1/2} \tag{f}$$

At x = 0, (e) takes the form of a Fourier series

$$\xi(y=0) = \sum_{n=1}^{\infty} -A_n \sin k_n b \cos \omega_o t \sin \frac{n\pi y}{a} \tag{g}$$

This function of (y,t) has the correct dependence on t. The dependence on y is made that of Fig. 10P.25 by adjusting the coefficients A_n as is usual in a Fourier series. Note that because of the symmetry of the excitation about y = a/2, only odd values of n give finite A_n. Thus

$$\int_o^{a/2} \frac{2\xi_o}{a} y \sin \frac{m\pi}{a} y\, dy + \int_{a/2}^{a} \frac{2\xi_o}{a}(a-y)\sin \frac{m\pi}{a} y\, dy \tag{h}$$

$$= -\int_o^a A_n \sin k_n b \sin \frac{n\pi y}{a} \sin \frac{m\pi y}{a}\, dy$$

Evaluation of the integrals gives

$$\frac{4\xi_o a}{(m\pi)^2} \sin \left(\frac{m\pi}{2}\right) = - \frac{A_n a \sin k_n b}{2} \tag{i}$$

Hence, the required function is (e) with k_n given by (f) and A_n given by solving (i)

$$A_n = - \frac{8\xi_o}{(m\pi)^2} \sin \left(\frac{m\pi}{2}\right)/\sin k_n b \tag{j}$$

PROBLEM 10.26

The force per unit length is $\mu_o \bar{I} \times \bar{H}$, where $\bar{H}$ is the magnetic field intensity evaluated at the position of the wire. That is,

$$\bar{S} = \mu_o I[H_x \bar{i}_y - H_y \bar{i}_x] \tag{a}$$

To evaluate H_x and H_y at $u\bar{i}_x + v\bar{i}_y$ note that $\bar{H}(0,0) = 0$. By symmetry $H_y(0,y) = 0$ and therefore $\partial H_y/\partial y\ (0,0) = 0$. Then, $\nabla\cdot\bar{B} = 0$ requires that $\partial H_x/\partial x(0,0) = 0$. Thus, an expansion of (a) about the origin gives

PROBLEM 10.26 (continued)

$$\bar{S} \cong \mu_o I \left[\frac{\partial H_x}{\partial y} v \bar{i}_y - \frac{\partial H_y}{\partial x} u \bar{i}_x\right] \tag{b}$$

Note that because $\nabla \times \bar{H} = 0$ at the origin, $\partial H_x/\partial y = \partial H_y/\partial x$. Thus, (b) becomes

$$\bar{S} = \mu_o I \frac{\partial H_y}{\partial x} [-u \bar{i}_x + v \bar{i}_y] \tag{c}$$

and $\partial H_y/\partial x$ (0,0) is easily computed because

$$H_y(x,0) = \frac{I_o}{2\pi}\left[\frac{1}{a-x} - \frac{1}{a+x}\right] \cong \frac{I_o}{2\pi}\left[\frac{2x}{a^2}\right] \tag{d}$$

Thus,

$$\bar{S} = \frac{\mu_o I I_o}{\pi a^2} [-u \bar{i}_x + v \bar{i}_y] \tag{e}$$

It is the fact that $\nabla \times \bar{H} = 0$ in the neighborhood of the origin that requires that the contributions to (e) be negatives.

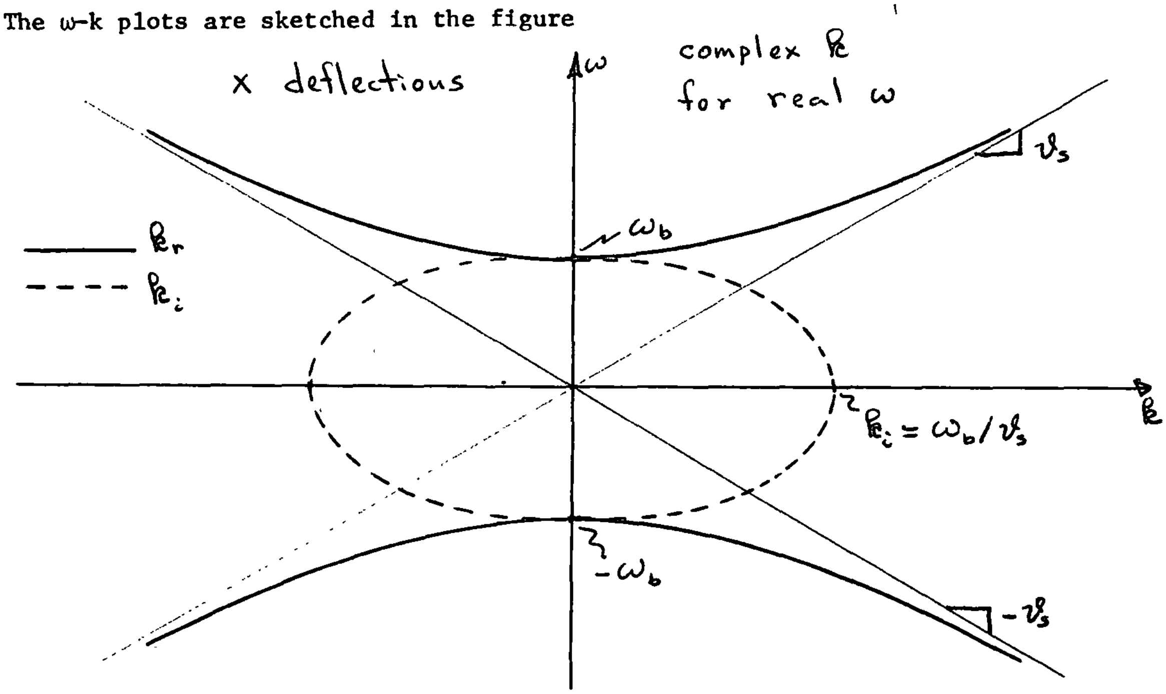
Part b

(i) Assume $u = Re[\hat{u}\, e^{j(\omega t - kz)}]$ (f)

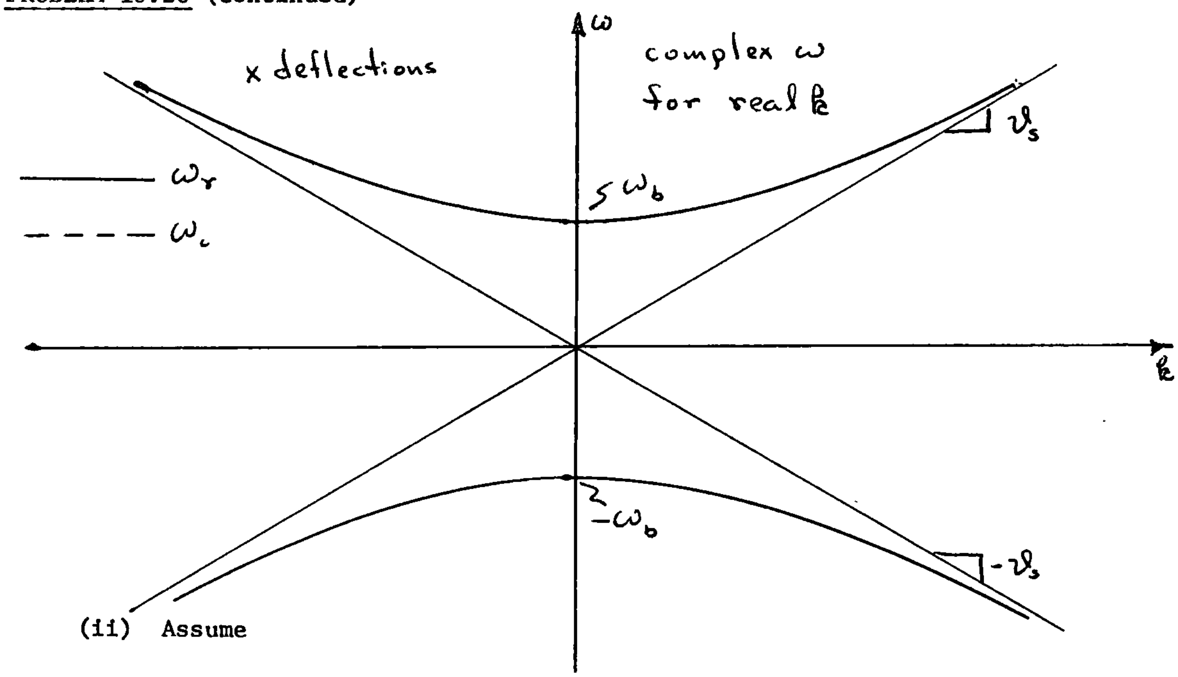
Then

$$\omega^2 = v_s^2 k^2 + \omega_b^2,\; v_s^2 = \frac{f}{m};\; \omega_b^2 = \frac{Ib}{m} \tag{g}$$

The ω-k plots are sketched in the figure



PROBLEM 10.26 (continued)



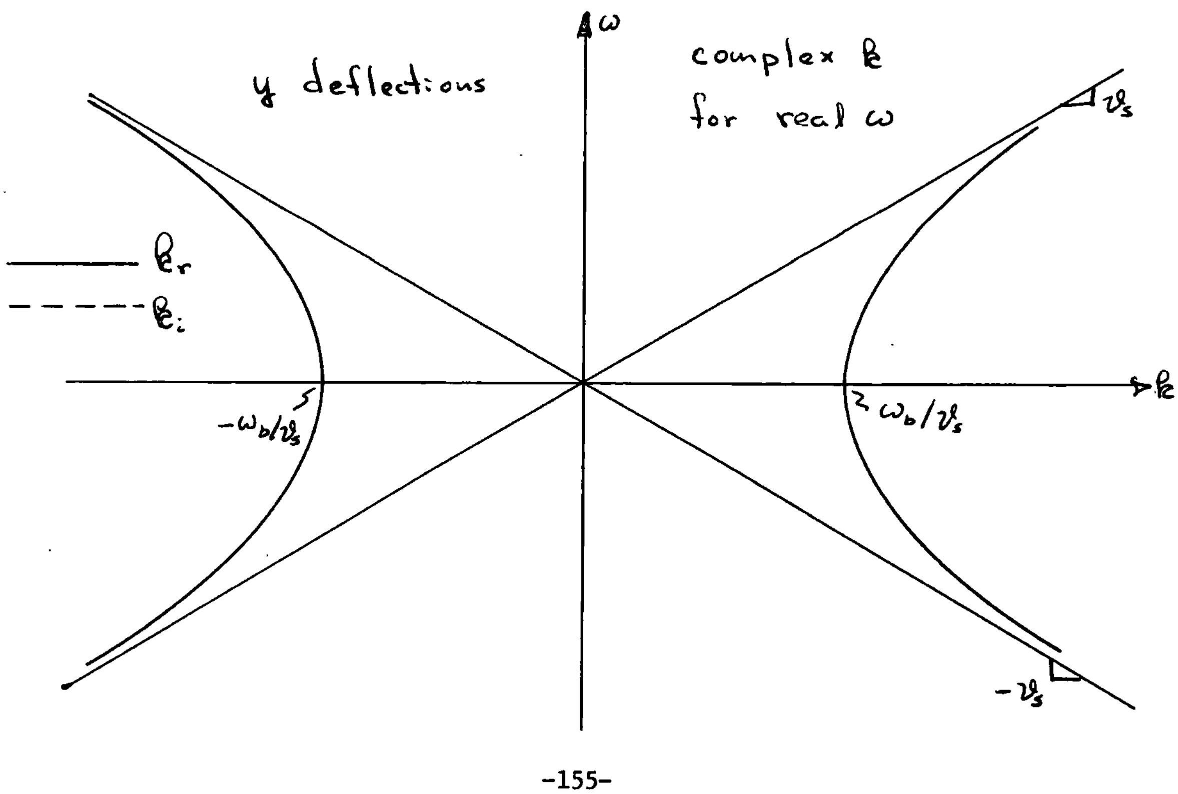
(ii) Assume

$$v = \text{Re}[\hat{v}\, e^{j(\omega t - kz)}] \tag{h}$$

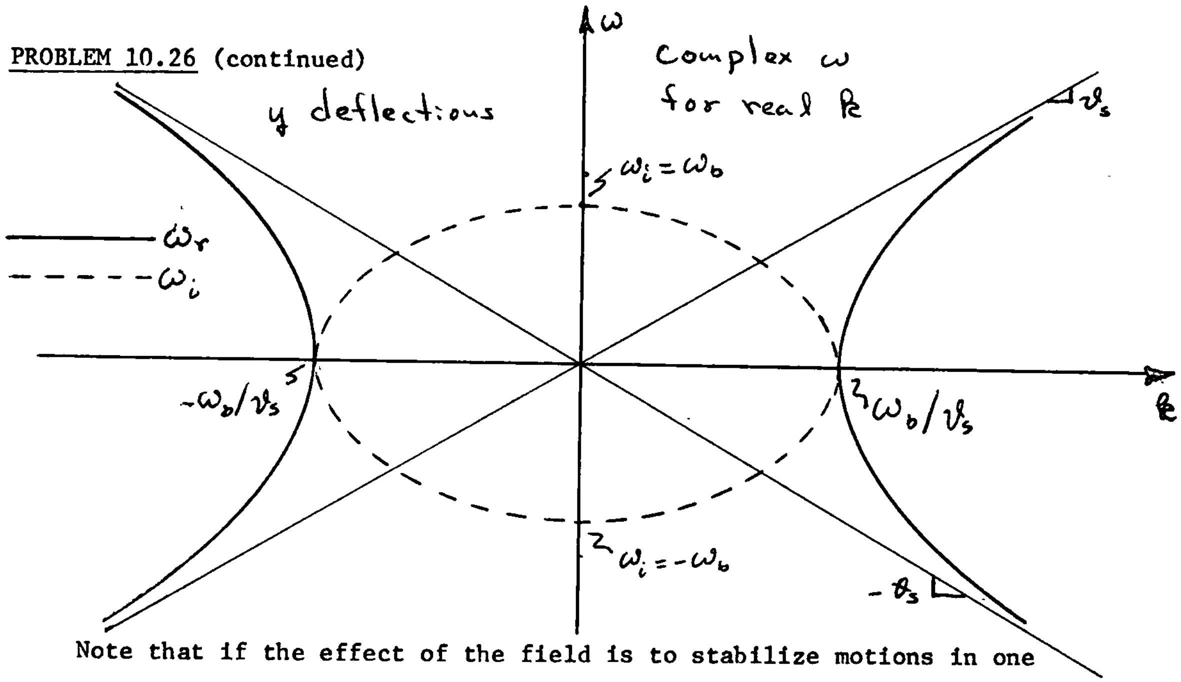
Then

$$\omega^2 = v_s^2 k^2 - \omega_b^2 \tag{i}$$

and the ω-k plots are as shown



PROBLEM 10.26 (continued)



Note that if the effect of the field is to stabilize motions in one direction, it must destabilize motions in the other direction.

Part c

Driven response is found in a manner similar to that for Prob. 10.2. Thus for

$\omega < \omega_b$ (cutoff)

$$u(z,t) = -\frac{u_o \sinh \alpha_u x}{\sinh \alpha \ell} \cos \omega_o t \tag{j}$$

$$v(z,t) = -\frac{v_o \sin\ k_r x}{\sin k_v \ell} \sin \omega_o t \tag{k}$$

$\omega > \omega_b$

$$u(z,t) = -\frac{u_o \sin k_u x}{\sin k_u \ell} \cos \omega_o t \tag{ℓ}$$

$$v(z,t) = -\frac{v_o \sin k_v x}{\sin k_v \ell} \sin \omega_o t \tag{m}$$

where
$$\alpha_u = \left[\frac{\omega_b^2 - \omega_o^2}{v_s^2}\right]^{1/2}$$

$$k_u = \left[\frac{\omega_o^2 - \omega_b^2}{v_s^2}\right]^{1/2}$$

PROBLEM 10.26 (continued)

$$k_v = \left[\frac{\omega_o^2 + \omega_b^2}{v_s^2}\right]^{1/2}$$

Part d

We must suppress instability of lowest natural mode in v.

$$v_s^2 \left(\frac{\pi}{\ell}\right)^2 > \omega_b^2 \qquad \text{(n)}$$

or

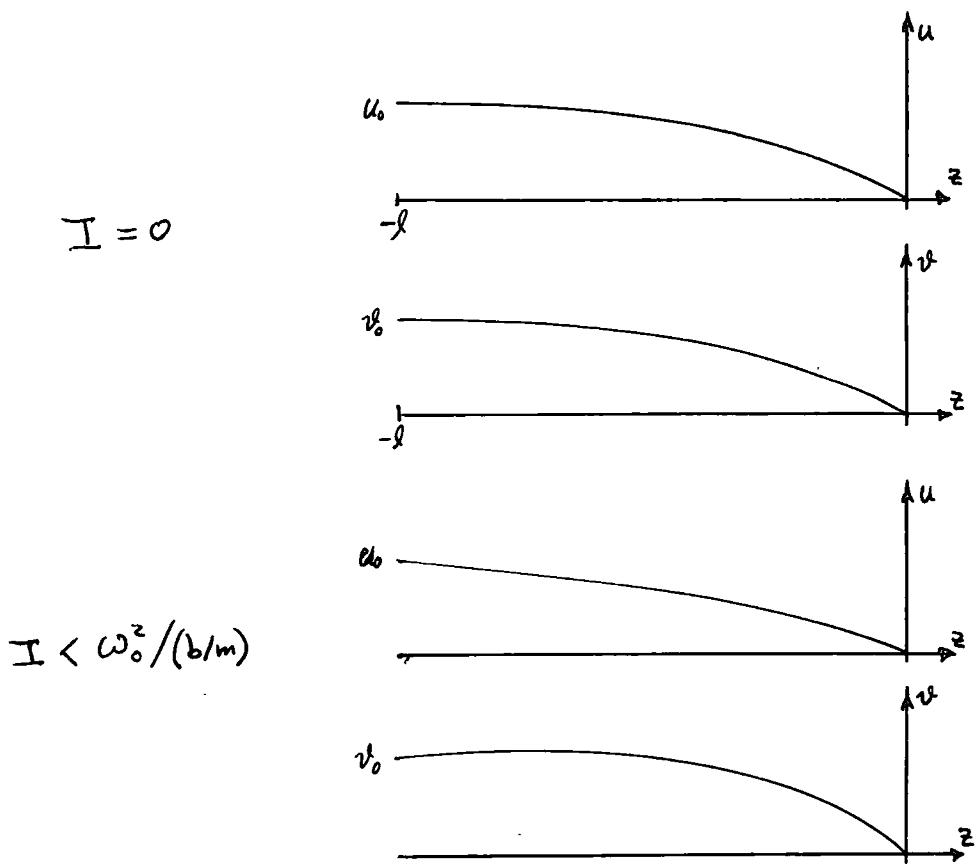
$$I\, I_o < \frac{f\pi a^2}{\mu_o}\left(\frac{\pi}{\ell}\right)^2 \qquad \text{(o)}$$

for evanescent waves

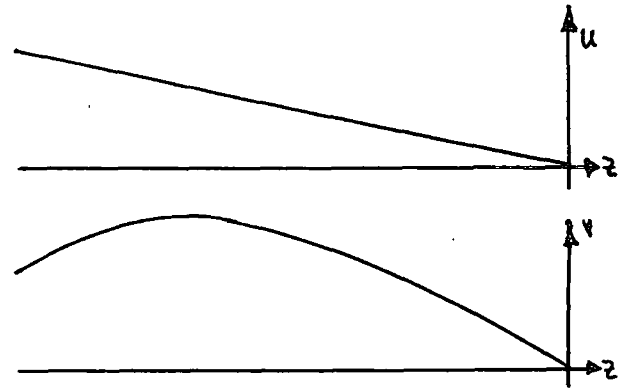
$$\omega_o^2 < \omega_b^2 \qquad \text{(p)}$$

Thus, from (n) and (p), $\omega_o^2 < v_s^2\ (\pi/\ell)^2$.

Part e

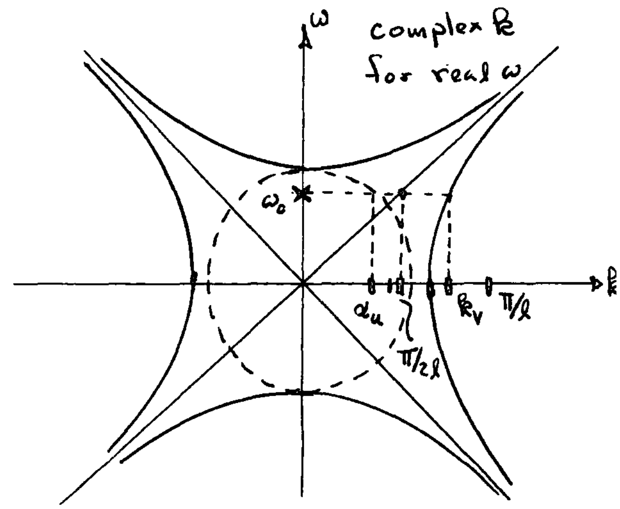


PROBLEM 10.26 (continued)



$$I = \frac{\omega_o^2}{b/m}$$

The effect of raising the current is summarized by the ω-k plot, with complex k plotted for real ω.



As I is raised the hyperbola moves outward. Thus, k_v increases and k_u decreases to zero and becomes imaginary. Thus, wavelengths for the v deflection shorten while those for u lengthen to infinity and then deflections decay. Note that v waves shorter than $\lambda=2\ell$ will not be observed because of instability.

PROBLEM 10.27

Part a

We may take the results of problem 10.26 and replace $\frac{\partial}{\partial t}$ by $\frac{\partial}{\partial t} + U\frac{\partial}{\partial z}$ in the differential equations, and ω by ω-kU in the dispersion equations. Therefore, the equations of motion are

$$m\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial z}\right)^2 u = f\frac{\partial^2 u}{\partial z^2} - Ibu \tag{a}$$

$$m\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial z}\right)^2 v = f\frac{\partial^2 v}{\partial z^2} + Ibv \tag{b}$$

Part b

For the x motions, the dispersion relation is

$$-m(\omega-kU)^2 = -fk^2 - Ib \tag{c}$$

We let

PROBLEM 10.27 (continued)

$$\frac{Ib}{m} = \omega_b^2$$

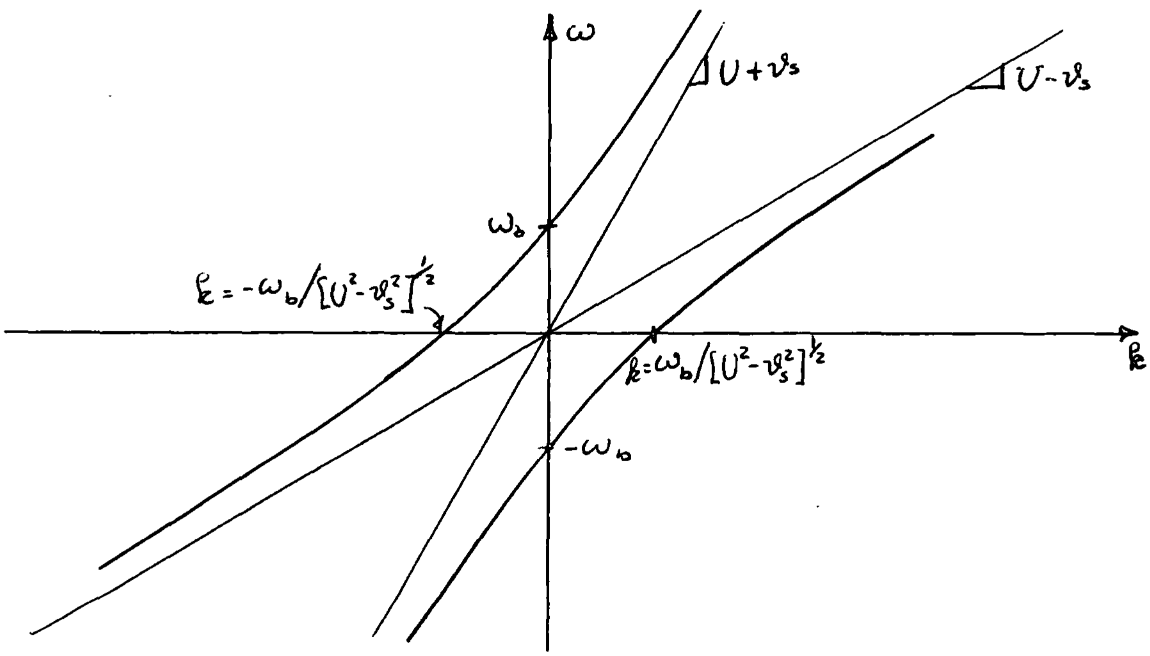
$$\frac{f}{m} = v_s^2$$

Therefore $\omega = kU \pm \sqrt{k^2v_s^2 + \omega_b^2}$ (d)

or solving for k

$$k = \frac{\omega U \pm \sqrt{\omega^2 v_s^2 + \omega_b^2(U^2 - v_s^2)}}{(U^2 - v_s^2)} \quad \text{(e)}$$

The ω-k plot for x motions is sketched as



For real ω, we have only k real. For real k, we have only real ω.

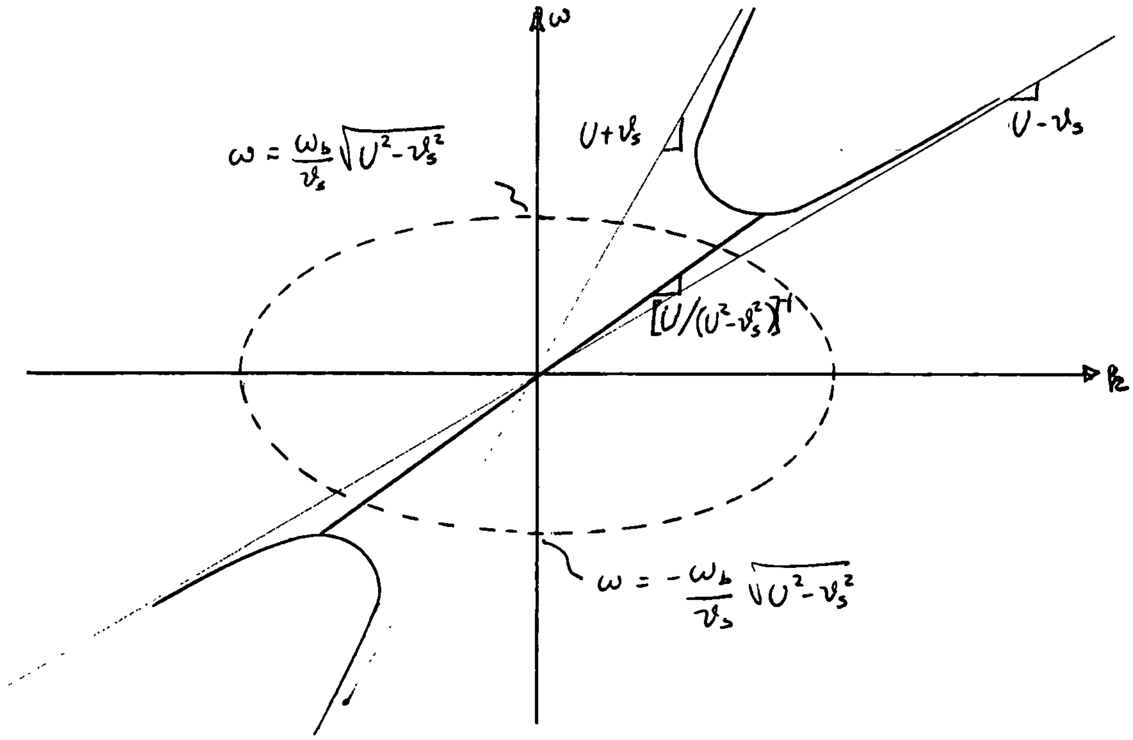
For the y motions, we obtain

$$\omega = kU \pm \sqrt{v_s^2k^2 - \omega_b^2} \quad \text{(f)}$$

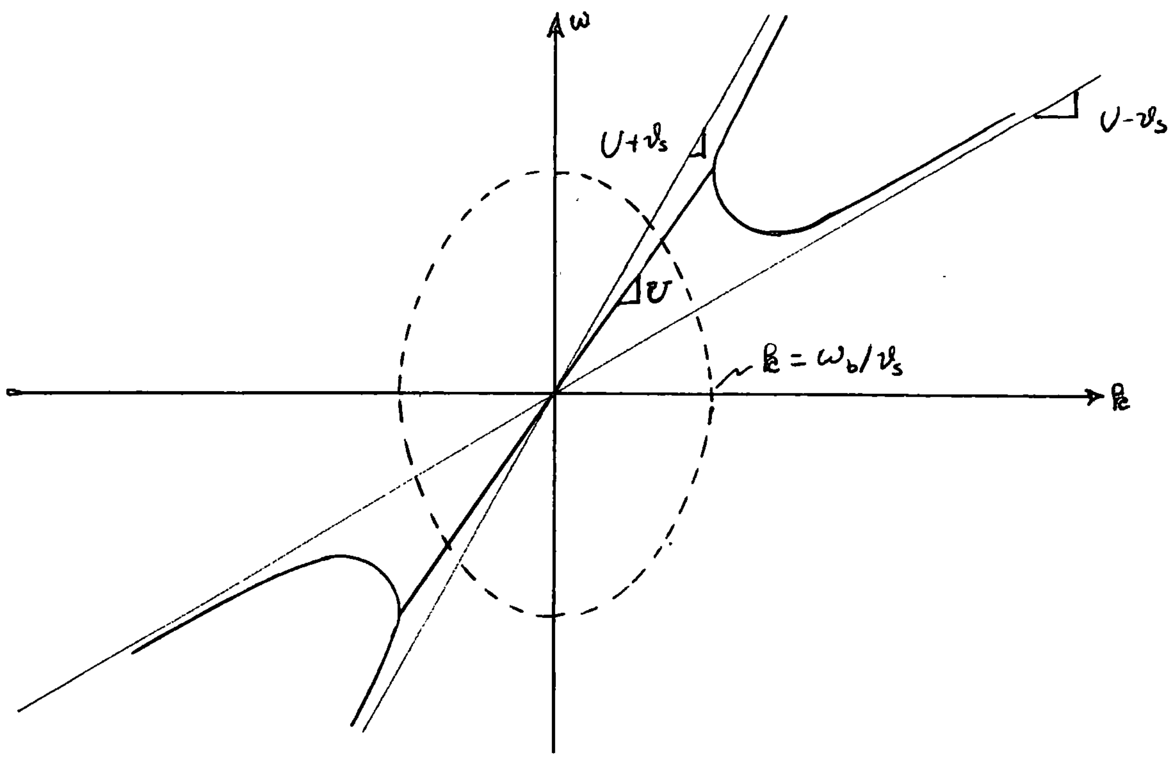
$$k = \frac{\omega U \pm \sqrt{\omega^2 v_s^2 - \omega_b^2(U^2 - v_s^2)}}{(U^2 - v_s^2)} \quad \text{(g)}$$

Thus for real ω, the sketch is

PROBLEM 10.27 (continued)



while for real k, the ω-k plot is



PROBLEM 10.27 (continued)

Part c

Since the wire is traveling at a "supersonic" velocity, we cannot impose a downstream boundary condition to determine upstream behavior.

We are given

$$u(0,t)\bar{i}_x + v(0,t)\bar{i}_y = u_o \cos\omega_o t\ \bar{i}_x + v_o \sin\omega_o t\ \bar{i}_y \tag{h}$$

and the boundary conditions

$$\frac{\partial u}{\partial z}(0,t) = 0; \frac{\partial v}{\partial z}(0,t) = 0 \tag{i}$$

We let

$$\alpha = \frac{\omega U}{U^2 - v_s^2}; \beta = \sqrt{\frac{\omega^2 v_s^2 + \omega_b^2(U^2 - v_s^2)}{(U^2 - v_s^2)}}$$

$$\gamma = \sqrt{\frac{\omega^2 v_s^2 - \omega_b^2(U^2 - v_s^2)}{(U^2 - v_s^2)}} \tag{j}$$

For the x motions, the allowed values of k are

$$k_1 = \alpha + \beta \qquad \text{with } \omega = \omega_o$$

$$k_2 = \alpha - \beta \tag{k}$$

Therefore

$$u = \mathrm{Re}\left\{\left[A_1 e^{-jk_1 z} + A_2 e^{-jk_2 z}\right] e^{j\omega_o t}\right\} \tag{ℓ}$$

Applying the boundary conditions and simplying, we obtain

$$u = u_o \mathrm{Re}[(\frac{j\alpha}{\beta} \sin\beta z + \cos\beta z)\ e^{j(\omega t - \alpha z)}] \tag{m}$$

For the y motions, the allowed values of k are

$$k_3 = \alpha + \gamma$$

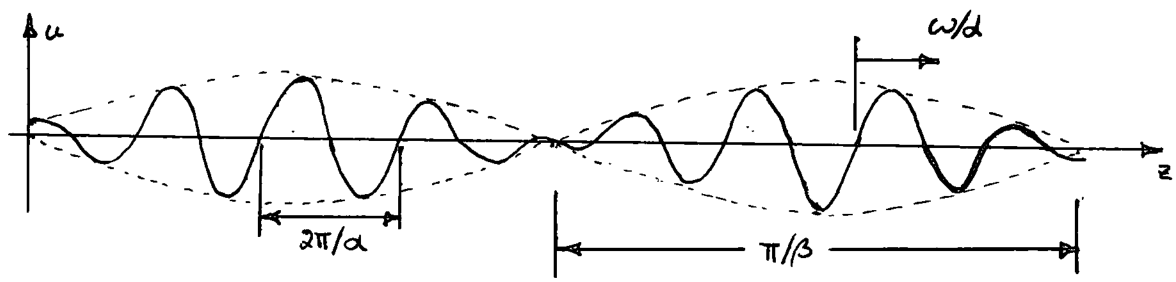
$$k_4 = \alpha - \gamma \tag{n}$$

Therefore

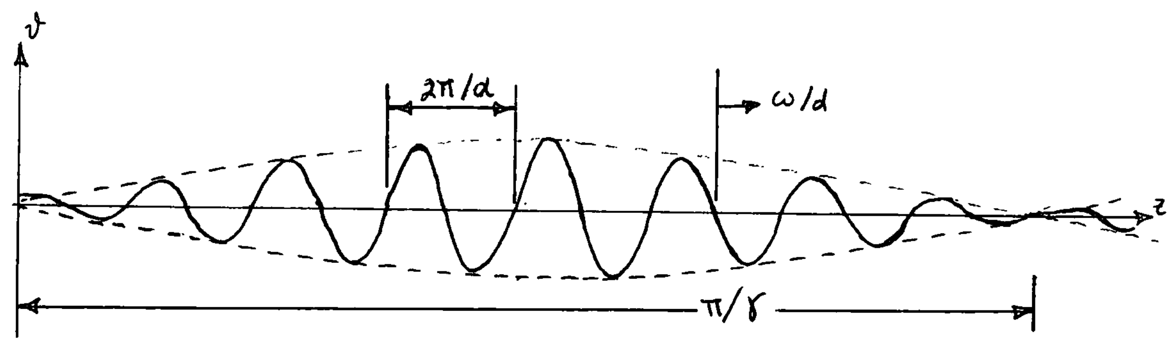
$$v = -v_o\ \mathrm{Re}[(-\frac{\alpha}{\gamma} \sin\gamma z + j\cos\gamma z) e^{j(\omega t - \alpha z)}] \tag{o}$$

PROBLEM 10.27 (continued)

Part d



As long as $U > v_s$ this is the form of u, no matter what the value of I (as long as $I > 0$). As the magnitude of I increases, β increases but α remains unchanged.

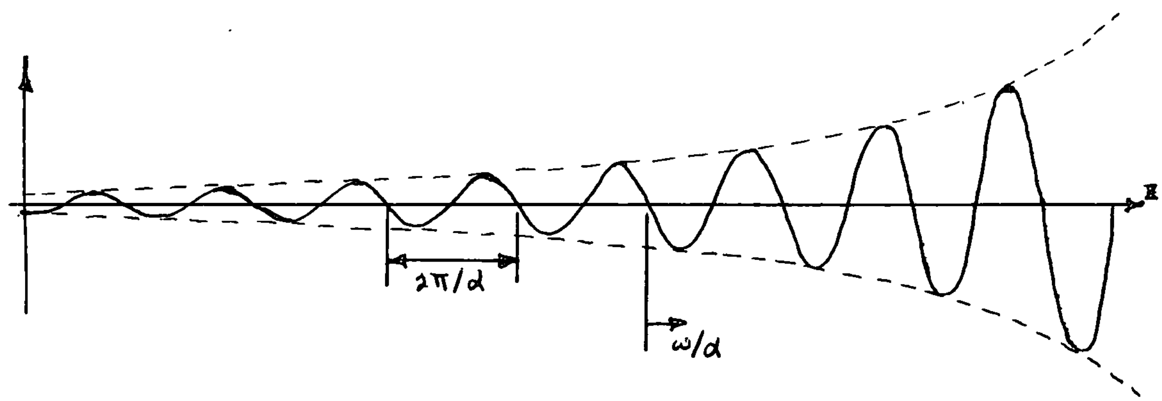


PROBLEM 10.27 (continued)

This is the form of v, as long as

$$\omega^2 v_s^2 - \omega_b^2(U^2 - v_s^2) > 0 \tag{p}$$

As I is increased, we reach a value whereby this inequality no longer holds. At this point γ becomes imaginary and we have spatial growth.



As I is increased beyond the critical value, v will begin to grow exponentially with z.

Part e

To simulate the moving wire, we could use a moving stream of a conducting liquid such as mercury. We would introduce current onto the stream at the nozzle and complete the circuit by having the stream strike a metal plate at some downstream postion.

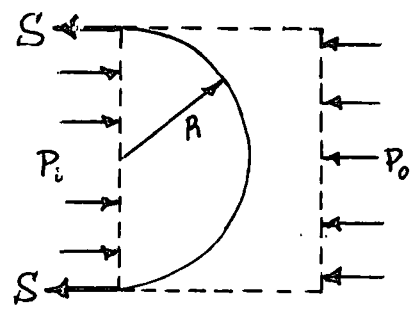
PROBLEM 10.28

Part a

A simple static argument establishes the required pressure difference. The pressure, as a mechanical stress that occurs in a fluid, always acts on a surface in the normal direction. The figure shows a section of length Δz from the membrane. Since the volume which encloses this section must be in force equilibrium, we can write

PROBLEM 10.28 (continued)

$$2R(\Delta z)[p_i - p_o] = 2S(\Delta z) \tag{a}$$



where we have summed the forces acting on the surfaces. It follows that the required pressure difference is

$$p_i - p_o = \frac{S}{R} \tag{b}$$

Part b

To answer this question, and other questions concerning the dynamics of the circular membrane, we must include in our description a perturbation displacement from the equilibrium at $r = R$. Hence, we define the membrane surface by the relation

$$r = R + \xi(\theta,z,t) \tag{c}$$

The pressure difference p_i-p_o is a force per unit area acting on the membrane in the normal direction. It is the surface force density necessary to counteract a mechanical force per unit area T_m

$$T_m = -\frac{S}{R} \tag{d}$$

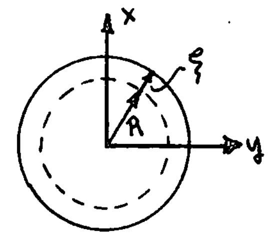
which acts on each section of the membrane in the radial direction. We wish now to determine the mechanical force acting on each section, when the surface is perturbed to the position given by (c). We can do this in steps. First, consider the case where ξ is independent of θ and z, as shown in the figure. Then from (d)

$$T_m = -\frac{S}{R+\xi} = -S[\frac{1}{R} - \frac{\xi}{R^2}] \tag{e}$$

where we have kept only the linear term in the expansion of T_m about $r = R$.

When the perturbation ξ depends on θ, the surface has a tilt, as shown. We can sum the components to S acting on the section in the radial direction as

PROBLEM 10.28 (continued)



$$\lim_{\Delta\theta\to 0} \frac{S}{R\Delta\theta}\left[\frac{1}{R}\frac{\partial\xi}{\partial\theta}\bigg|_{\theta+\frac{\Delta\theta}{2}} - \frac{1}{R}\frac{\partial\xi}{\partial\theta}\bigg|_{\theta-\frac{\Delta\theta}{2}}\right] = \frac{S}{R}\frac{\partial^2\xi}{\partial\theta^2} \tag{f}$$

Similarly, a dependence on z gives rise to a radial force on the section due to the mechanical tension S,

$$\lim_{\Delta z\to 0} \frac{S}{\Delta z}\left[\frac{\partial\xi}{\partial z}\bigg|_{z+\frac{\Delta z}{2}} - \frac{\partial\xi}{\partial z}\bigg|_{z-\frac{\Delta z}{2}}\right] = S\frac{\partial^2\xi}{\partial z^2} \tag{g}$$

In general, the force per unit area exerted on a small section of membrane under the constant tension S from the adjacent material is the sum of the forces given by (e), (f) and (g),

$$T_m = S\left(-\frac{1}{R} + \frac{\xi}{R^2} + \frac{1}{R}\frac{\partial^2\xi}{\partial\theta^2} + \frac{\partial^2\xi}{\partial z^2}\right) \tag{h}$$

It is now possible to write the dynamic force equation for radial motions. In addition to the pressure difference $p_i - p_o$ acting in the radial direction, we will include the inertial force density $\sigma_m/(\partial^2\xi/\partial t^2)$ and a surface force density T_r due to electric or magnetic fields. Hence,

$$\sigma_m\frac{\partial^2\xi}{\partial t^2} = S\left(-\frac{1}{R} + \frac{\xi}{R^2} + \frac{1}{R^2}\frac{\partial^2\xi}{\partial\theta^2} + \frac{\partial^2\xi}{\partial z^2}\right) + T_r + p_i - p_o \tag{i}$$

Consider now the case where there is no electromechanical interaction. Then $T_r = 0$, and static equilibrium requires that (b) hold. Hence, the constant terms in (i) cancel, leaving the perturbation equation

$$\sigma_m\frac{\partial^2\xi}{\partial t^2} = S\left(\frac{\xi}{R^2} + \frac{1}{R^2}\frac{\partial^2\xi}{\partial\theta^2} + \frac{\partial^2\xi}{\partial z^2}\right) \tag{j}$$

PROBLEM 10.28 (continued)
Parts c & d

This equation is formally the same as those that we have encountered previously (see Sec. 10.1.3). However, the cylindrical geometry imposes additional requirements on the solutions. That is, if we assume solutions having the form,

$$\xi = \text{Re } \hat{\xi}(z)e^{j(\omega t + m\theta)} \tag{k}$$

the assumed dependence on θ is a linear combination of sin $m\theta$ and cos $m\theta$. If the displacement is to be single valued, m must have integer values. Otherwise we would not have $\xi(\theta,z,t) = \xi(\theta + 2\pi,z,t)$.

With the assumed dependence on θ and t, (j) becomes,

$$\frac{d^2\hat{\xi}}{dz^2} + k^2\hat{\xi} = 0 \tag{ℓ}$$

where

$$k^2 = \frac{1}{R^2}(1-m^2) + \frac{\omega^2\sigma_m}{S}$$

The membrane is attached to the rigid tubes at $z = 0$ and $z = \ell$. The solution to (ℓ) which satisfies this condition is

$$\hat{\xi} = A \sin k_n x \tag{m}$$

where

$$k_n = \frac{n\pi}{\ell}, \quad n = 1,2,3,...$$

The eigenvalue k_n determines the eigenfrequency, because of (ℓ).

$$\omega_n^2 = \left[\left(\frac{n\pi}{\ell}\right)^2 + \frac{(m^2 - 1)}{R^2}\right]\frac{S}{\sigma_m} \tag{n}$$

To obtain a picture of how these modes appear, consider the case where A is real, and (m) and (k) become

$$\xi(\theta,z,t) = A \sin \frac{n\pi}{\ell} x \cos m\theta \cos \omega t \tag{o}$$

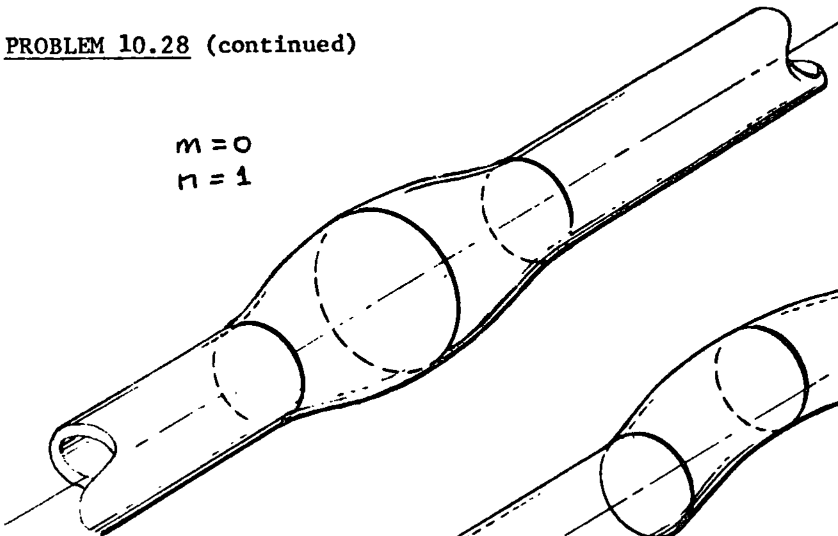
The instantaneous displacements for the first four modes are shown in the figure.

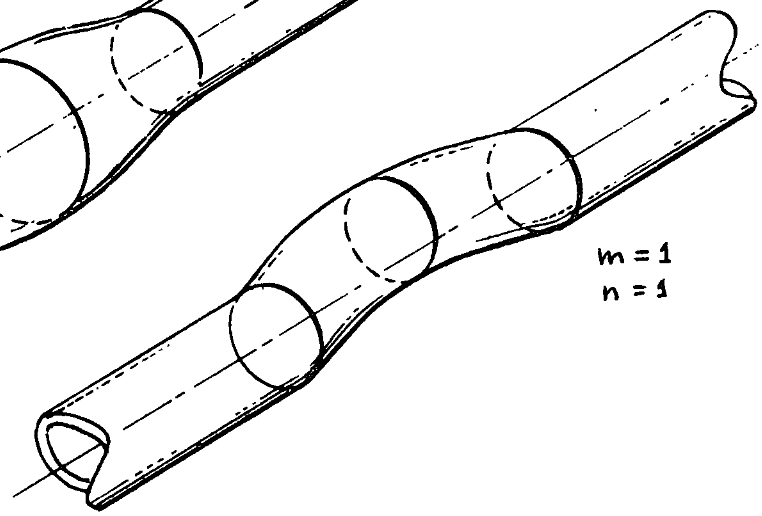
There is the possibility that the $m = 0$ mode is unstable, as can be seen from (n), where if

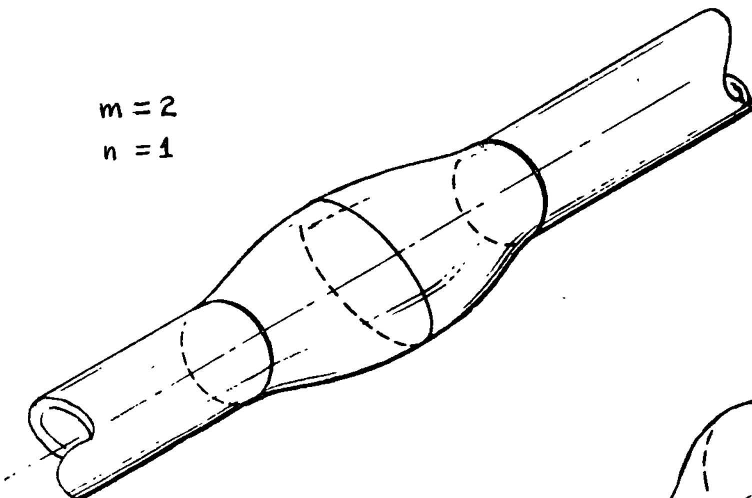
$$\left(\frac{n\pi}{\ell}\right)^2 < \frac{1}{R^2} \tag{p}$$

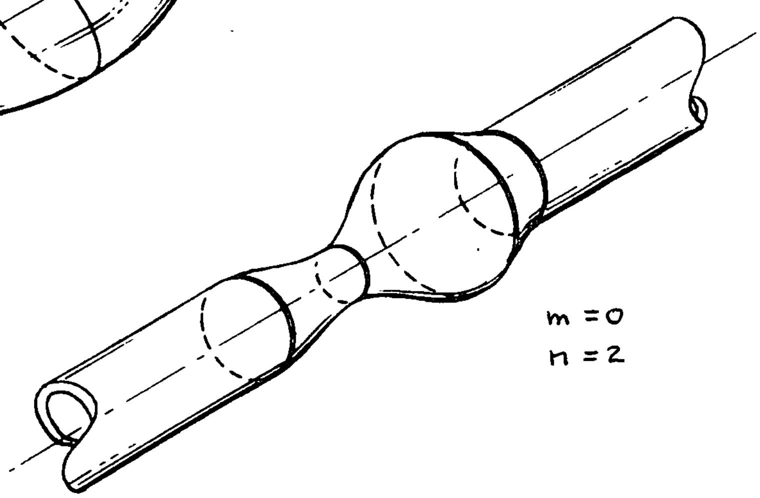
we find that the time dependence has the form exp $\pm |\omega| t$. The first mode to meet this condition for instability is the $n = 1$ mode. Hence, it is not possible

PROBLEM 10.28 (continued)









PROBLEM 10.28 (continued)

to maintain the uniform cylindrical shape of the static equilibrium if

$$R\pi/\ell < 1 \tag{q}$$

This condition for static instability is easily understood if we remember that in the $m = 0$, $n = 1$ mode, there are two perturbation surface forces on a small section of the membrane surface. One of these is the perturbation part of (e) and arises because of the curvature in static equilibrium. This force acts in the same direction as the displacement, and hence tends to produce static instability. It is counteracted by a restoring force proportional to the second derivative in the z direction, as given by (g). Condition (q) is satisfied when the effect of the initial curvature predominates the stiffness from the boundaries.

Part e

With rotation, the dispersion becomes:

$$(\omega - m\Omega)^2 = \Omega_s^2[m^2 - 1 - m_c^2]$$

with

$$\Omega_s^2 = \frac{S}{\sigma_m R^2}$$

$$m_c^2 = \frac{2\varepsilon_o V_o^2}{a^3} \frac{R^2}{S}$$

Because there is no z dependence (no surface curvature in the z direction) the equilibrium is unstable in the $m = 0$ mode even in the absence of an applied voltage.

PROBLEM 10.29

The solution is of the form

$$\xi = \xi_+(\alpha) + \xi_-(\beta) \tag{a}$$

where

$$\alpha = x - y$$

$$\beta = x + y$$

We are given that at $x = 0$

$$\frac{\partial \xi}{\partial x} = \frac{d\xi_+}{d\alpha} + \frac{d\xi_-}{d\beta} = \Delta[u_{-1}(y) - u_{-1}(y-a)] \tag{b}$$

and that

PROBLEM 10.29 (continued)

$$\xi = 0$$

which implies that

$$\frac{\partial \xi}{\partial y} = 0 = -\frac{d\xi_+}{d\alpha} + \frac{d\xi_-}{d\beta} \qquad \text{(c)}$$

We therefore have

$$\frac{d\xi_+}{d\alpha} = \frac{\Delta}{2}\,[u_{-1}(-\alpha) - u_{-1}(-\alpha-a)]$$

and

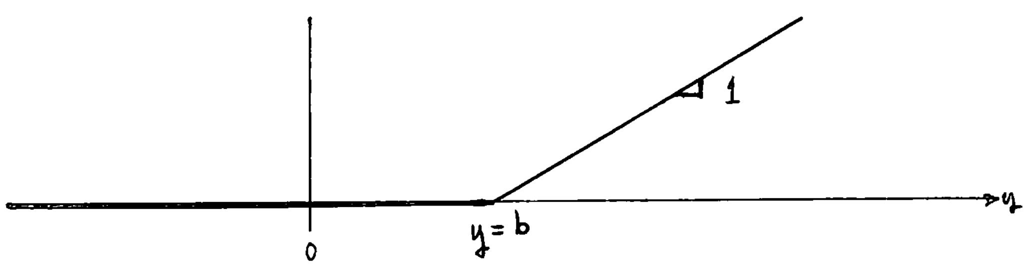
$$\frac{d\xi_-}{d\beta} = \frac{\Delta}{2}\,[u_{-1}(\beta) - u_{-1}(\beta-a)] \qquad \text{(e)}$$

Then $\frac{\partial \xi}{\partial y} = -\frac{d\xi_+}{d\alpha} + \frac{d\xi_-}{d\beta} = \frac{\Delta}{2}\left\{u_{-1}(y-x) - u_{-1}(y-x-a) + u_{-1}(x+y) - u_{-1}(x+y-a)\right\}$ (f)

Integrating with respect to y, we obtain

$$\xi = \frac{\Delta}{2}\,\{-u_{-2}(y-x) + u_{-2}(y-x-a) + u_{-2}(y+x) - u_{-2}(y+x-a) \qquad \text{(g)}$$

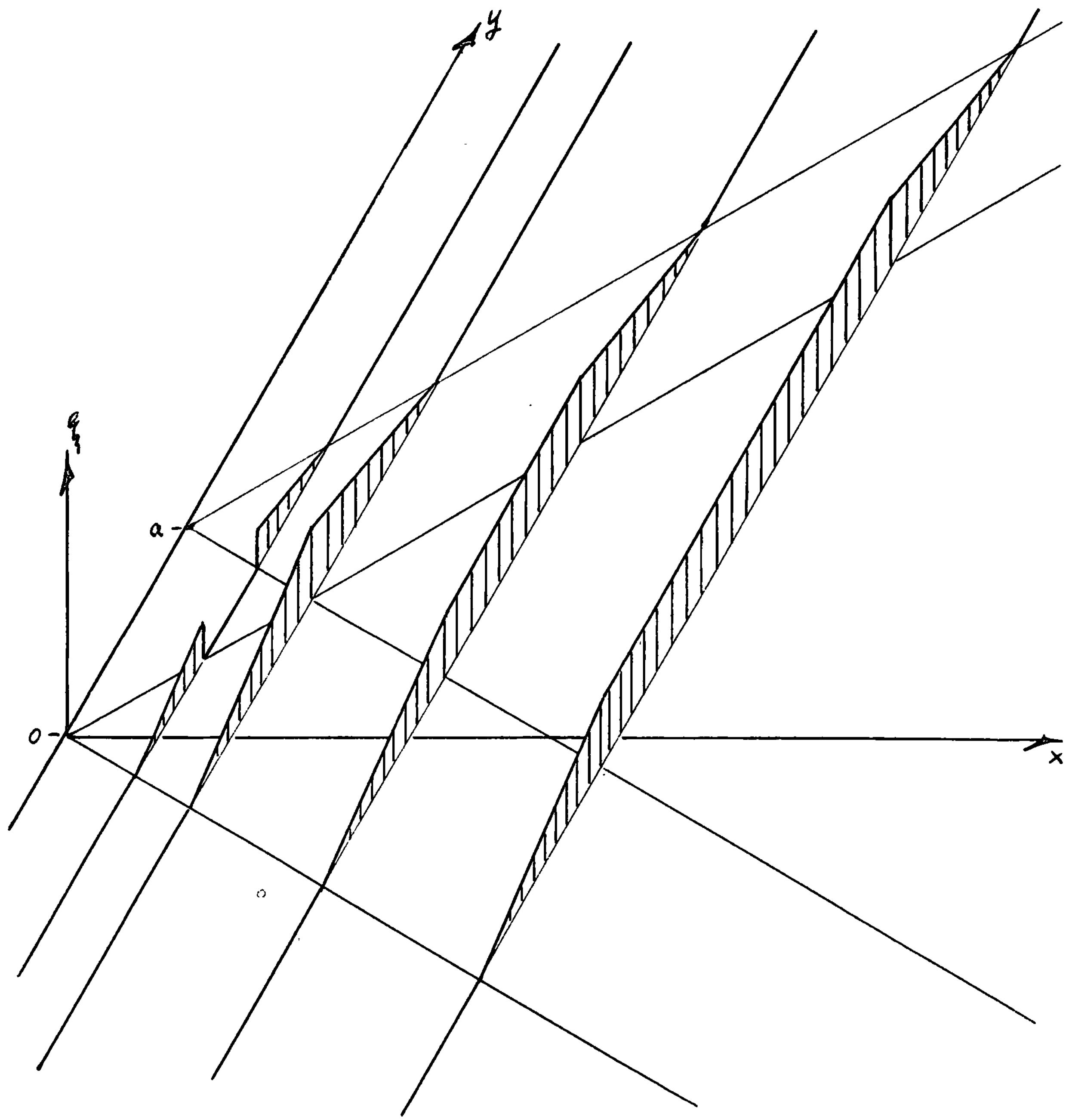
where u_{-2} is a ramp function; that is $u_{-2}(y-b)$ is defined as



Part b

The constraint represented by Fig. 10P.29 could be obtained by ejecting the membrane from a slit at $x = 0$ that is planar, but tilted over the range $0 < y < a$. Thus, the membrane would have no deflection ξ at $x = 0$, but would have the required constant slope Δ over the range $0 < y < a$, and zero slope elsewhere.

PROBLEM 10.29 (continued)



PROBLEM 10.30

For this situation, the governing equation is (10.4.15) of the text.

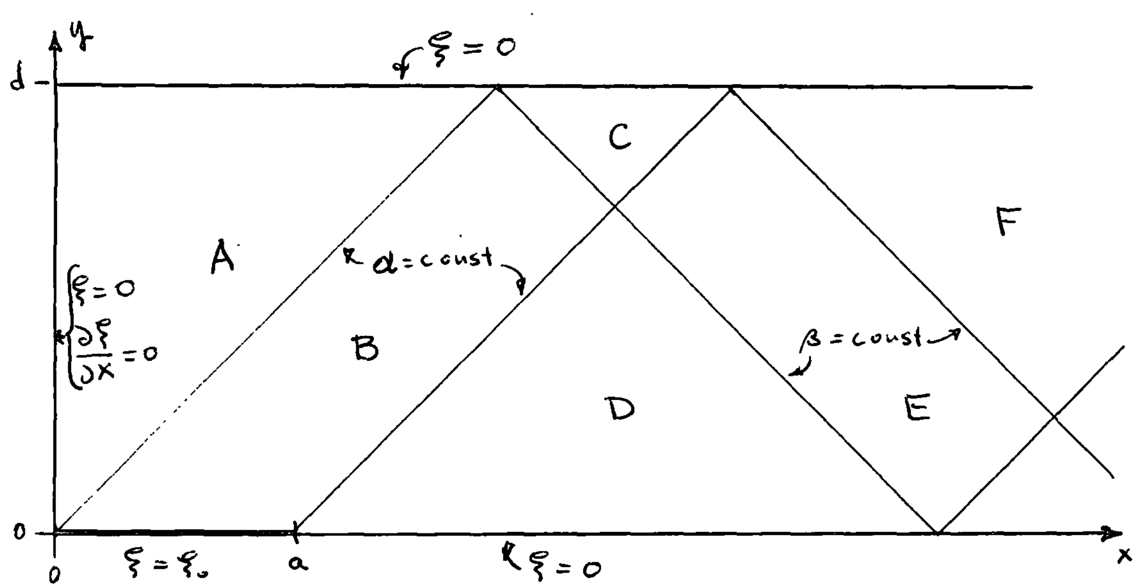
$$(M^2-1)\frac{\partial^2\xi}{\partial x^2} = \frac{\partial^2\xi}{\partial y^2} \tag{a}$$

Here $M^2 = 2$; so we have the equation:

$$\frac{\partial^2\xi}{\partial x^2} = \frac{\partial^2\xi}{\partial y^2} \tag{b}$$

The characteristics are determined from equations (10.4.17) and (10.4.18) to be:

$$\begin{aligned}\alpha &= x - y\\ \beta &= x + y\end{aligned} \tag{c}$$



The x-y plane divides into regions A...F, as shown in the sketch. Tracing back on the characteristics from points in regions A, D, F... shows that in these regions $\xi = 0$; the characteristics originate on "zero" boundary conditions. At points in region B, only the C^+ characteristic originates on finite data; $\xi_+(\alpha) = \xi_o$, $\xi_-(\beta) = 0$ and hence

$$\xi = \xi_o \text{ in region B}$$

PROBLEM 10.30 (continued)

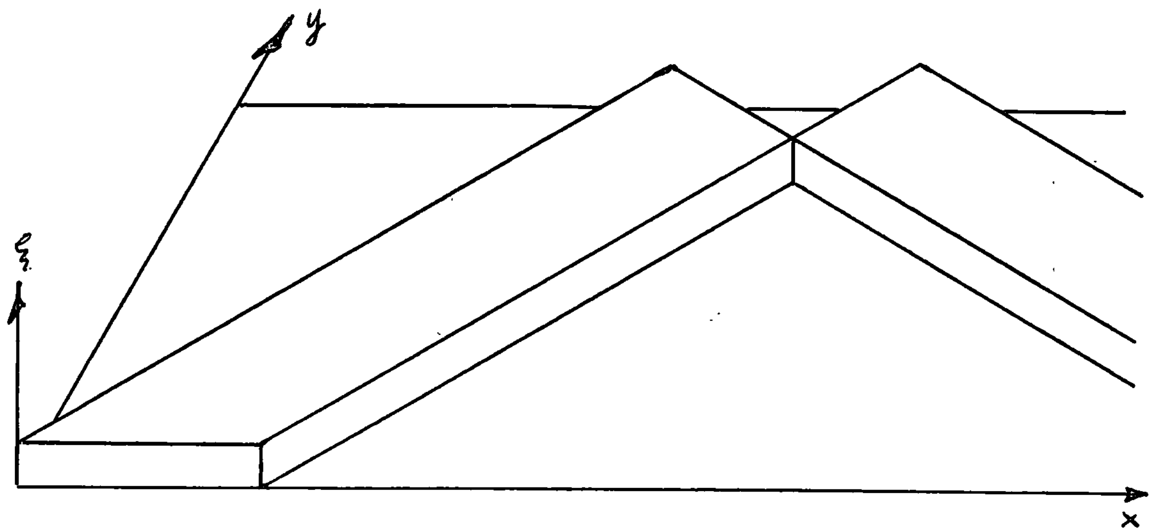
In C, deflections are determined by waves, both originating from the initial data. Hence $\xi_+(\alpha) = \xi_o$, but $\xi_-(\beta)$ is determined by the reflection of an incident wave on the boundary at y = d. Hence $\xi_-(\beta) = -\xi_o$ and

$$\xi = 0 \text{ in region C}$$

In region E only the $\xi_-(\beta)$ wave is finite because the $\xi_+(\alpha)$ wave originates on zero conditions and

$$\xi = -\xi_o \text{ in region E}$$

The deflection has the stationary appearance shown in the figure.



PROBLEM 10.31

From equation 10.4.30, we have

$$\omega^2 = k^2 v_s^2 \pm \frac{kB_o I}{m} \tag{a}$$

We define

$$\alpha = + \frac{IB_o}{2mv_s^2} \tag{b}$$

and

$$\beta = \sqrt{\left(\frac{B_o I}{2mv_s^2}\right)^2 + \frac{\omega^2}{v_s^2}} \tag{c}$$

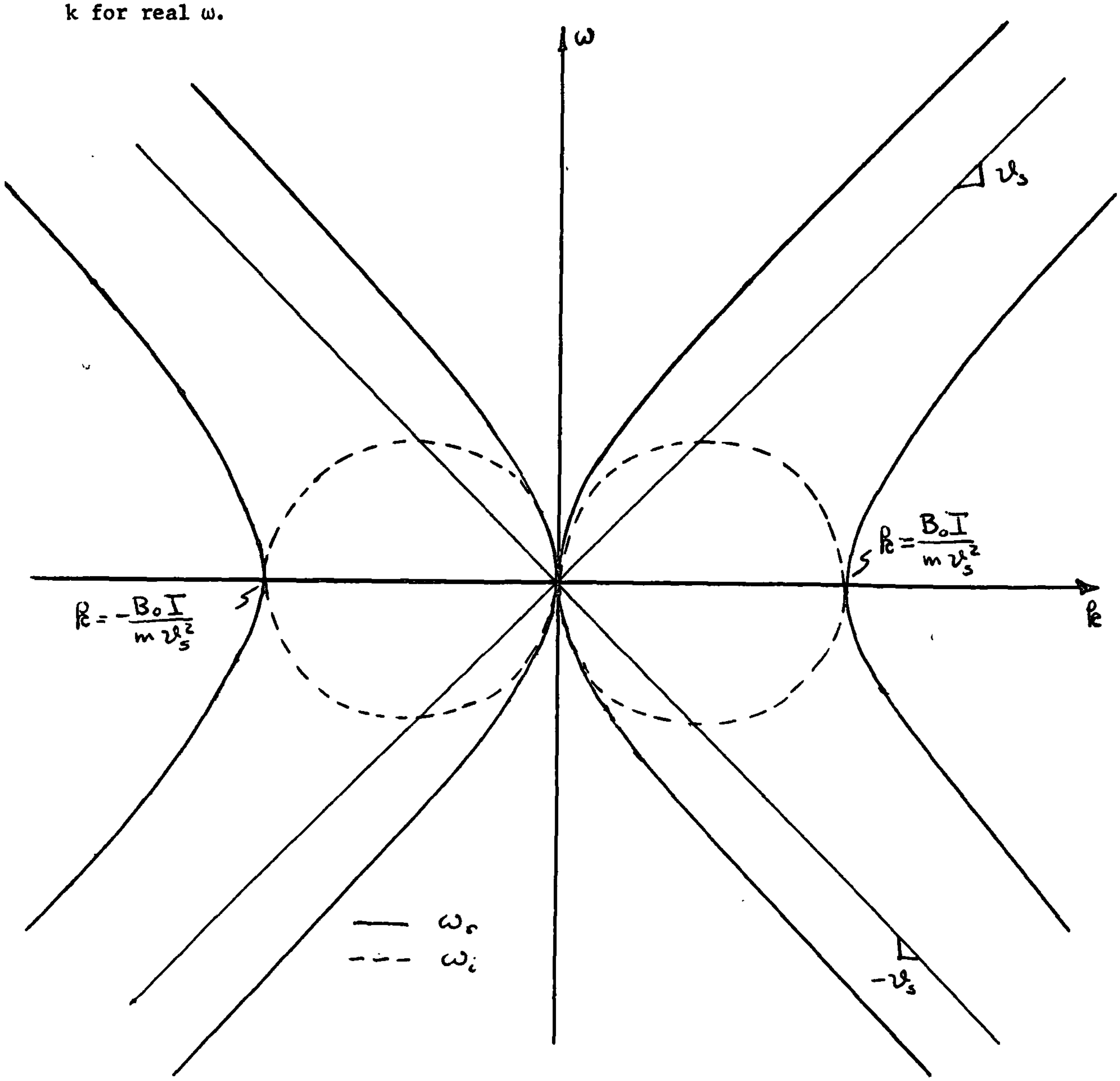
PROBLEM 10.31 (continued)

The four allowed branches of k as a function of ω are therefore $k = \pm k_1$, and $\pm k_2$, where

$$k_1 = \alpha + \beta \tag{d}$$

$$k_2 = -\alpha + \beta \tag{e}$$

The sketch shows complex ω for real k. Note however, that only real values of k are given if ω is real and hence the solid lines represent the plot of complex k for real ω.



PROBLEM 10.32

The effect of the longitudinal convection is accounted for by replacing ω in Eq. 10.4.3 by ω-kU (see for example page 594). Thus,

$$(\omega - kU)^2 = k^2 v_s^2 \pm \frac{kB_o I}{m} \tag{a}$$

This expression can be solved to give

$$k = \frac{\left(2\omega U \pm \frac{B_o I}{m}\right) \pm \sqrt{4v_s^2\omega^2 + 4\omega U\left(\pm \frac{B_o I}{m}\right) + \left(\frac{B_o I}{m}\right)^2}}{2(U^2 - v_s^2)} \tag{b}$$

The sketch of complex k for real ω is made with the help of the following observations: Consider the modes that are represented by $-B_o$.

1) Asymptotes for branches are $k = \omega/(U \mp v_s)$ as $\omega \to \infty$.

2) As ω is lowered, the $(-B_o)$ branches become complex as

$$4v_s^2\omega^2 + 4\omega U\left(\frac{-B_o I}{m}\right) + \left(\frac{B_o I}{m}\right)^2 = 0$$

or at the frequencies

$$\omega = \frac{B_o I}{2v_s^2 m} \left(U \pm \sqrt{U^2 - v_s^2}\right)$$

Thus, for $U > v_s$ there is a lower as well as an upper positive frequency at which k switches from real to complex values.

In this range of complex k, real k is

$$k = \left(2\omega U - \frac{B_o I}{m}\right)/2(U^2 - v_s^2)$$

or a straight line intercepting the k = 0 axis at

$$\omega = \frac{B_o I}{2Um}$$

3) as $\omega \to 0$,

$$k \to 0 \text{ and } k \to \pm B_o I/m(U^2 - v_s^2)$$

where the - sign goes with the unstable branches.

4) As $\omega \to -\infty$ the values of k are real and approach the asymptotes $k = \omega/(U \mp v_s)$.

PROBLEM 10.32 (continued)

Similar reasoning gives the modes represented in (b) by $+B_o$. Note that these modes have a plot obtained by replacing $\omega \rightarrow -\omega$ and $k \rightarrow -k$ in the figure.