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Electromechanical Dynamics

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CHAPTER 6

FIELDS AND MOVING MEDIA

6.0 INTRODUCTION

In Chapter 1 we reviewed the basic postulates and definitions of electromagnetic theory. We defined the quasi-static electromagnetic field theory suitable for efficient analysis of low-frequency, low-velocity electromechanical systems. In Chapter 2 we used the quasi-static electromagnetic equations to calculate lumped parameters for important classes of electromechanical systems. The effect of mechanical motion on the electric or magnetic fields was accounted for by allowing the electrical lumped parameters to have a dependence on the mechanical displacements. In Chapter 3 this lumped-parameter model was used to determine the electric or magnetic forces on the mechanical system. We were then prepared for the study in Chapters 4 and 5 of the dynamics of lumped-parameter electromechanical systems.

In this chapter we return to the field description introduced in Chapter 1. This is necessary if we are to extend the class of electromechanical situations with which we can deal beyond the lumped-parameter systems of Chapters 2 to 5. In subsequent chapters we shall be treating continuum systems, that is, those in which both the electrical and mechanical parts of the system are described, at least in part, by partial differential equations. This necessitates a field description of the electromechanical coupling.

Even if we are concerned only with lumped-parameter systems, a field description of the electromechanical interaction provides a useful alternative to the lumped-parameter models of Chapters 2 to 5. In many cases forces of electrical origin can be most easily deduced from the fields themselves, thus bypassing the computation of lumped parameters and an energy function. Similarly, the effects of material motion on the electrical system can be deduced from field considerations. On the other hand, an understanding of lumped-parameter systems, based on the viewpoint of Chapters 2 to 5,

provides considerable insight into what is required of a field description that includes effects of moving media. By the time this chapter is completed, the reader should be acquainted with Appendix B, in which the subject of quasi-statics is reviewed.

A simple example explains how the following sections lead to a generalization of electromagnetic theory to include the effects of material motion. A disk of copper is shown in Fig. 6.0.1*a* as it moves with velocity \mathbf{v} into a region of flux density \mathbf{B} imposed by a magnet. (We could do this experiment by attaching a handle to the disk so that we could wield it through the magnetic field.) Our lumped-parameter model provides a qualitative description of what happens. Suppose we model the disk as a one-turn, perfectly conducting loop shorted through a resistance, as shown in Fig. 6.0.1*b*. Then, as the loop enters the magnetic field region, it links an increasing flux λ from the imposed magnetic field \mathbf{B} , and an induced current i flows in a direction that induces a magnetic field which tends to cancel the flux of the imposed field \mathbf{B} . As the loop enters the field, this current i , interacting with \mathbf{B} , gives rise to a force tending to retard the motion. As we know from Section 5.1.3, the nature of this force depends on the resistance R of the loop. For a copper disk of reasonable size, as we move it through the magnetic field, it is likely that there would be an impression of passing it through a viscous liquid. The important point is that there is a magnetic force on the disk, hence a current within the disk.

Suppose that we are to analyze this problem in terms of fields. As discussed in Chapter 1, we are concerned with solutions to field equations for a quasi-static magnetic field system (equations summarized in Table 1.2).

$$\nabla \times \mathbf{H} = \mathbf{J}_f, \quad (6.0.1)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (6.0.2)$$

$$\nabla \cdot \mathbf{J}_f = 0, \quad (6.0.3)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (6.0.4)$$

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \quad (\mathbf{M} \simeq 0 \text{ for copper}). \quad (6.0.5)$$

In addition, there are boundary conditions on the surfaces of the disk and magnet. Also, we need the continuum equivalent of Ohm's law, for that was used in the lumped parameter model to explain the experiment. The necessary constituent relation was introduced in Chapter 1 as (1.1.9) and is

$$\mathbf{J}_f = \sigma \mathbf{E} \quad (6.0.6)$$

where σ is the electrical conductivity of the disk material. One approach to solving the obviously difficult problem at hand is to guess a reasonable

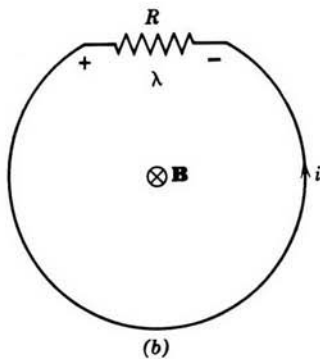
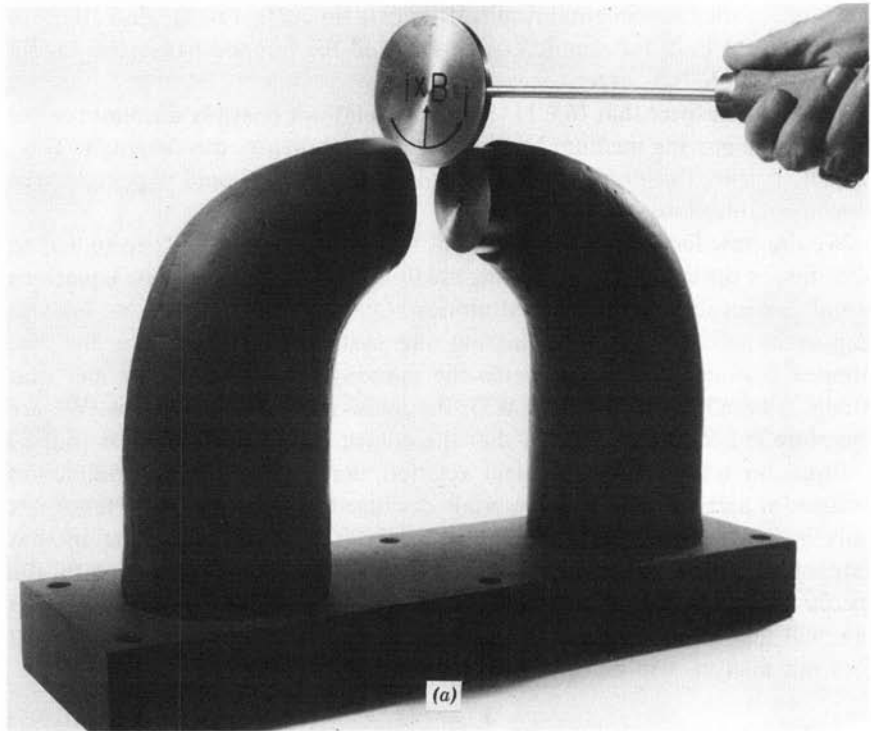


Fig. 6.0.1 (a) As the copper disk moves into the field \mathbf{B} , there is an induced current i which interacts with \mathbf{B} to retard the motion; (b) conducting loop equivalent to the disk of copper.

solution and show that it satisfies all of the equations and boundary conditions. In the absence of the moving conductor we have the fields $\mathbf{H} = \mathbf{H}_s$, $\mathbf{E} = 0$, $\mathbf{B} = \mu_0 \mathbf{H}_s$ and $\mathbf{J}_f = 0$, where \mathbf{H}_s is a function of position but is independent of time. By definition this solution satisfies (6.0.1) to (6.0.6). In fact, it satisfies these equations and boundary conditions even as the disk passes through the magnetic field!

What we have found is that our field equations in the form given do not

account for the experimental result. If there is no current in the disk, there is no force. Yet both the simple experiment and the lumped parameter model show that there is a force.

We might suspect that (6.0.1) to (6.0.6) could not possibly account for the effect of the moving medium because they do not involve the velocity \mathbf{v} . This, in fact, is true, but note that the boundary conditions could depend on the mechanical motions.

We are now faced with the question, how do we alter (6.0.1) to (6.0.6) to account for the effects of the moving medium? At least one of these equations is not correct if there is material motion. One possibility is that we left out important effects by approximating the system as quasi-static; but the lumped parameter model explains the induced force on the disk and that model is based on (6.0.1) to (6.0.5), the quasi-static field equations. We are therefore led to the conclusion that the culprit in our description is (6.0.6.)

Equation 6.0.6 is a constituent relation that represents the conduction process in a certain class of materials discussed in Section 1.1.1. Hence not only is it a law deduced from experiments [(6.0.1) to (6.0.6) are in that category] but it is found to hold for certain media in a particular state: the media are at rest. To analyze our experiment we must know what form this law will take when the material is in motion. Suffice it to say at this point that our analysis will be correct when we rewrite (6.0.6) as

$$\mathbf{J}'_f = \sigma \mathbf{E}', \quad (6.0.7)$$

where \mathbf{J}'_f and \mathbf{E}' are the current density and field intensity that would be measured by an observer moving with the material. Of course, we wish to formulate the problem in the laboratory reference frame in which the current density and electric field intensity are \mathbf{J}_f and \mathbf{E} , respectively. Hence our first objective is to relate the field variables measured in a frame of reference moving with a constant velocity to the field variables measured in the laboratory frame. We then discuss constituent relations for moving media.

We confine our attention to quasi-static electric and magnetic field systems. As discussed in the following sections, this means that we consider Galilean transformations that are appropriate also for Newtonian mechanics. This approach is the logical extension of a division of electromechanics into electric and magnetic field systems. The relationship of Galilean and Lorentz transformations and the relevance of Einstein relativity are discussed at the appropriate points.

Our treatment of relative motion is based on two postulates: (a) the equations of motion, including Maxwell's equations, are always written for an inertial coordinate system, that is, a coordinate system that is traveling with a velocity of constant magnitude and fixed direction; and (b) the laws of physics (e.g., Newton's laws and Maxwell's equations) are the same in every

inertial coordinate system. These postulates are normally associated with Einstein relativity but they are also valid for Galilean systems.*

In Sections 6.3 and 6.4 examples that demonstrate how field transformations, boundary conditions, and constituent relations (the subjects of Sections 6.1 to 6.3) are used in the analysis of practical problems are considered. Emphasis is given to the magnetic field system; Section 6.4 is devoted to a class of rotating machines we were not prepared to discuss in detail in Chapter 4. These commutator-type machines, which are also of considerable practical significance, illustrate the fundamental point of this chapter.

6.1 FIELD TRANSFORMATIONS

In this treatment we are interested in field phenomena that occur in systems with material media in relative motion. The Lorentz force was introduced as the definition of the \mathbf{E} and \mathbf{B} fields in (1.1.28) of Chapter 1:

$$\mathbf{f} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}. \quad (6.1.1)$$

This expression states that a charge q in motion with velocity \mathbf{v} with respect to an observer will experience the force \mathbf{f} when subjected to the fields \mathbf{E} and \mathbf{B} . Because Newton's laws must be the same in all inertial reference systems, another observer in a different reference frame will measure the same force on the charge but the charge will have a different velocity. It should be clear then that the two observers will measure different values of electric field intensity and magnetic flux density. The object of this section is to find the relations between electromagnetic quantities that are measured by two observers in uniform relative motion.

We have already stated that when an observer defines electromagnetic quantities he does so with the understanding that they are defined in his reference frame (coordinate system). Hence they are related by Maxwell's equations written in his coordinate system. It is a postulate of special relativity that physical laws, such as Maxwell's equations, must be the same in all inertial coordinate systems. We use this postulate to determine the relations between electromagnetic quantities measured in different inertial coordinate systems.

We define two inertial coordinate systems \mathbf{r} and \mathbf{r}' which are moving with a constant relative velocity \mathbf{v}' . The times t and t' measured by observers in the two coordinate systems are assumed to be the same

$$t = t'. \quad (6.1.2)$$

* For discussions of the postulates and consequences of the special theory of relativity see, for example, J. D. Jackson, *Classical Electrodynamics*, Wiley, New York, 1962, Chapters 11 and 12; L. Landau and E. Lifshitz, *The Classical Theory of Fields*, Addison-Wesley, Reading, Mass., 1951, Chapters 1 and 2.

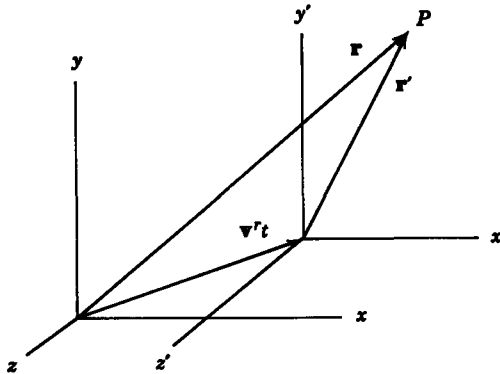


Fig. 6.1.1 Two inertial coordinate systems in relative motion.

We select the origins of the coordinate systems to coincide at $t = 0$. The relative geometry at a time t is illustrated in Fig. 6.1.1, from which we can obtain the relation between \mathbf{r} and \mathbf{r}' , the instantaneous position vectors of point P , as measured in the two coordinate systems

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}'t. \quad (6.1.3)$$

Equations 6.1.2 and 6.1.3 define a *Galilean coordinate transformation* between inertial coordinate systems. We now show that this transformation is consistent with the quasi-static Maxwell equations. An analogous procedure can be used to show that the Galilean transformation is also consistent with all mechanical equations of motion introduced in Chapter 2 or in later chapters.

It is worthwhile to interject at this point that the general form of Maxwell's equations cannot be transformed consistently by means of the Galilean transformation. This, in fact, is the basis for a relativistic treatment, which demands that the transformation be consistent (that the equations be invariant) and results in the Lorentz transformation.* Because the relativistic terms usually make no significant contribution to the electromechanics, it is most convenient to work with the quasi-static equations from the outset, as is done here. This avoids our having to discuss effects that we would end up neglecting in a practical context.

We obtain our transformations for field variables from the differential equations; consequently, before we derive the transformations, we need to consider the differential operators in the two coordinate systems and how they are related. The space differential operator ∇ for cartesian coordinates

* For the relativistic treatment see, for example, J. A. Stratton, *Electromagnetic Theory*, McGraw-Hill, New York, 1941, pp. 59-82.

x, y, z of the unprimed coordinate system \mathbf{r} is

$$\nabla = \left(\mathbf{i}_x \frac{\partial}{\partial x} + \mathbf{i}_y \frac{\partial}{\partial y} + \mathbf{i}_z \frac{\partial}{\partial z} \right). \quad (6.1.4)$$

The space differential operator ∇' for coordinates x', y', z' of the primed coordinate system \mathbf{r}' is

$$\nabla' = \left(\mathbf{i}_x \frac{\partial}{\partial x'} + \mathbf{i}_y \frac{\partial}{\partial y'} + \mathbf{i}_z \frac{\partial}{\partial z'} \right). \quad (6.1.5)$$

To determine the relation between ∇ and ∇' we first write out the three coordinates of the vector equation (6.1.3).

$$x' = x - v_x^r t, \quad (6.1.6a)$$

$$y' = y - v_y^r t, \quad (6.1.6b)$$

$$z' = z - v_z^r t. \quad (6.1.6c)$$

Consider a function $f'(x, y, z, t)$ which can also be written as $f'(x', y', z', t')$ by making substitutions from (6.1.2) and (6.1.6). The gradient of this function in the primed coordinate system is

$$\nabla' f' = \mathbf{i}_x \frac{\partial f'}{\partial x'} + \mathbf{i}_y \frac{\partial f'}{\partial y'} + \mathbf{i}_z \frac{\partial f'}{\partial z'}. \quad (6.1.7)$$

The chain rule of differentiation* is used to write

$$\frac{\partial f'}{\partial x} = \frac{\partial f'}{\partial x'} \frac{\partial x'}{\partial x} + \frac{\partial f'}{\partial y'} \frac{\partial y'}{\partial x} + \frac{\partial f'}{\partial z'} \frac{\partial z'}{\partial x} + \frac{\partial f'}{\partial t'} \frac{\partial t'}{\partial x}. \quad (6.1.8)$$

It is evident from (6.1.2) and (6.1.6) that

$$\frac{\partial x'}{\partial x} = 1; \quad \frac{\partial y'}{\partial x} = \frac{\partial z'}{\partial x} = \frac{\partial t'}{\partial x} = 0, \quad (6.1.9a)$$

$$\frac{\partial y'}{\partial y} = 1; \quad \frac{\partial x'}{\partial y} = \frac{\partial z'}{\partial y} = \frac{\partial t'}{\partial y} = 0, \quad (6.1.9b)$$

$$\frac{\partial z'}{\partial z} = 1; \quad \frac{\partial x'}{\partial z} = \frac{\partial y'}{\partial z} = \frac{\partial t'}{\partial z} = 0. \quad (6.1.9c)$$

We now use (6.1.8) and (6.1.9) with (6.1.7) to establish that

$$\nabla' f' = \nabla f'. \quad (6.1.10)$$

The scalar function f' may be a component of a vector; therefore we can use

* P. Franklin, *Methods of Advanced Calculus*, McGraw-Hill, New York, 1944, Chapter 2.

this same formalism to establish that for any vector $\mathbf{A}'(x', y', z', t')$ the space derivatives can be written as

$$\nabla' \cdot \mathbf{A}' = \nabla \cdot \mathbf{A}', \quad (6.1.11)$$

$$\nabla' \times \mathbf{A}' = \nabla \times \mathbf{A}'. \quad (6.1.12)$$

The same mathematical techniques are used to establish the relation between time derivatives. We again assume a function $f'(x', y', z', t')$ and write the time derivative in the unprimed system as

$$\frac{\partial f'}{\partial t} = \frac{\partial f'}{\partial t'} \frac{\partial t'}{\partial t} + \frac{\partial f'}{\partial x'} \frac{\partial x'}{\partial t} + \frac{\partial f'}{\partial y'} \frac{\partial y'}{\partial t} + \frac{\partial f'}{\partial z'} \frac{\partial z'}{\partial t}. \quad (6.1.13)$$

From (6.1.2) and (6.1.6) it follows that

$$\begin{aligned} \frac{\partial t'}{\partial t} &= 1; & \frac{\partial x'}{\partial t} &= -v_x^r, \\ \frac{\partial y'}{\partial t} &= -v_y^r; & \frac{\partial z'}{\partial t} &= -v_z^r. \end{aligned} \quad (6.1.14)$$

Substitution of these results into (6.1.13) yields

$$\frac{\partial f'}{\partial t} = \frac{\partial f'}{\partial t'} - \left(v_x^r \frac{\partial}{\partial x'} + v_y^r \frac{\partial}{\partial y'} + v_z^r \frac{\partial}{\partial z'} \right) f'. \quad (6.1.15)$$

The term in parentheses can be written as $\mathbf{v}^r \cdot \nabla'$; thus (6.1.15) is written in the form

$$\frac{\partial f'}{\partial t} = \frac{\partial f'}{\partial t'} - (\mathbf{v}^r \cdot \nabla') f'. \quad (6.1.16)$$

We use (6.1.10) to write this result in the alternative form

$$\frac{\partial f'}{\partial t'} = \frac{\partial f'}{\partial t} + (\mathbf{v}^r \cdot \nabla) f'. \quad (6.1.17)$$

The function f' can be a component of a vector; thus, if we define a vector $\mathbf{A}'(x', y', z', t')$, the same mathematical process leads to

$$\frac{\partial \mathbf{A}'}{\partial t'} = \frac{\partial \mathbf{A}'}{\partial t} + (\mathbf{v}^r \cdot \nabla) \mathbf{A}'. \quad (6.1.18)$$

Suppose that the unprimed frame is the fixed or laboratory frame. Then, from the left-hand side of (6.1.18) it is clear that the right-hand side is *the rate of change with respect to time of \mathbf{A}' for an observer moving with velocity \mathbf{v}^r* . This derivative, written in terms of the coordinates (x, y, z, t) of the fixed

frame, is used not only in this chapter but in many of the chapters that follow. Hence it is designated

$$\frac{DA'}{Dt} \equiv \frac{\partial A'}{\partial t} + (\mathbf{v}^r \cdot \nabla)A' \tag{6.1.19}$$

and is called the *substantial* or *convective derivative*. An example will help to clarify the significance of this derivative.

Example 6.1.1. To illustrate the significance of (6.1.19) consider an example in which A' is the displacement of a surface from the x - z plane given by

$$A' = \xi(x, t)\mathbf{i}_y, \tag{a}$$

as shown in Fig. 6.1.2a. The function ξ gives the y -coordinate of the surface. At a given position on the surface this y -coordinate has the same value, no matter whether it is viewed from the fixed frame or from a (primed) frame moving in the x -direction with velocity $\mathbf{v}^r = V\mathbf{i}_x$; that is, $A' = A$ and $\xi' = \xi$ for this particular case. If we evaluate (6.1.19), using (a), it follows that

$$\frac{DA'}{Dt} = \left(\frac{\partial \xi}{\partial t} + V \frac{\partial \xi}{\partial x} \right) \mathbf{i}_y. \tag{b}$$

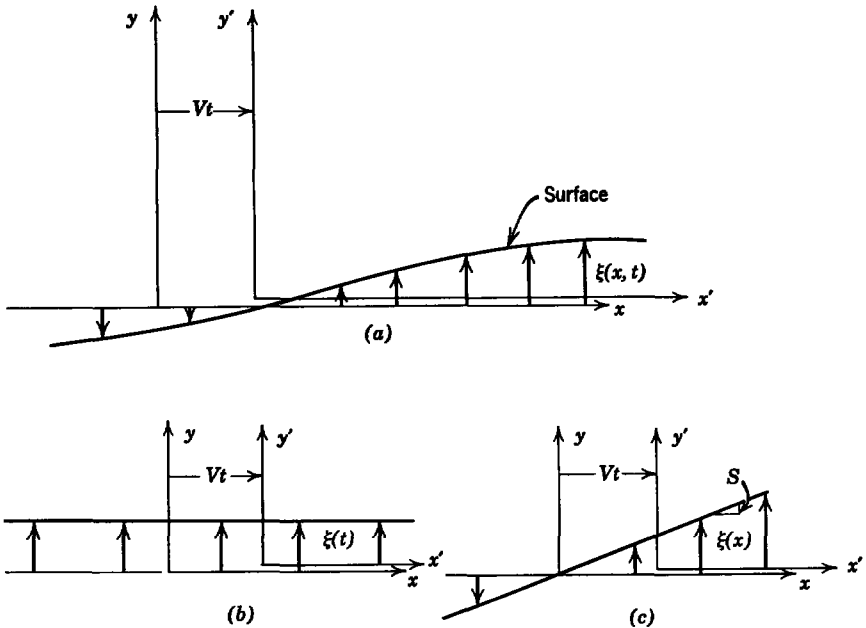


Fig. 6.1.2 (a) A surface described by $y = \xi(x, t)$ has an elevation above the x - z plane which is the same whether viewed from the moving (primed) frame or the fixed frame ($\xi' = \xi$); (b) ξ is independent of position so that only the first term in (6.1.19) makes a contribution to $D|Dt$; (c) ξ is independent of time and only the second term in (6.1.19) makes a contribution.

The significance of the two terms in this expression can be understood by considering limiting cases in which they, individually, make the sole contribution to the substantial derivative.

Figure 6.1.2*b* shows a surface ξ that is independent of position (x or x') but has a displacement that varies with time. Then the second term in (b) is zero and the rate of change for an observer moving with velocity V is the same as for a stationary observer. This we would have known from Fig. 6.1.2*b* without recourse to a mathematical equation.

In Fig. 6.1.2*c* the surface elevation is independent of time, since $\xi = \xi(x)$ and the rate of change of ξ with respect to time for the fixed observer is zero. By contrast the time rate of change for an observer moving with the velocity V is

$$\frac{DA'}{Dt} = V \frac{\partial \xi}{\partial x} i_y. \quad (c)$$

This result is not surprising either, because an observer in the moving frame travels to the right with a velocity V and sees a deflection ξ that increases in proportion to the slope of the surface $\partial \xi / \partial x$ and in proportion to the velocity V . In particular, if

$$\xi = Sx, \quad (d)$$

then from (c)

$$\frac{DA'}{Dt} = VS i_y. \quad (e)$$

This result could be obtained by inspection of Fig. 6.1.2*c*.

We shall find it useful later to write (6.1.18) in a different form. Because \mathbf{v}^r is constant, a vector identity* makes it possible to write (6.1.18) in the form

$$\frac{\partial \mathbf{A}'}{\partial t'} = \frac{\partial \mathbf{A}'}{\partial t} + \mathbf{v}^r (\nabla \cdot \mathbf{A}') - \nabla \times (\mathbf{v}^r \times \mathbf{A}'). \quad (6.1.20)$$

We are now in a position to obtain transformations for electromagnetic quantities from the field equations for magnetic and electric field systems.

6.1.1 Transformations for Magnetic Field Systems

The differential equations that define the relations of the field quantities to sources in quasi-static magnetic field systems were given in Section 6.0 (6.0.1) to (6.0.5) and are repeated here for convenience:

$$\nabla \times \mathbf{H} = \mathbf{J}_f, \quad (6.1.21)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (6.1.22)$$

$$\nabla \cdot \mathbf{J}_f = 0, \quad (6.1.23)$$

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t}, \quad (6.1.24)$$

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}). \quad (6.1.25)$$

* $\nabla \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \nabla) \mathbf{a} - (\mathbf{a} \cdot \nabla) \mathbf{b} + \mathbf{a} (\nabla \cdot \mathbf{b}) - \mathbf{b} (\nabla \cdot \mathbf{a})$.

These equations describe the field quantities measured by an observer who is fixed in the unprimed inertial coordinate system of Fig. 6.1.1.

It is a postulate of special relativity that physical laws must be the same in any inertial coordinate system. Consequently, we write the equations to describe the field quantities measured by an observer who is fixed in the primed inertial coordinate system of Fig. 6.1.1 as

$$\nabla' \times \mathbf{H}' = \mathbf{J}'_f, \quad (6.1.26)$$

$$\nabla' \cdot \mathbf{B}' = 0, \quad (6.1.27)$$

$$\nabla' \cdot \mathbf{J}'_f = 0, \quad (6.1.28)$$

$$\nabla' \times \mathbf{E}' = -\frac{\partial \mathbf{B}'}{\partial t'}, \quad (6.1.29)$$

$$\mathbf{B}' = \mu_0(\mathbf{H}' + \mathbf{M}'). \quad (6.1.30)$$

Use is now made of (6.1.11), (6.1.12), and (6.1.20) to express (6.1.26) to (6.1.29) in the equivalent forms

$$\nabla \times \mathbf{H}' = \mathbf{J}'_f, \quad (6.1.31)$$

$$\nabla \cdot \mathbf{B}' = 0, \quad (6.1.32)$$

$$\nabla \cdot \mathbf{J}'_f = 0, \quad (6.1.33)$$

$$\nabla \times (\mathbf{E}' - \mathbf{v}' \times \mathbf{B}') = -\frac{\partial \mathbf{B}'}{\partial t}. \quad (6.1.34)$$

We have made use of (6.1.32) to simplify the form of (6.1.34).

It has been postulated that (6.1.31) to (6.1.34) describe the same physical laws as (6.1.21) to (6.1.24). A comparison of the two sets of equations shows that a consistent set of transformations which satisfies this requirement is

$$\mathbf{H}' = \mathbf{H}, \quad (6.1.35)$$

$$\mathbf{J}'_f = \mathbf{J}_f, \quad (6.1.36)$$

$$\mathbf{B}' = \mathbf{B}, \quad (6.1.37)$$

$$\mathbf{E}' = \mathbf{E} + \mathbf{v}' \times \mathbf{B}. \quad (6.1.38)$$

We also use (6.1.35) and (6.1.37) with (6.1.25) and (6.1.30) to obtain the transformation for magnetization density

$$\mathbf{M}' = \mathbf{M}. \quad (6.1.39)$$

The transformations of (6.1.35) to (6.1.39) relate the values of electromagnetic quantities in a quasi-static, magnetic field system that would be

measured by two observers in relative motion with constant relative velocity \mathbf{v}^r at a particular point in space at a given instant of time. Note that there is no contradiction or inconsistency among these transformations as there would have been had we kept terms such as the displacement current in Ampère's law. Note also that the transformation for free current density (6.1.36) indicates that current flow by the convection of net free charge is consistently neglected in a magnetic field system. We can now return to the integral form of the magnetic system equations (Table 1.2) to see that the definition of \mathbf{E}' postulated there is consistent with what we have found here. Still another derivation of the integral form of Ampère's law for deforming contours of integration is given in Section B.4.1.

It is interesting to interpret (6.1.38) in terms of the Lorentz force (6.1.1). Consider a charge q at rest in the primed coordinate system. The force measured by an observer in that system is simply

$$\mathbf{f}' = q\mathbf{E}'.$$

An observer in the unprimed system who measures fields \mathbf{E} and \mathbf{B} will see the charge moving with a velocity \mathbf{v}^r and will therefore describe the force as

$$\mathbf{f} = q\mathbf{E} + q\mathbf{v}^r \times \mathbf{B}.$$

The transformation of (6.1.38) is just the relation between \mathbf{E} and \mathbf{E}' that must exist if the force on the charge is to be independent of the coordinate system in which it is expressed. Some writers actually use the Lorentz force to obtain the transformation for the electric field rather than the differential equations as we have.* Although this can be done, it is important to see that there is a close connection between the field transformations and the field equations. The field equations for the magnetic field systems do not include the displacement current, and it would be *inconsistent* to use field transformations based on equations that did not include this same approximation. For this reason it is not surprising that in the next section a *different* set of field transformations is found for the electric field systems.

Example 6.1.2. The most interesting of the field transformations introduced in this section is given by (6.1.38) and it is important to understand the close connection between this expression for \mathbf{E}' in terms of the fields in the fixed frame and the lumped parameter models of preceding chapters. For this purpose consider the idealized problem shown in Fig. 6.1.3, in which a pair of perfectly conducting plates are shorted by a conducting bar. The bar moves to the right with the velocity V and there is a uniform magnetic flux density \mathbf{B} imposed in the z -direction by an external source. We assume that the plates are terminated at the left in an essentially open circuit so that no currents flow to make additions to the field \mathbf{B} .

* R. M. Fano, L. J. Chu, and R. B. Adler, *Electromagnetic Fields, Energy, and Forces*, Wiley, New York, 1960, p. 390.

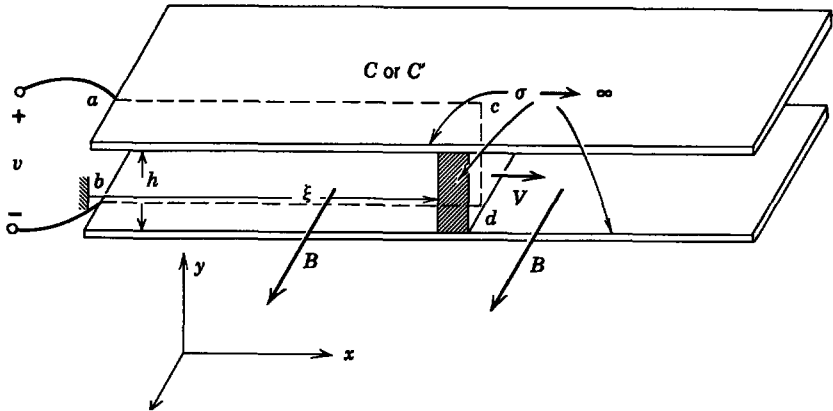


Fig. 6.1.3 A pair of parallel perfectly conducting plates are short-circuited by a moving perfectly conducting bar. Because of the magnetic field B , a voltage v is induced which can be computed either by integrating the induction equation around the fixed loop C' that passes through the bar or by integrating the induction equation around a loop C that expands in area as the bar moves to the right. The field transformation of (6.1.38) guarantees that both integrations will give the same result.

First recall how the voltage v is computed in Chapter 2. A contour C , as shown in Fig. 6.1.3, passes through the perfectly conducting bar. Then the induction equation is written in the form (2.1.6) and (2.1.7)

$$\oint_C \mathbf{E}' \cdot d\mathbf{l} = - \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} da, \quad (\text{a})$$

where \mathbf{E}' is the electric field in the frame of the conductor. Hence the integral of \mathbf{E}' along the contour $a-c-d-b$ makes no contribution and (a) reduces to the familiar form

$$v = \frac{d\lambda}{dt}, \quad (\text{b})$$

where

$$\lambda = \int_S \mathbf{B} \cdot \mathbf{n} da = -\xi h B. \quad (\text{c})$$

In the viewpoint represented by this derivation the voltage v arises because the contour C is expanding, thus enclosing more magnetic flux. In particular (b) and (c) give

$$v = -hBV. \quad (\text{d})$$

The field transformations make it possible to take an alternative approach to this problem. The integral form of the induction equation can also be written for a contour that is fixed in space

$$\oint_{C'} \mathbf{E} \cdot d\mathbf{l} = - \frac{d}{dt} \int_{S'} \mathbf{B} \cdot \mathbf{n} da \quad (\text{e})$$

This expression has the same form as (a), but now C' and S' are fixed and \mathbf{E} is the electric field intensity evaluated in the fixed frame. In the present example we can consider the

contour C' shown in Fig. 6.1.3, but even though this contour has the same instantaneous position as before it is now fixed in space rather than moving. As a direct consequence the right-hand side of (e) vanishes (remember, we assume that B is constant). If we further recognize that the integral of \mathbf{E} through the perfectly conducting plates from a - c and d - b makes no contribution, (e) reduces to

$$\int_b^a \mathbf{E} \cdot d\mathbf{l} + \int_c^d \mathbf{E} \cdot d\mathbf{l} = 0. \quad (\text{f})$$

In the region in which the terminals are located we assume (as in the preceding approach) that there is no time-varying magnetic field so that $\mathbf{E} = -\nabla\phi$ and

$$\int_b^a \mathbf{E} \cdot d\mathbf{l} = -(\phi_a - \phi_b) = -v. \quad (\text{g})$$

Hence (f) reduces to

$$v = \int_c^d \mathbf{E} \cdot d\mathbf{l}. \quad (\text{h})$$

The remaining integration from c - d must provide the voltage v . Note that this "speed voltage" is given by the term on the right in (a), but is now accounted for by the term on the left in (e). This term can be evaluated by recognizing that because $\mathbf{E}' = 0$ in the bar (6.1.38)

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B} = V B \mathbf{i}_y. \quad (\text{i})$$

This result can be incorporated into (h) to give

$$v = -hBV, \quad (\text{j})$$

which will be recognized as the same result obtained with the deforming contour of integration (d).

6.1.2 Transformations for Electric Field Systems

The differential equations that define the fields and their relations to sources in quasi-static, electric field systems were given in Table 1.2.

$$\nabla \times \mathbf{E} = 0, \quad (6.1.40)$$

$$\nabla \cdot \mathbf{D} = \rho_f, \quad (6.1.41)$$

$$\nabla \cdot \mathbf{J}_f = -\frac{\partial \rho_f}{\partial t}, \quad (6.1.42)$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}, \quad (6.1.43)$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}. \quad (6.1.44)$$

These equations describe the field quantities measured by an observer who is fixed in the unprimed inertial coordinate system of Fig. 6.1.1.

Our procedure here is analogous to that of the preceding section. We

recognize, by postulate, that these physical laws must be the same in any other inertial coordinate system. We write them for the primed inertial coordinate system of Fig. 6.1.1 as

$$\nabla' \times \mathbf{E}' = 0, \quad (6.1.45)$$

$$\nabla' \cdot \mathbf{D}' = \rho'_f, \quad (6.1.46)$$

$$\nabla' \cdot \mathbf{J}'_f = -\frac{\partial \rho'_f}{\partial t'}, \quad (6.1.47)$$

$$\nabla' \times \mathbf{H}' = \mathbf{J}'_f + \frac{\partial \mathbf{D}'}{\partial t'}, \quad (6.1.48)$$

$$\mathbf{D}' = \epsilon_0 \mathbf{E}' + \mathbf{P}'. \quad (6.1.49)$$

We now use (6.1.11), (6.1.12), (6.1.17)*, and (6.1.20) to express (6.1.45) to (6.1.48) in the forms

$$\nabla \times \mathbf{E}' = 0, \quad (6.1.50)$$

$$\nabla \cdot \mathbf{D}' = \rho'_f, \quad (6.1.51)$$

$$\nabla \cdot (\mathbf{J}'_f + \rho'_f \mathbf{v}') = -\frac{\partial \rho'_f}{\partial t}, \quad (6.1.52)$$

$$\nabla \times (\mathbf{H}' + \mathbf{v}' \times \mathbf{D}') = \mathbf{J}'_f + \rho'_f \mathbf{v}' + \frac{\partial \mathbf{D}'}{\partial t}. \quad (6.1.53)$$

We have used (6.1.51) to obtain (6.1.53).

Using the postulate that (6.1.40) to (6.1.43) express the same physical laws as (6.1.50) to (6.1.53) we obtain the following consistent set of transformations:

$$\mathbf{E}' = \mathbf{E}, \quad (6.1.54)$$

$$\mathbf{D}' = \mathbf{D}, \quad (6.1.55)$$

$$\rho'_f = \rho_f, \quad (6.1.56)$$

$$\mathbf{H}' = \mathbf{H} - \mathbf{v}' \times \mathbf{D}, \quad (6.1.57)$$

$$\mathbf{J}'_f = \mathbf{J}_f - \rho_f \mathbf{v}'^r. \quad (6.1.58)$$

We use (6.1.54) and (6.1.55) with (6.1.44) and (6.1.49) to obtain the transformation for polarization density

$$\mathbf{P}' = \mathbf{P}. \quad (6.1.59)$$

Note that these transformations are consistent with those postulated in

* Remember \mathbf{v}'^r is constant, so $(\mathbf{v}'^r \cdot \nabla) f' = \nabla \cdot (\mathbf{v}'^r f)$.

Section 1.1.2*b* to express the integral form of the equations for an electric field system. Yet another derivation of these integral laws is given in Section B.4.2.

Example 6.1.3. The simple significance of the field transformations for the electric field systems can be illustrated by means of the parallel-plate capacitor shown in Fig. 6.1.4. Here a battery is used to induce surface charges on the plates, as shown. Hence in the laboratory (unprimed) frame there is an electric field intensity between the plates related to the surface charge density by

$$\mathbf{E} = \frac{\sigma_f}{\epsilon_0} \mathbf{i}_y, \tag{a}$$

whereas there is no magnetic field \mathbf{H} . (We assume here that there are no external currents that would induce a magnetic field in the laboratory frame.) For the purpose of the example consider that the plates have infinite extent in the x -direction. Then, according to the electric field transformations, an observer in the moving frame of Fig. 6.1.4*a* would measure the magnetic field intensity (6.1.57)

$$\mathbf{H}' = -V\epsilon_0 E_y \mathbf{i}_z = -V\sigma_f \mathbf{i}_z. \tag{b}$$

This magnetic field is present in the moving frame because in that frame of reference the surface charges give rise to surface currents. These currents induce the field \mathbf{H}' . To see this,

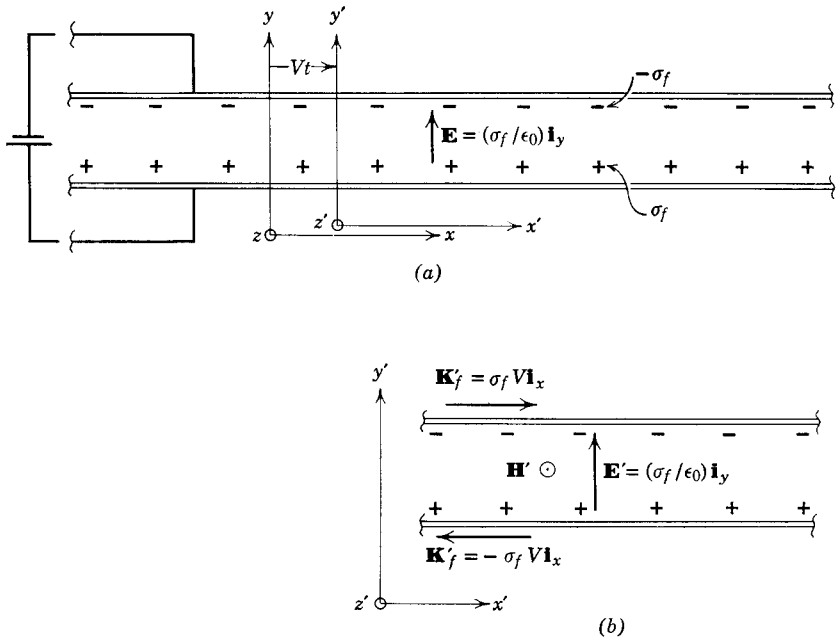


Fig. 6.1.4 (a) A parallel-plate capacitor is biased by a voltage source so that surface charges of opposite polarity are induced; (b) the fields in the moving frame can be found by computing the magnetic field induced by the convection of the surface charges or by using the field transformation of (6.1.57).

consider that the plates are being viewed from the moving frame of reference, as shown in Fig. 6.1.4*b*. In this frame surface currents flow in the x -direction, thus giving rise to the magnetic field of (b). Note that implicit to this reasoning is the transformation for the free current density (6.1.58).

In this and the preceding section we have obtained transformations that describe the relations between the field quantities measured by two observers in relative motion with a constant relative velocity. These transformations have been obtained for quasi-static systems and are valid only for such systems. We have stated that these transformations are consistent. By this we mean that we can use our transformation relations to transform field quantities repeatedly back and forth between two inertial reference frames without generating inconsistencies. A summary of transformations is given in Table 6.1.

6.2 BOUNDARY CONDITIONS

It is often found that electrical properties change significantly over distances that are infinitesimal with respect to significant dimensions of an electromechanical system. Such changes occur at the surface of a medium or at an interface between two media. In such cases we can represent the abrupt changes mathematically as spatial discontinuities in the electromagnetic variables. It is these discontinuities that provide boundary conditions on the electromagnetic variables.

The conventional treatment in electromagnetic theory considers conditions at stationary boundaries.* Because we are interested here in electromechanics we require boundary conditions at moving boundaries. The conditions derived are correct only for quasi-static systems.

First, we define the surface Σ , illustrated in Fig. 6.2.1, which separates medium a from medium b . Media a and b move with velocities \mathbf{v}^a and \mathbf{v}^b with respect to the inertial coordinate system \mathbf{r} in which all field and source quantities (\mathbf{E} , \mathbf{B} , \mathbf{P} , \mathbf{M} , \mathbf{J}_f , ρ_f) are defined. Superscripts a and b indicate the medium in which a quantity exists. The normal vector \mathbf{n} is defined as normal to the surface Σ and has a positive direction from medium b to medium a , as shown.

In order that the surface Σ may be a well-defined boundary between the two media, the normal components of the two velocities \mathbf{v}^a and \mathbf{v}^b must be the same at the surface; thus

$$\mathbf{n} \cdot (\mathbf{v}^a - \mathbf{v}^b) = 0. \quad (6.2.1)$$

If this condition is not satisfied, the two materials are diffusing through each other or moving apart, leaving a vacuum between them. In either case a

* Fano et al., *op. cit.*, pp. 86–89.

Table 6.1 Differential Equations, Transformations, and Boundary Conditions for Quasi-static Electromagnetic Systems with Moving Media

| | Differential Equations | | Transformations | | Boundary Conditions | |
|---|--|----------------------------|---|----------|---|----------|
| Magnetic field systems | $\nabla \times \mathbf{H} = \mathbf{J}_f$ | (1.1.1) | $\mathbf{H}' = \mathbf{H}$ | (6.1.35) | $\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b) = \mathbf{K}_f$ | (6.2.14) |
| | $\nabla \cdot \mathbf{B} = 0$ | (1.1.2) | $\mathbf{B}' = \mathbf{B}$ | (6.1.37) | $\mathbf{n} \cdot (\mathbf{B}^a - \mathbf{B}^b) = 0$ | (6.2.7) |
| | $\nabla \cdot \mathbf{J}_f = 0$ | (1.1.3) | $\mathbf{J}'_f = \mathbf{J}_f$ | (6.1.36) | $\mathbf{n} \cdot (\mathbf{J}_f^a - \mathbf{J}_f^b) + \nabla_{\Sigma} \cdot \mathbf{K}_f = 0$ | (6.2.9) |
| | $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ | (1.1.5) | $\mathbf{E}' = \mathbf{E} + \mathbf{v}^r \times \mathbf{B}$ | (6.1.38) | $\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b) = v_n(\mathbf{B}^a - \mathbf{B}^b)$ | (6.2.22) |
| | $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ | (1.1.4) | $\mathbf{M}' = \mathbf{M}$ | (6.1.39) | | |
| Electric field systems | $\nabla \times \mathbf{E} = 0$ | (1.1.11) | $\mathbf{E}' = \mathbf{E}$ | (6.1.54) | $\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b) = 0$ | (6.2.31) |
| | $\nabla \cdot \mathbf{D} = \rho_f$ | (1.1.12) | $\mathbf{D}' = \mathbf{D}$ | (6.1.55) | $\mathbf{n} \cdot (\mathbf{D}^a - \mathbf{D}^b) = \sigma_f$ | (6.2.33) |
| | | | $\rho'_f = \rho_f$ | (6.1.56) | | |
| | $\nabla \cdot \mathbf{J}_f = -\frac{\partial \rho_f}{\partial t}$ | (1.1.14) | $\mathbf{J}'_f = \mathbf{J}_f - \rho_f \mathbf{v}^r$ | (6.1.58) | $\mathbf{n} \cdot (\mathbf{J}_f^a - \mathbf{J}_f^b) + \nabla_{\Sigma} \cdot \mathbf{K}_f = v_n(\rho_f^a - \rho_f^b) - \frac{\partial \sigma_f}{\partial t}$ | (6.2.36) |
| | $\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$ | (1.1.15) | $\mathbf{H}' = \mathbf{H} - \mathbf{v}^r \times \mathbf{D}$ | (6.1.57) | $\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b) = \mathbf{K}_f + v_n \mathbf{n} \times [\mathbf{n} \times (\mathbf{D}^a - \mathbf{D}^b)]$ | (6.2.38) |
| $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$ | (1.1.13) | $\mathbf{P}' = \mathbf{P}$ | (6.1.59) | | | |

well-defined boundary of the type postulated does not exist. From the electromagnetic theory viewpoint no requirement is necessary on the tangential velocities of the media at the boundary. Consequently, the media can slide past each other at the boundary with no restrictions on the tangential components of the velocities. We shall find that the boundary conditions depend on the normal component of the velocity of the boundary. One or both media may be vacuum.

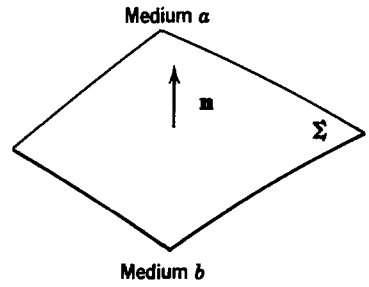


Fig. 6.2.1 Surface separating two media.

In general, the surface Σ is not plane and it is moving and deforming. When a mathematical description of the surface is given, we must be able to evaluate the normal vector \mathbf{n} . In the coordinate system \mathbf{r} a surface can be described by the general functional form

$$f(\mathbf{r}, t) = 0. \quad (6.2.2)$$

The normal vector \mathbf{n} can then be evaluated as

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|}. \quad (6.2.3)$$

This statement is familiar from electromagnetic field theory; that is, if we assume that (6.2.2) defines one equipotential surface of the set

$$f(\mathbf{r}, t) = \phi,$$

where ϕ is the potential, the electric field is the negative gradient of the potential and is normal to an equipotential surface. Hence we can think of the normal vector \mathbf{n} , defined by (6.2.3) as the negative of the normalized electric field, evaluated at the zero-potential surface.

Example 6.2.1. To illustrate the manner in which a surface is represented by an expression like that in (6.2.2), consider the surface defined in Fig. 6.2.2 in which the height of the surface above the x - z plane is given by

$$y = A \sin \omega t \cos \frac{2\pi x}{l} + B. \quad (a)$$

where A , B , ω , and l are positive constants. This represents a surface whose position is independent of z and whose height varies as the cosine function with x . The amplitude of the variation in height at a fixed position x is a sinusoidal function of time.

To obtain the equation for the surface in the form of (6.2.2) we write

$$f(x, y, t) = y - A \sin \omega t \cos \frac{2\pi x}{l} - B = 0. \quad (b)$$

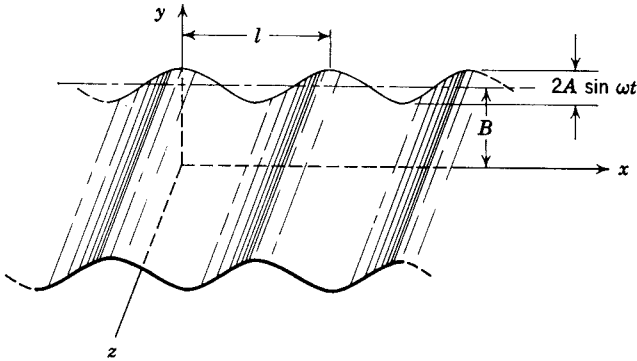


Fig. 6.2.2 Example of surface that varies in space and time.

Taking the gradient of this expression yields

$$\nabla f = \mathbf{i}_x \frac{2\pi A}{l} \sin \omega t \sin \frac{2\pi x}{l} + \mathbf{i}_y.$$

The magnitude of this gradient is

$$|\nabla f| = \left(1 + \frac{4\pi^2 A^2}{l^2} \sin^2 \omega t \sin^2 \frac{2\pi x}{l} \right)^{1/2}$$

and the normal vector is

$$\mathbf{n} = \frac{\mathbf{i}_x (2\pi A/l) \sin \omega t \sin (2\pi x/l) + \mathbf{i}_y}{\sqrt{1 + (4\pi^2 A^2/l^2) \sin^2 \omega t \sin^2 (2\pi x/l)}} \quad (c)$$

This normal vector becomes \mathbf{i}_y at

$$\frac{2\pi x}{l} = \pm n\pi; \quad n = 0, 1, 2, \dots;$$

that is, on the crests and in the troughs of the corrugations the normal vector is vertical. At other values of x the normal vector is not vertical and its direction can be determined from (c). The direction of the normal can be reversed by defining f as the negative of (b). Consequently, we can label media a and b and make sure that the definition of f yields a normal vector as described in Fig. 6.2.1 or we can define f and label the materials after the direction of the normal has been determined.

6.2.1 Boundary Conditions for Magnetic Field Systems

For studying boundary conditions in a quasi-static, magnetic field system we assume that the surface Σ carries a free surface current density \mathbf{K}_f (amperes per meter) and a free surface charge density σ_f (coulombs per square meter). The free surface current density \mathbf{K}_f is part of the primary excitation, whereas the surface charge density is a quantity that can be determined from auxiliary relations after the fields have been determined.

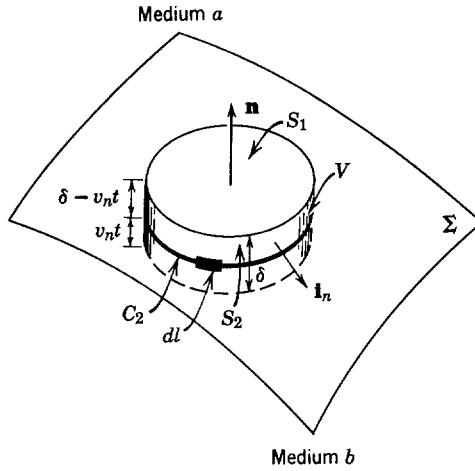


Fig. 6.2.3 Geometry for calculating discontinuities in normal components of field vectors.

Consider first the equations in integral form that determine how the sources excite the fields. They were given in (1.1.20) and (1.1.21) and are

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J}_f \cdot \mathbf{i}_n da, \quad (6.2.4)$$

$$\oint_S \mathbf{B} \cdot \mathbf{i}_n da = 0. \quad (6.2.5)$$

Here we have introduced \mathbf{i}_n as the unit vector perpendicular to the area of integration so that it can be distinguished from the vector \mathbf{n} normal to the surface Σ . To find the boundary condition imposed by (6.2.5) we define a small, right-circular cylindrical volume V , enclosed by a surface S consisting of the top and bottom surfaces of areas S_1 and a lateral surface of height δ and area S_2 , as shown in Fig. 6.2.3. The volume V is fixed in the inertial coordinate system \mathbf{r} and is so oriented that it intersects the boundary Σ as shown in Fig. 6.2.3. The surface S_1 is small enough that the boundary Σ can be assumed plane in its vicinity and the top and bottom surfaces S_1 are parallel to the boundary Σ . Hence the vector \mathbf{n} is normal to both Σ and S_1 . We assume that the height δ of the pillbox is so small that the lateral area S_2 is much smaller than the area of surface S_1 . When we integrate (6.2.5) over the surface S and assume that S_1 is so small that \mathbf{B} does not change appreciably over S_1 , we obtain

$$(\mathbf{B}^a \cdot \mathbf{n})S_1 - (\mathbf{B}^b \cdot \mathbf{n})S_1 = 0. \quad (6.2.6)$$

We cancel S_1 from this expression to obtain

$$\mathbf{n} \cdot (\mathbf{B}^a - \mathbf{B}^b) = 0. \quad (6.2.7)$$

Equation 6.2.7 states that the normal component of \mathbf{B} must be continuous at the boundary. This is the same result obtained in the electromagnetic theory of stationary systems; thus the motion has not affected this boundary condition.

Like the flux density the free-current density \mathbf{J}_f in a quasi-static magnetic field system has no divergence [see (1.1.3) of Table 1.2]. However, when deriving the boundary condition on \mathbf{J}_f , the integral expression of (1.1.22) of Table 1.2 must be used with due regard for surface currents at the discontinuity. The current density \mathbf{J}_f , unlike \mathbf{B} , can be singular. If we use (1.1.22) ($\oint \mathbf{J}_f \cdot \mathbf{i}_n da = 0$) with the pillbox in Fig. 6.2.3 and neglect the contribution of volume current density \mathbf{J}_f over the lateral surface S_2 , we obtain

$$(\mathbf{J}_f^a \cdot \mathbf{n})S_1 - (\mathbf{J}_f^b \cdot \mathbf{n})S_1 + \oint_{C_2} \mathbf{K}_f \cdot \mathbf{i}_n dl = 0. \quad (6.2.8)$$

This is simply an expression of the fact that current into the pillbox from the two media must equal surface current across the contour C_2 because no appreciable free charge density (volume or surface) can exist in a magnetic field system. Dividing (6.2.8) by S_1 and taking the limit as $S_1 \rightarrow 0$ yields the desired boundary condition

$$\mathbf{n} \cdot (\mathbf{J}_f^a - \mathbf{J}_f^b) + \nabla_\Sigma \cdot \mathbf{K}_f = 0, \quad (6.2.9)$$

where

$$\nabla_\Sigma \cdot \mathbf{K}_f = \lim_{S_1 \rightarrow 0} \frac{\oint_{C_2} \mathbf{K}_f \cdot \mathbf{i}_n dl}{S_1} \quad (6.2.10)$$

is the surface (two-dimensional) divergence of \mathbf{K}_f applied in the plane of the surface Σ at the point in question.*

We consider next the boundary condition imposed by (6.2.4). For this purpose we use the contour C which encloses the open surface S and is fixed in the coordinate system \mathbf{r} . The contour instantaneously intersects the boundary, as illustrated in Fig. 6.2.4. The surface S is a plane rectangle and is small enough for the boundary Σ to be assumed plane in its vicinity. The surface S is perpendicular to Σ and the height δ of the contour is much smaller than the length L .

$$\delta \ll L$$

* The two-dimensional divergence is simply the sum of the derivatives of the two orthogonal components of a vector in the surface with respect to the distance in the component directions; for example, assume a surface Σ with normal vector $\mathbf{n} = \mathbf{i}_z$. A vector \mathbf{A} lying in the surface Σ will have only x - and y -components $\mathbf{A} = A_x \mathbf{i}_x + A_y \mathbf{i}_y$ and the surface divergence of this vector is $\nabla_\Sigma \cdot \mathbf{A} = [\mathbf{i}_x(\partial/\partial x) + \mathbf{i}_y(\partial/\partial y)] \cdot \mathbf{A} = \partial A_x/\partial x + \partial A_y/\partial y$. Note that the surface divergence in this case is just two terms of the volume divergence.

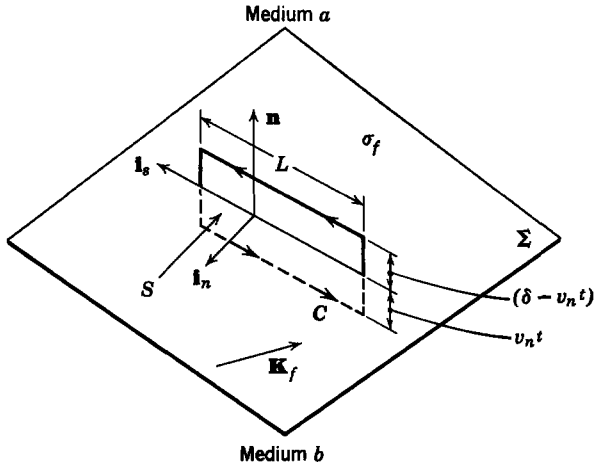


Fig. 6.2.4 Contour and surface for determining discontinuities in tangential components of field vectors.

The three unit vectors \mathbf{n} , \mathbf{i}_n , and \mathbf{i}_s , shown in Fig. 6.2.4, are mutually orthogonal. We assume that the contour C is small enough that fields do not vary appreciably over its length L . With these assumptions, and ignoring contributions from the ends (δ), we integrate (6.2.4) to obtain (after canceling out the length L)

$$(\mathbf{H}^a - \mathbf{H}^b) \cdot \mathbf{i}_s = \mathbf{K}_f \cdot \mathbf{i}_n. \quad (6.2.11)$$

This expression states that the discontinuity in the tangential component of \mathbf{H} in the direction of \mathbf{i}_s is equal to the component of \mathbf{K}_f perpendicular to \mathbf{i}_s .

Equation 6.2.11 can be put into a more useful form in the following way. We substitute

$$\mathbf{i}_s = \mathbf{i}_n \times \mathbf{n} \quad (6.2.12)$$

into (6.2.11) and use a vector identity* to obtain

$$[\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b)] \cdot \mathbf{i}_n = \mathbf{K}_f \cdot \mathbf{i}_n. \quad (6.2.13)$$

By definition, the vector \mathbf{K}_f lies in the boundary Σ . The vector $[\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b)]$ also lies in the boundary Σ . The vector \mathbf{i}_n has an arbitrary direction except that it also lies in the boundary Σ . Therefore from (6.2.13) we obtain the result

$$\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b) = \mathbf{K}_f. \quad (6.2.14)$$

Note once again that this is the same boundary condition obtained for stationary systems (it is independent of the boundary velocity).

* $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$.

We now derive the boundary condition for the electric field by starting with (1.1.23) written for a fixed contour C :

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = - \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{i}_n da. \quad (6.2.15)$$

We integrate (6.2.15) by using the contour C and surface S defined in Fig. 6.2.4. The restrictions on orientation and size in the derivation of (6.2.14) also apply here. By using the unit vectors and dimensions defined in Fig. 6.2.4 we obtain the contour integral

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = (\mathbf{E}^a - \mathbf{E}^b) \cdot \mathbf{i}_s L. \quad (6.2.16)$$

In this expression we have neglected the contribution to this integral from the sides perpendicular to Σ because we require that $\delta \ll L$.* The time origin is defined as the instant in which the surface Σ coincides with the lower edge of the contour. Thus, defining the normal component of velocity of the boundary as

$$v_n = \mathbf{n} \cdot \mathbf{v}^a = \mathbf{n} \cdot \mathbf{v}^b, \quad (6.2.17)$$

we write the surface integral in (6.2.15) as

$$\int_S \mathbf{B} \cdot \mathbf{i}_n da = [\mathbf{B}^a(\delta - v_n t) + \mathbf{B}^b(v_n t)] \cdot \mathbf{i}_n L. \quad (6.2.18)$$

In this expression we have assumed that δ is small enough that v_n does not change appreciably as the boundary Σ passes the contour C . Note that this does not require that v_n be a constant, for we shall shortly take the limit in which $\delta \rightarrow 0$.

We take the time derivative of (6.2.18)

$$\begin{aligned} \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{i}_n da &= -v_n(\mathbf{B}^a - \mathbf{B}^b) \cdot \mathbf{i}_n L \\ &+ \left[\frac{\partial v_n}{\partial t} t(\mathbf{B}^b - \mathbf{B}^a) + \frac{\partial \mathbf{B}^a}{\partial t} (\delta - v_n t) + \frac{\partial \mathbf{B}^b}{\partial t} (v_n t) \right] \cdot \mathbf{i}_n L. \end{aligned} \quad (6.2.19)$$

The time derivatives of the fields are finite in the two media: consequently, in the limit as $\delta \rightarrow 0$ the term in brackets on the right of (6.2.19) vanishes. (Note that by definition $\delta \geq v_n t$ so that the interval of time during which the surface is within the volume V is the largest time t with which we are concerned. Hence, as $\delta \rightarrow 0$, so also does the largest value of t .) We equate the negative of (6.2.19) to (6.2.16) and cancel the length L to obtain

$$(\mathbf{E}^a - \mathbf{E}^b) \cdot \mathbf{i}_s = v_n(\mathbf{B}^a - \mathbf{B}^b) \cdot \mathbf{i}_n. \quad (6.2.20)$$

* There is the implicit assumption here that although \mathbf{E} can be discontinuous at the boundary it must be finite. Otherwise we could not ignore the contribution to the integral along the sides of length δ .

By using the relation

$$\mathbf{i}_s = \mathbf{i}_n \times \mathbf{n}$$

and a vector identity,* we put (6.2.20) in the form

$$[\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b)] \cdot \mathbf{i}_n = v_n (\mathbf{B}^a - \mathbf{B}^b) \cdot \mathbf{i}_n. \quad (6.2.21)$$

The vector \mathbf{i}_n lies in the boundary Σ but otherwise it has an arbitrary direction. The vector $\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b)$ also lies in the boundary Σ . The normal component of \mathbf{B} is continuous [see (6.2.7)]; consequently, the vector $(\mathbf{B}^a - \mathbf{B}^b)$ lies in the boundary Σ . Therefore we conclude from (6.2.21) that

$$\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b) = v_n (\mathbf{B}^a - \mathbf{B}^b). \quad (6.2.22)$$

This is the desired boundary condition on the electric field.

We indicate an alternative method of deriving (6.2.22) by putting it in a different form. We define the velocity \mathbf{v} as

$$\mathbf{v} = \mathbf{n}v_n, \quad (6.2.23)$$

which is simply the normal velocity of the boundary Σ . We now write (6.2.22) in the form

$$\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b) = (\mathbf{n} \cdot \mathbf{v})(\mathbf{B}^a - \mathbf{B}^b) \quad (6.2.24)$$

and use a vector identity† with the boundary condition on the normal component of \mathbf{B} (6.2.7) to write (6.2.24) as

$$\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b) = -\mathbf{n} \times [\mathbf{v} \times (\mathbf{B}^a - \mathbf{B}^b)]. \quad (6.2.25)$$

When we define $\mathbf{E}^{a'}$ and $\mathbf{E}^{b'}$ as

$$\mathbf{E}^{a'} = \mathbf{E}^a + \mathbf{v} \times \mathbf{B}^a, \quad (6.2.26)$$

$$\mathbf{E}^{b'} = \mathbf{E}^b + \mathbf{v} \times \mathbf{B}^b, \quad (6.2.27)$$

we can rewrite (6.2.25) as

$$\mathbf{n} \times (\mathbf{E}^{a'} - \mathbf{E}^{b'}) = 0. \quad (6.2.28)$$

From the transformation of (6.1.38) we recognize that $\mathbf{E}^{a'}$ and $\mathbf{E}^{b'}$ are the electric fields that an observer will measure when he is in a coordinate system moving with the normal velocity of the boundary. In this coordinate system the boundary is at rest; consequently, as (6.2.24) indicates, the tangential component of electric field must be continuous, as it must be in any stationary system.‡ This idea can be used as the basis for an alternative derivation of the boundary condition on the tangential component of electric field, once the condition on a fixed boundary has been obtained. Note, however, that our

* $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$.

† $(\mathbf{a} \cdot \mathbf{b})\mathbf{c} = -\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b}(\mathbf{a} \cdot \mathbf{c})$.

‡ At least any stationary system in which \mathbf{E} is finite everywhere.

transformations were derived for an inertial coordinate system. The boundary condition just derived is not restricted to boundaries that move with a constant velocity.

Example 6.2.2. Consider the system illustrated in Fig. 6.2.5 in which a surface Σ that is perpendicular to the y -axis moves with a speed v in the y -direction

$$\mathbf{v} = \mathbf{i}_y v.$$

We assume that the surface Σ has an infinite extent in the x - and z -directions so we can write its instantaneous position y_s as

$$y_s = vt.$$

We have chosen $t = 0$ as the instant when the surface contains the origin.

The surface Σ is immersed in vacuum and carries a uniform surface current density

$$\mathbf{K}_f = \mathbf{i}_z K_f.$$

The system is constrained so that to the right of the surface ($y > y_s$) the fields are zero. A simple experiment in which this physical situation arises is shown in Fig. 6.2.5b, in which a moving conductor shorts parallel electrodes driven by a current source. Here the moving short is modeled as being very thin and carrying a surface current K_f . In practice, the moving short could be a sheet of highly ionized gas moving down a shock tube.

The electric and magnetic fields in the region ($y < y_s$) to the left of the surface Σ are to be found.

We select as the normal vector \mathbf{n} the vector \mathbf{i}_y ,

$$\mathbf{n} = \mathbf{i}_y,$$

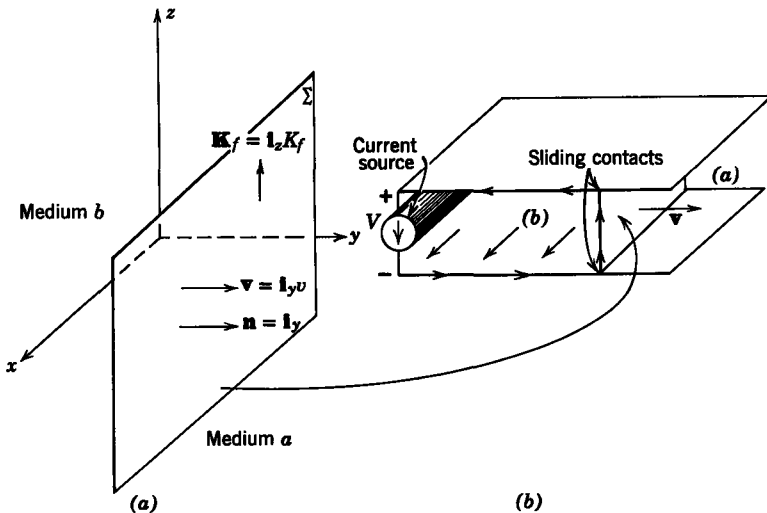


Fig. 6.2.5 (a) Plane surface in motion and carrying a surface current density; (b) the moving surface could constitute the moving conductor that short-circuits parallel plates excited by a current source distributed along the x -axis.

and thus medium b is $y < y_s$ and medium a is $y > y_s$. Our problem is specified such that

$$\mathbf{E}^a = 0 \quad \text{and} \quad \mathbf{H}^a = 0.$$

We first use (6.2.14) to find the magnetic field intensity \mathbf{H}^b :

$$\mathbf{i}_y \times (-\mathbf{H}^b) = \mathbf{i}_z K_f.$$

If we write \mathbf{H}^b in component form,

$$\mathbf{H}^b = \mathbf{i}_x H_x^b + \mathbf{i}_y H_y^b + \mathbf{i}_z H_z^b,$$

we can write the boundary condition as

$$\mathbf{i}_z H_x^b - \mathbf{i}_x H_z^b = \mathbf{i}_z K_f.$$

Equating components in this expression yields

$$H_x^b = K_f,$$

$$H_z^b = 0.$$

Thus in vacuum $\mathbf{B} = \mu_0 \mathbf{H}$ and we can use the boundary condition on the normal component of \mathbf{B} (6.2.7) with the given information that $\mathbf{H}^a = 0$ to arrive at the result

$$H_y^b = 0.$$

Thus the magnetic field intensity adjacent to the surface in region b is completely determined.

To find the electric field intensity to the left of the moving current sheet we use (6.2.22) to obtain

$$\mathbf{i}_y \times [-\mathbf{E}^b] = -v\mathbf{B}^b,$$

from which

$$\mathbf{E}^b = \mathbf{i}_z v \mu_0 K_f.$$

This is the electric field intensity to the left of the current sheet generated by the moving discontinuity in magnetic field intensity.

The discontinuity in electric field at the moving interface is necessary if concepts introduced earlier in this and preceding chapters are to remain consistent. We have already pointed out that the same boundary condition follows from the field transformation. If the moving surface is placed in the context of the problem shown in Fig. 6.2.5b, it is also possible to find the electric field behind the surface by using lumped parameter ideas to compute the voltage V , hence the electric field \mathbf{E} between the plates.

6.2.2 Boundary Conditions for Electric Field Systems

For studying the boundary conditions in a quasi-static electric field system we assume a boundary surface Σ that carries a surface charge density σ_f and a surface current density \mathbf{K}_f (see Figs. 6.2.3 and 6.2.4). The surface charge density σ_f is part of the primary excitation, whereas the surface current density \mathbf{K}_f simply accounts for the conduction or convection of charge. The magnetic field generated by \mathbf{K}_f can be computed once all of the other fields are known.

Two of the integral equations that determine the fields in a quasi-static electric field system are (see Table 1.2)

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0, \quad (6.2.29)$$

$$\oint_S \mathbf{D} \cdot \mathbf{i}_n \, da = \int_V \rho_f \, dV. \quad (6.2.30)$$

The results derived in the preceding section can be used to obtain the boundary conditions implied by these equations.

In the preceding section the boundary condition on the tangential component of \mathbf{E} was derived from (6.2.15) by using the contour defined in Fig. 6.2.4 to obtain the result of (6.2.22). We note that (6.2.29) is simply (6.2.15) with the right side set equal to zero; consequently, by setting the right side of (6.2.22) equal to zero, we obtain

$$\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b) = 0. \quad (6.2.31)$$

To derive the boundary condition on the normal component of \mathbf{D} we use the pillbox-shaped surface of Fig. 6.2.3 with the same restrictions on relative geometry that were used in deriving (6.2.7) in the preceding section. By performing the integration (6.2.30) and taking the limit as $\delta \rightarrow 0$, we obtain the result

$$(\mathbf{D}^a \cdot \mathbf{n})S_1 - (\mathbf{D}^b \cdot \mathbf{n})S_1 = \sigma_f S_1, \quad (6.2.32)$$

where S_1 is the area of the top and bottom of the closed surface S . Division of both sides by S_1 yields the desired boundary condition

$$\mathbf{n} \cdot (\mathbf{D}^a - \mathbf{D}^b) = \sigma_f, \quad (6.2.33)$$

which is the same as the corresponding boundary condition for stationary systems.

When the conductivity of a material is uniform, the free charges have only a transient existence in the bulk of the material. Surface charges play an important role in such cases. As we shall see in Chapter 7, the conduction process in the region of an interface is an important factor in many electric field systems. For this reason the boundary condition associated with the conservation of charge equation (Table 1.2)

$$\oint_S \mathbf{J}_f \cdot \mathbf{i}_n \, da = - \frac{d}{dt} \int_V \rho_f \, dV \quad (6.2.34)$$

assumes primary significance.

The evaluation of this boundary condition is much like the evaluation of the boundary condition on current density derived for magnetic field systems in (6.2.8) to (6.2.10), except that now we must include the effects of volume

and surface charge densities. Using the pillbox-shaped volume of Fig. 6.2.3 and neglecting the contribution of volume current density \mathbf{J}_f across the lateral surface S_2 , we find that (6.2.34) becomes

$$S_1 \mathbf{n} \cdot (\mathbf{J}_f^a - \mathbf{J}_f^b) + \oint_{C_2} \mathbf{K}_f \cdot \mathbf{i}_n dl = - \frac{d}{dt} [\rho_f^a S_1 (\delta - v_n t) + \rho_f^b S_1 v_n t + S_1 \sigma_f]. \quad (6.2.35)$$

In the limit, as $\delta \rightarrow 0$, then $S_1 \rightarrow 0$, this expression becomes

$$\mathbf{n} \cdot (\mathbf{J}_f^a - \mathbf{J}_f^b) + \nabla_\Sigma \cdot \mathbf{K}_f = v_n (\rho_f^a - \rho_f^b) - \frac{\partial \sigma_f}{\partial t}, \quad (6.2.36)$$

where we have used the definition of the surface divergence of surface current density ($\nabla_\Sigma \cdot \mathbf{K}_f$) given in (6.2.10). It should be pointed out here that surface current density occurs in electric field systems most often as the convection of free surface charge density, as indicated by the application of the transformation in (6.1.58) to surface current density and surface charge density.

Equation 6.2.36 is the boundary condition implied by the conservation of charge equation. An example will help to clarify the significance of the terms.

Example 6.2.3. An application of the conservation of charge boundary condition that is considered in Section 7.2 is shown in Fig. 6.2.6. Here two slightly conducting materials form a common boundary that moves to the right with the velocity $U \mathbf{i}_x$. There are no free charges in the bulk of the materials in which $\mathbf{J}_f = \sigma \mathbf{E}$ (see Section 7.2.2). Hence in this particular case the boundary condition (6.2.36) becomes

$$\sigma_a E_y^a - \sigma_b E_y^b + \frac{\partial}{\partial x} K_{fx} + \frac{\partial}{\partial z} K_{fz} = - \frac{\partial \sigma_f}{\partial t}. \quad (a)$$

On the interface the only surface current is due to the convection of free charge σ_f ; that is,

$$K_{fz} = 0; \quad K_{fx} = U \sigma_f. \quad (b)$$

Moreover, σ_f is related to the electric field through the boundary condition (6.2.33) (we assume that $\mathbf{D} = \epsilon \mathbf{E}$ in both materials):

$$\sigma_f = \epsilon_a E_y^a - \epsilon_b E_y^b. \quad (c)$$

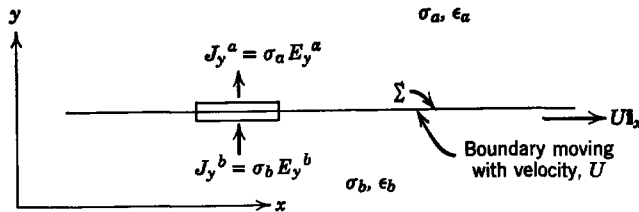


Fig. 6.2.6 A boundary between materials with conductivities σ_a and σ_b and permittivities ϵ_a and ϵ_b moves to the right with velocity U . Boundary condition (6.2.36) accounts for conservation of charge in a small section of the boundary.

It follows that (a) can be written as

$$\sigma_a E_y^a - \sigma_b E_y^b + \left(\frac{\partial \sigma_f}{\partial t} + U \frac{\partial \sigma_f}{\partial x} \right) = 0. \quad (d)$$

Note that (c) and (d) together constitute a single boundary condition on the electric field intensity at the moving surface. Remember that $\partial \sigma_f / \partial t + U \partial \sigma_f / \partial x$ is the rate of change with respect to time for an observer traveling with the velocity U [from (6.1.56) $\sigma_f = \sigma'_f$]. Hence (d) simply states that for such an observer the net current into a small section of the interface goes into an increase in the surface charge σ_f .

Equations 6.2.31, 6.2.33, and 6.2.36 are the only boundary conditions needed to solve problems for the electric fields in most quasi-static electric field systems. In these systems magnetic fields are generated by time-changing electric fields. Boundary conditions for these magnetic fields can be obtained from the integral form of Ampère's law (see Table 1.2)

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J}_f \cdot \mathbf{i}_n da + \frac{d}{dt} \int_S \mathbf{D} \cdot \mathbf{i}_n da, \quad (6.2.37)$$

where C and S are fixed as shown in Fig. 6.2.4. The process is analogous to that used in deriving (6.2.22) in the preceding section and leads to the boundary condition*

$$\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b) = \mathbf{K}_f + v_n \mathbf{n} \times [\mathbf{n} \times (\mathbf{D}^a - \mathbf{D}^b)]. \quad (6.2.38)$$

Note that this boundary condition is essentially that of (6.2.14) for the magnetic field system, with an added term to account for displacement current.

A summary of field transformations and boundary conditions is given in Table 6.1, which is arranged so that the correspondence of transformations and boundary conditions with differential equations is emphasized. One of the most important concepts related in this chapter is the consistency that must exist among differential equations, transformations, and boundary conditions.

The most obvious effects on boundary conditions from material motion are brought in through the normal velocity v_n . It must be remembered, however, that the boundary is itself part of a mechanical system that can often deform in the presence of magnetic or electric forces. This geometric effect of the boundary conditions is represented by the normal vector \mathbf{n} and illustrated by Example 6.2.4.

Example 6.2.4. Boundary conditions provide a mechanism by which mechanical motions can alter electrostatic field solutions. In Fig. 6.2.7 one of a pair of perfectly conducting electrodes is plane, whereas the other has the sinusoidal dependence on (x, t)

$$y = A \sin \omega t \cos kx + B, \quad (a)$$

* To show this note that $-\mathbf{n} \times (\mathbf{n} \times \mathbf{A})$ is the component of \mathbf{A} in the surface to which \mathbf{n} is normal.

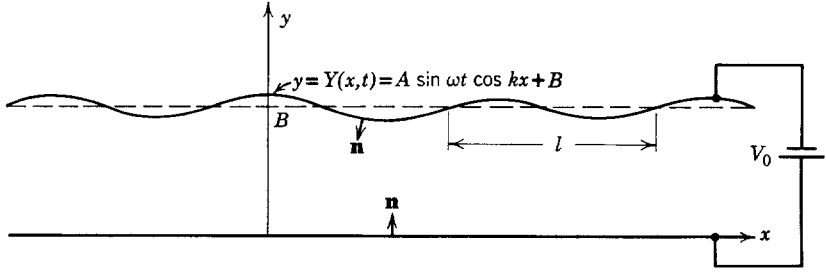


Fig. 6.2.7 Perfectly conducting electrodes at $y = 0$ and $y = A \sin \omega t \cos kx + B$ (where $k = 2\pi/l$) constrained to a constant potential difference.

where the dimension $A \ll B$, ω is the angular frequency, and $k = 2\pi/l$ (see Example 6.2.1). We wish to compute the electric field in the region between the electrodes when they are at the potential difference V_0 and to find the surface charge density on the lower electrode. It is assumed that the plates have infinite extent in the x - and z -directions.

The important boundary condition is (6.2.31). Because the electrodes are perfectly conducting, they can support no internal electric field. Hence at the surface of the electrodes

$$\mathbf{n} \times \mathbf{E} = 0, \quad (\text{b})$$

where we have set $\mathbf{E}^a = \mathbf{E}$ in (6.2.31). On the lower electrode $\mathbf{n} = \mathbf{i}_y$ and (b) reduces to

$$E_x(x, 0, t) = 0. \quad (\text{c})$$

Here we have used the fact that the upper surface position does not vary with z to set

$$E_z(x, y, t) = 0.$$

The normal vector \mathbf{n} on the upper electrode is given by the negative of (c) of Example 6.2.1. We assume that the amplitude A is small enough ($4\pi^2 A^2 / l^2 \ll 1$) to justify setting the denominator of this expression equal to 1. Then

$$\mathbf{n} = -\mathbf{i}_x k A \sin \omega t \sin kx - \mathbf{i}_y. \quad (\text{d})$$

The boundary condition (b), applied to the upper electrode, reduces to

$$n_y E_x(x, Y, t) = n_x E_y(x, Y, t), \quad (\text{e})$$

where $Y = A \sin \omega t \cos kx + B$ is the position of the upper electrode. If A were zero (two parallel flat plates) the electric field would be $\mathbf{E} = (V_0/B)\mathbf{i}_y$. Hence we define a perturbation electric field $\mathbf{e}(x, y, t)$ and let

$$\mathbf{E} = e_x \mathbf{i}_x + \left(\frac{V_0}{B} + e_y \right) \mathbf{i}_y. \quad (\text{f})$$

The perturbations e_x and e_y are proportional to the amplitude A . Introducing (f) into (e) [with n_x and n_y defined by (d)], we have

$$e_x(x, Y, t) = kA \sin \omega t \sin kx \left[\frac{V_0}{B} + e_y(x, Y, t) \right] \quad (\text{g})$$

and now, if we ignore terms that are proportional to A^2 (compared with terms proportional to A),

$$e_x(x, B, t) = \frac{kAV_0}{B} \sin \omega t \sin kx. \quad (\text{h})$$

Here e_x has been evaluated at $y = B$ rather than $y = Y$ because the difference in e_x evaluated at these points is proportional to A^2 . The approximate effect of the corrugated surface on the electric field is found by using (h) and (c) as boundary conditions.

Between the plates the divergence and curl of the electric field are zero [(1.1.11) and (1.1.12) with no free charge]. For the two-dimensional case under consideration this gives two expressions for the perturbation components.

$$\frac{\partial e_x}{\partial x} + \frac{\partial e_y}{\partial y} = 0, \quad (\text{i})$$

$$\frac{\partial e_x}{\partial y} - \frac{\partial e_y}{\partial x} = 0. \quad (\text{j})$$

If boundary condition (h) is to be satisfied, e_x must have the (x, t) dependence $\sin \omega t \sin kx$. Hence we assume (and later justify) that

$$e_x(x, y, t) = f(y) \sin \omega t \sin kx. \quad (\text{k})$$

Then (i) and (j) will be satisfied for all values of t and x only if e_y has the (x, t) dependence $\sin \omega t \cos kx$:

$$e_y(x, y, t) = g(y) \sin \omega t \cos kx. \quad (\text{l})$$

The dependence on (x, t) assumed for e_x and e_y is justified when we substitute (k) and (l) into (i) and (j) and find that the functions of x and t cancel out. After carrying out this process there remain the equations

$$fk + \frac{dg}{dy} = 0, \quad (\text{m})$$

$$\frac{df}{dy} + kg = 0. \quad (\text{n})$$

This pair of ordinary differential equations has the solution

$$f = C \sinh ky + D \cosh ky. \quad (\text{o})$$

where C and D are arbitrary constants. Remember that e_x is proportional to f [see (k)]. Hence, if (c) is to hold for *all* values of y , $D = 0$. From (k) and (o)

$$e_x(x, y, t) = C \sinh ky \sin \omega t \sin kx, \quad (\text{p})$$

where the constant C follows from boundary condition (h) as

$$C = \frac{kAV_0}{B \sinh kB}. \quad (\text{q})$$

Note that it was our foresight in guessing the (x, t) dependence of (k) that allowed us to satisfy condition (h) for all values of x and t . We now know f and therefore can find g from (n). Hence (l) becomes

$$e_y = -C \cosh ky \sin \omega t \cos kx. \quad (\text{r})$$

In the limiting case in which $B \ll l$ we can calculate these fields much more easily. This limiting case occurs when the wavelength l of the corrugation on the upper plate is long compared with the average spacing B between the plates. We expect a small section to have the same field as a parallel plate capacitor; that is,

$$E_y = \frac{V_0}{B + A \sin \omega t \cos kx} \cong \frac{V_0}{B} - \frac{V_0 A}{B^2} \sin \omega t \cos kx \quad (s)$$

and

$$E_x = 0. \quad (t)$$

If $B \ll l$, then $kB \ll 1$; and in this limit our solution [(f), (p), and (r)] reduces to the "long-wave" approximation of (s) and (t). In this limit all of the perturbation charges (those additional charges due to the corrugation) on one plate have image charges on the opposite plate. In the opposite extreme in which $kB \gg 1$ ($l \ll B$) there are few perturbation charges on the flat plate. To see this we can compute the surface charge density on the lower plate as

$$\sigma_f = \epsilon_0 \left[\frac{V_0}{B} + e_y(x, 0, t) \right], \quad (u)$$

which [from (q) and (r)] is

$$\sigma_f = \epsilon_0 \left[\frac{V_0}{B} - \frac{kA V_0 \sin \omega t \cos kx}{B \sinh kB} \right]. \quad (v)$$

As kB becomes large, the perturbation part of σ_f becomes small (as $kB \rightarrow \infty$, $\sinh kB \rightarrow \infty$).

A scheme for finding the deflection of a conducting surface would measure the charge on an electrode imbedded in the flat plate. Equation v shows that the perturbation surface charge density provides the (x, t) dependence of the deflection. The amplitude of σ_f however, would be inversely proportional to the wavelengths l to be detected.

6.3 CONSTITUENT RELATIONS FOR MATERIALS IN MOTION

Constituent relations, which are mathematical models of the electromagnetic properties of matter, were discussed briefly for stationary media in Section 1.1. At that point specific models that describe materials in a way that is useful in this study of electromechanical interactions were presented. As indicated in Section 6.0, however, the constituent relations expressed for stationary material may not be correct when the material is moving. In the next two sections we recast the constituent relations in forms that are correct when describing material that is in motion with respect to the reference frame in which electromagnetic quantities are measured. In general, a medium may be in motion relative to a particular inertial coordinate system in which we wish to define field and source quantities. We postulate that constituent relations, as conventionally defined for stationary media, still hold for moving media, provided they use source and field quantities defined in an inertial coordinate system with the same velocity as the material at the instant of time in question. It is therefore assumed that acceleration and rate of deformation do not affect local material properties. Constituent relations

obtained with this postulate yield predictions that agree to a high degree of accuracy with experimental results.* To apply this postulate we shall use the transformations of Sections 6.1.1 and 6.1.2 with the constituent relations for stationary media given in Section 1.1. Because the transformations are different for quasi-static electric and magnetic field systems we consider constituent relations for the two systems separately.

6.3.1 Constituent Relations for Magnetic Field Systems

Reference to the differential equations of Section 1.1 (see Table 1.2) and the boundary conditions of Section 6.2.1 shows that the fields in a quasi-static magnetic field system are excited by free current density \mathbf{J}_f , free surface current density \mathbf{K}_f , and magnetization density \mathbf{M} . Thus we need consider here only how \mathbf{J}_f , \mathbf{K}_f , and \mathbf{M} are affected by field quantities in the presence of material motion.

Consider first a linear isotropic conducting medium, which, when stationary, has the constituent relation introduced as (1.1.9),

$$\mathbf{J}_f = \sigma \mathbf{E}, \quad (6.3.1)$$

where σ is the electrical conductivity.

We now define an inertial coordinate system \mathbf{r} in which we measure the quantities \mathbf{E} , \mathbf{B} , and \mathbf{J} , as functions of space (\mathbf{r}) and time (t). The material medium moves with respect to this coordinate system with a velocity $\mathbf{v}(\mathbf{r}, t)$. In general, the velocity \mathbf{v} is different for each point within the material because it can be translating, rotating, and deforming. We wish to express the constituent relation describing electrical conduction in terms of quantities measured in the coordinate system \mathbf{r} . To do this we use the postulate given in the preceding section which states that the constituent relation for the material at rest is applicable in an inertial coordinate system with respect to which the material is instantaneously at rest. Thus to express the constituent relation for the material occupying position \mathbf{r} at time t we define an inertial coordinate system having the velocity

$$\mathbf{v}^r = \mathbf{v}(\mathbf{r}, t), \quad (6.3.2)$$

that is, \mathbf{v}^r is a constant with a value equal to the material velocity at position \mathbf{r} at time t . We denote electromagnetic quantities as measured in this moving coordinate system with primes and apply our postulate along with (6.3.1) to write

$$\mathbf{J}'_f(\mathbf{r}, t) = \sigma(\mathbf{r}, t) \mathbf{E}'(\mathbf{r}, t). \quad (6.3.3)$$

* A case in which acceleration effects on conduction are computed is discussed in L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, Addison-Wesley, Reading, Mass., 1960, pp. 210–212. These effects are usually ignorable.

We now use the transformations of (6.1.36) and (6.1.38) to rewrite (6.3.3) as

$$\mathbf{J}_f(\mathbf{r}, t) = \sigma(\mathbf{r}, t)[\mathbf{E}(\mathbf{r}, t) + \mathbf{v}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t)]. \quad (6.3.4)$$

This equation is the desired result in that it is the relation imposed among electromagnetic variables expressed in the fixed frame by linear isotropic electrical conduction in a moving medium.

The functional notation (\mathbf{r}, t) has been included in the preceding equations to make the meanings of the terms more explicit. Equation 6.3.4 is more compactly expressed without the functional notation as

$$\mathbf{J}_f = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \quad (6.3.5)$$

Next consider surface conduction on a material with surface conductivity σ_s as described for stationary materials by (1.1.10).

$$\mathbf{K}_f = \sigma_s[-\mathbf{n} \times (\mathbf{n} \times \mathbf{E})], \quad (6.3.6)$$

where \mathbf{n} is the normal to the surface and $[-\mathbf{n} \times (\mathbf{n} \times \mathbf{E})]$ is the component of \mathbf{E} tangent to the surface. When the surface is moving, we use a process analogous to that used for volume conduction to obtain the result

$$\mathbf{K}_f = \sigma_s\{-\mathbf{n} \times [\mathbf{n} \times (\mathbf{E} + \mathbf{v} \times \mathbf{B})]\}. \quad (6.3.7)$$

The velocity \mathbf{v} is the velocity of the surface with respect to the coordinate system in which the electromagnetic quantities are measured.

The final constituent relation that must be defined for magnetic field systems is the relation between magnetization density \mathbf{M} and magnetic field intensity \mathbf{H} .

The constituent relation for an isotropic, linear, magnetic material was written as (1.1.6)

$$\mathbf{M} = \chi_m \mathbf{H}, \quad (6.3.8)$$

where χ_m is the magnetic susceptibility. Using (1.1.4) we wrote

$$\mathbf{B} = \mu \mathbf{H}, \quad (6.3.9)$$

where the permeability μ is defined as

$$\mu = \mu_0(1 + \chi_m). \quad (6.3.10)$$

Although the constituent relation as expressed by (6.3.8) or (6.3.9) was written for stationary material, the transformations of (6.1.35), (6.1.37), and (6.1.39) show that in a quasi-static magnetic field system \mathbf{B} , \mathbf{H} , and \mathbf{M} are unaffected by relative motion. Consequently, (6.3.8) and (6.3.9) hold also when the material is moving with respect to the coordinate system in which the electromagnetic quantities are to be measured.

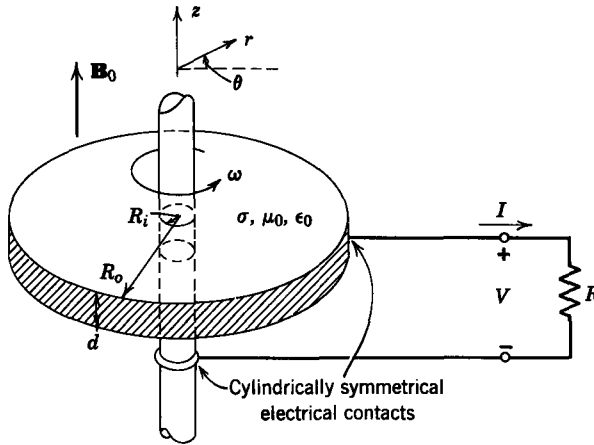


Fig. 6.3.1 A homopolar generator.

Example 6.3.1. As an example of transformations and constituent relations for moving materials, consider the device illustrated schematically in Fig. 6.3.1. This is a Faraday disk (also called a homopolar machine or an acyclic machine). Machines with the basic configuration of Fig. 6.3.1 or alternative configurations that operate physically in the same way are manufactured for supplying dc power at low voltage and high current.* A cutaway view of one such configuration is shown in Fig. 6.3.2.

With reference to Fig. 6.3.1, the device consists essentially of a right circular cylinder of conducting material that is rotated about its axis. Electrical contacts, usually made of liquid metal (see Fig. 6.3.2), are made symmetrically at inner and outer radii. Not shown in the figure is the electromagnet which produces a uniform axial flux density \mathbf{B}_0 .

We specify that the applied flux density \mathbf{B}_0 is constant and that the shaft is driven by a constant angular velocity source ω . The electrical terminals are loaded by a resistance R . The material of the rotating disk is homogeneous, isotropic, and electrically linear with the material constants σ , μ_0 , ϵ_0 . The dimensions are defined in the figure.

We wish to find the terminal voltage and current for all values of load resistance R and steady-state operation.

It should be clear from an inspection of Fig. 6.3.1 that the current in the disk is radial and the current density is uniform around the periphery at any radius. Thus the magnetic field generated by this current density is tangential and has no effect on the terminal voltage. Hence we neglect the field due to current in the disk. The validity of this assumption becomes clearer in the analysis to follow.

We select the cylindrical coordinate system r , θ , z shown in Fig. 6.3.1. The cylindrical symmetry and the uniformity of variables in the z -direction indicate that we can assume

$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial z} = 0,$$

* D. A. Watt, "Development and Operation of a 10KW Homopolar Generator with Mercury Brushes," *Proc. I.E.E. (London)*, **105A**, 33-40, (June 1958). A. K. Das Gupta, "Design of Self-Compensated High-Current Comparatively Higher Voltage Homopolar Generators," *Trans. AIEE*, **80**, Part III, 567-573, 1961-1962.

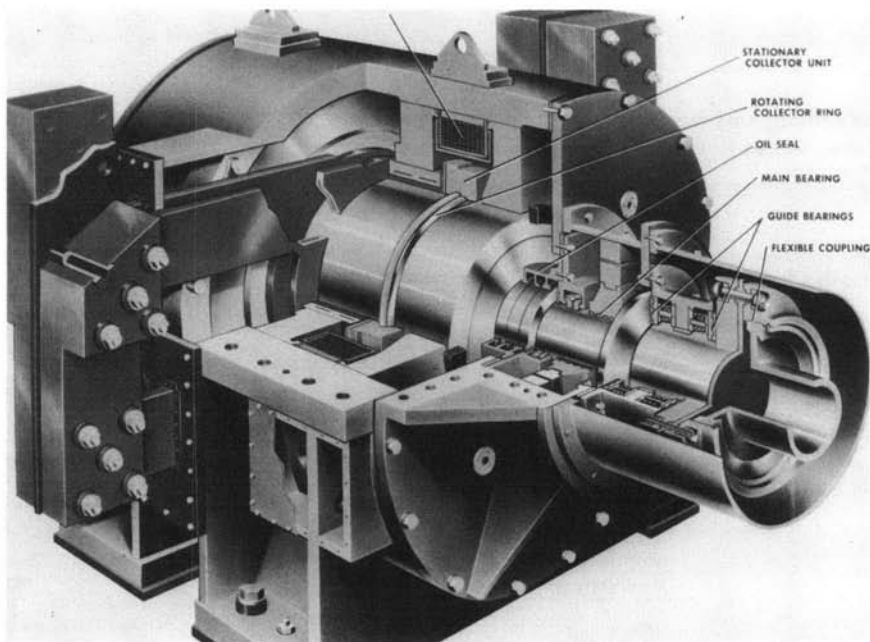


Fig. 6.3.2 Cutaway view of an acyclic generator. The solid rotor is made of magnetically soft steel, the flux density is radial, and current is axial between two liquid metal collector rings, one of which is shown. (Courtesy of General Electric Company.)

so that electromagnetic quantities of interest will vary with radius only. The electromagnetic equations for this quasi-static magnetic field system are those of Section 1.1.1a (see Table 1.2).

We first use the conservation of charge in integral form (1.1.22)

$$\oint \mathbf{J}_r \cdot \mathbf{n} \, da = 0, \quad (\text{a})$$

to establish that the radial component of current density is related to the terminal current by

$$J_r = \frac{I}{2\pi r d}. \quad (\text{b})$$

We next write Ohm's law for a grain of matter at the radius r by writing the r component of (6.3.5).

$$J_r = \sigma(E_r + \omega r B_0), \quad (\text{c})$$

where B_0 is the magnitude (z -component) of \mathbf{B}_0 and E_r is the radial component of electric field intensity. A tangential (θ) component of flux density is parallel to the material velocity and does not contribute a $\mathbf{v} \times \mathbf{B}$ term. Thus the neglect of the field generated by current in the disk is justified.

We now use (b) in (c) to find E_r :

$$E_r = \frac{I}{2\pi\sigma dr} - \omega B_0 r. \quad (d)$$

Recognizing that there is no time rate of change of magnetic field in the fixed reference frame, we can write the terminal voltage as

$$V = - \int_{R_i}^{R_o} E_r dr = - \frac{I}{2\pi\sigma d} \ln \frac{R_o}{R_i} + \frac{\omega B_0}{2} (R_o^2 - R_i^2). \quad (e)$$

Use of the terminal relation required by the resistance R yields

$$I = \frac{V_{oc}}{R + R_{int}} \quad (f)$$

where $V_{oc} = (\omega B_0/2)(R_o^2 - R_i^2)$ is the open circuit ($R \rightarrow \infty$) voltage of the generator, and $R_{int} = [\ln(R_o/R_i)]/2\pi\sigma d$ is the internal resistance of the generator.

To obtain some idea of the kinds of numbers obtainable with real materials consider a copper disk with the following parameters and dimensions:

$$\begin{array}{ll} \sigma = 5.9 \times 10^7 \text{ mhos/m} & \omega = 400 \text{ rad/sec} \\ d = 0.005 \text{ m} & B_0 = 1 \text{ Wb/m}^2 \\ R_i = 0.01 \text{ m} & R_o = 0.1 \text{ m} \end{array}$$

The open-circuit voltage is then

$$V_{oc} = 2 \text{ V}$$

and the internal resistance is

$$R_{int} = 1.25 \times 10^{-6} \Omega.$$

The short-circuit current is

$$I_{sc} = \frac{V_{oc}}{R_{int}} = 1.6 \times 10^6 \text{ A.}$$

The maximum power that can theoretically be delivered by this generator is

$$P_{max} = \frac{V_{oc} I_{sc}}{4} = 8 \times 10^5 \text{ W.}$$

For steady-state operation, however, the output power would be limited to a much lower value by allowable $I^2 R_{int}$ heating of the rotating disk. These figures indicate, though, that this device is suitable for supplying large pulses of power.*

We now use (d) and (f) to write the radial component of electric field intensity as

$$E_r = \left[\frac{R_{int} V_{oc}}{(R + R_{int}) \ln(R_o/R_i)} \right] \frac{1}{r} - \omega B_0 r. \quad (g)$$

In the spirit of the discussion of quasi-static systems in Section B.2.2 we can calculate the volume charge density necessary to satisfy Gauss's law:

$$\rho_f = \nabla \cdot \epsilon_0 \mathbf{E}. \quad (h)$$

* T. J. Crawford, "Kinetic Energy Storage for Resistance Welding," *Welding Engineer*, 33, 36 (1948).

Using the equation for the divergence in cylindrical coordinates,* we have

$$\rho_f = -2\epsilon_0\omega B_0. \quad (i)$$

This charge density arises from the second term in (g); the first term is divergenceless. The finite volume charge density results because the electric field is generated by nonuniform motion in a uniform magnetic field. There is, however, no net charge on the disk because an equal amount of charge of opposite sign occurs as a surface charge density at $r = R_0$ and $r = R_i$.

As discussed in Section B.2.2 this charge density was derived *after* the field problem was solved and its presence has negligible effect on the field solutions. To illustrate, consider the current density that results from the convection of this charge density by the rotating disk. The result is a θ -component of current density that has the value

$$J_\theta = -2\epsilon_0\omega^2 B_0 r.$$

We use the θ -component of Ampere's law in cylindrical coordinates to find the change in B_z caused by this current:

$$\frac{\partial B_z}{\partial r} = -\mu_0 J_\theta = 2\mu_0\epsilon_0\omega^2 B_0 r.$$

Integrating this expression, we find the maximum possible fractional change in B_z as

$$\frac{\Delta B_z}{B_0} = \omega^2 R_0^2 \mu_0 \epsilon_0 = \frac{\omega^2 R_0^2}{c^2},$$

where c is the speed of light (Section B.2.1). For any disk made of real material the peripheral speed (ωR_0) must be much smaller than the speed of light; thus the change in B_z due to convection current is negligible.

We reconsider the homopolar machine and complete its terminal description as an electromechanical coupling device in Section 6.4.

6.3.2 Constituent Relations for Electric Field Systems

The differential equations in Section 1.1.1*b* (see Table 1.2) and the boundary conditions of Section 6.2.2 indicate that fields in quasi-static, electric field systems are excited by free charge density ρ_f , free surface charge density σ_f , and polarization density \mathbf{P} . The constituent relations for these source quantities are given for stationary media in Section 1.1.1*b*. We generalize those constituent relations appropriate for electric field systems to include the effects of material motion.

In Section 1.1.1*b* the conduction process in a stationary medium was modeled by (1.1.16):

$$\mathbf{J}_f = (\rho_{f+}\mu_+ + \rho_{f-}\mu_-)\mathbf{E}, \quad (6.3.11)$$

where ρ_{f+} and ρ_{f-} are the densities of free charge and μ_+ and μ_- are the mobilities of the free charges defined in Section B.3.3. When the material is moving, (6.3.11) must be modified according to the transformations of

* $[d(\epsilon_0 r E_r)/dr]/r$

(6.1.56) and (6.1.58). Thus, if the material is moving with velocity \mathbf{v} , we must write (6.3.11) in the form

$$\mathbf{J}_f = (\rho_{f+}\mu_+ + \rho_{f-}\mu_-)\mathbf{E} + \rho_f\mathbf{v}, \tag{6.3.12}$$

where the net charge density ρ_f is given by

$$\rho_f = \rho_{f+} + \rho_{f-}. \tag{6.3.13}$$

The first term $[(\rho_{f+}\mu_+ + \rho_{f-}\mu_-)\mathbf{E}]$ of (6.3.12) describes the motion of charge carriers with respect to the material and the second term $(\rho_f\mathbf{v})$ describes the convection of net charge by the motion of the material.

The transformations of (6.1.54), (6.1.55), and (6.1.59) show that in a quasi-static electric field system \mathbf{D} , \mathbf{E} , and \mathbf{P} are unaffected by relative motion. Consequently, (1.1.17) and (1.1.19), which were written for stationary material, are still valid when the material is moving with respect to the coordinate system in which electromagnetic quantities are measured. For convenience we repeat these two equations here:

$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E}, \tag{6.3.14}$$

$$\mathbf{D} = \epsilon\mathbf{E}, \tag{6.3.15}$$

where the dielectric susceptibility χ_e and permittivity ϵ are related by

$$\epsilon = \epsilon_0(1 + \chi_e). \tag{6.3.16}$$

We conclude this section with an example that involves a particularly simple form of (6.3.12). Other examples of the use of these constituent relations are given in Chapter 7.

Example 6.3.2. A simple example in which the constituent relation (6.3.12) is used, is shown in Fig. 6.3.3. Here a cylindrical beam of charge carriers moves with the velocity V in the z -direction. We assume that the charge density is uniform throughout the beam and that the carriers (for example electrons) have zero mobility.

In a primed frame moving to the right at the velocity V the current is zero (6.3.12) and we have the simple fields associated with a uniform cylinder of charge density ρ_0 . Note that we have assumed $\rho_f = \rho'_f$, as given by (6.1.56). In this moving frame there is no magnetic field because there is no current \mathbf{J}'_f . Because the beam is uniform in the z -direction,

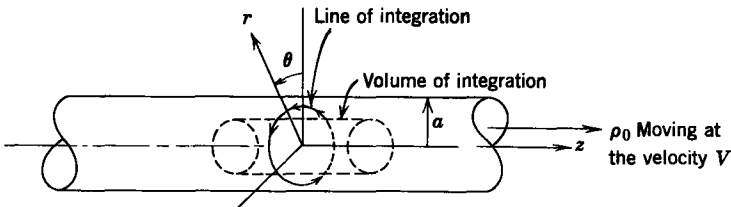


Fig. 6.3.3 Charged beam moving at velocity V to the right.

the electric field follows from Gauss's law (1.1.25) integrated over a cylinder of radius r and unit length in the z -direction:

$$\epsilon_0 E_r'(2\pi r) = \int_0^r \rho_f 2\pi r \, dr. \quad (a)$$

Hence inside the beam

$$E_r' = \frac{\rho_0 r}{2\epsilon_0}, \quad r < a, \quad (b)$$

and outside the beam

$$E_r' = \frac{\rho_0 a^2}{2\epsilon_0 r} \quad r > a. \quad (c)$$

To find the fields in the fixed frame is a simple matter, for (6.1.54) requires that $\mathbf{E} = \mathbf{E}'$ and (6.1.58) gives the current as

$$\mathbf{J}_f = \rho_0 V \mathbf{i}_z, \quad r < a. \quad (d)$$

The magnetic field follows from (6.1.57) and (6.1.54) as $\mathbf{H} = V \mathbf{i}_z \times \epsilon_0 \mathbf{E}$ or, by use of (b) and (c),

$$\mathbf{H} = \mathbf{i}_\theta \frac{V \rho_0 r}{2}, \quad r < a, \quad (e)$$

$$\mathbf{H} = \mathbf{i}_\theta \frac{V \rho_0 a^2}{2r}, \quad r > a.$$

The last result could be found alternatively by using the current density \mathbf{J}_f from (d) in the integral form of Ampère's law (1.1.20). A line integral of \mathbf{H} around the beam at a radius r gives

$$2\pi r H_\theta = \int_0^r J_z 2\pi r \, dr \quad (f)$$

or

$$H_\theta = \frac{V \rho_0 r}{2}, \quad r < a, \quad (g)$$

$$H_\theta = \frac{V \rho_0 a^2}{2r}, \quad r > a.$$

in agreement with (e).

6.4 DC ROTATING MACHINES

As stated in Section 4.1.6c, the dc machine is the most widely used rotating machine for control applications, especially when precise and versatile control of mechanical power or torque is required. Control can be achieved with high efficiency so that dc motors are used widely in high-power systems such as traction applications for driving locomotives and subway trains, rolling mills in steel plants, and ship propulsion. In electrically propelled ships and in diesel-electric locomotives the prime mover produces mechanical power and an electrical system is used for control rather than a mechanical

system of gears. The primary reason for this substitution is the versatility of the electrical power transmission system from the viewpoint of control.

The use of dc motors for controllable drives requires dc generators to supply the necessary power. Moreover, there are many applications of electric power in which direct current is necessary; for example, the production of aluminum is accomplished by the use of large quantities of direct current. Direct-current generators driven by alternating-current motors are used to supply power in many of these cases.

Because of the wide and extensive use of dc machines, we consider some of the important features of the more common types in use. Our treatment is introductory and intended to provide an understanding of the essential physics of dc machine operation and to indicate how the terminal behavior can be analyzed. In spite of our special attention to the topic at this point the reader should remember that although the dc machines we treat illustrate the basic material introduced in this chapter they are also specific examples of lumped-parameter, magnetic field-type, electromechanical devices introduced in Chapters 2 and 3. As we complete the analyses, we shall indicate the relation to the earlier chapters.

In the next two sections we treat two configurations of dc machines: commutator machines discussed briefly in Section 4.1.6c and homopolar machines, an example of which was introduced in Example 6.3.1.

6.4.1 Commutator Machines

6.4.1a Physical Characteristics

As discussed in Section 4.1.6c, a commutator can be viewed as a mechanically controlled frequency changer that causes rotor-current frequencies to satisfy automatically the condition for average power conversion (4.1.18) when rotor and stator electrical sources are at the same frequency (usually zero). To analyze the terminal behavior of a commutator machine, other viewpoints are used. (In some cases the techniques of Chapter 3 are employed, but this kind of treatment makes physical insight difficult.) We use a field approach to obtain equations of motion and indicate how the connection is made to the techniques of Chapter 3.*

To develop the equations of motion for a commutator machine from a field viewpoint we need to specify the geometry of the windings, the commutator, the brushes, and the magnetic material. To do this we use simplified schematic drawings. To put these representations in perspective a cutaway view

* For some alternative viewpoints on analytical techniques to be used with commutators see, for example, D. C. White and H. H. Woodson, *Electromechanical Energy Conversion*, Wiley, New York, 1959, Chapter 4; A. J. Thaler and M. I. Wilcox, *Electric Machines*, Wiley, New York, 1966, Chapters 3 and 4; A. E. Fitzgerald and C. Kingsley, Jr., *Electric Machinery*, 2nd ed., McGraw-Hill, New York, 1961, Chapter 3.

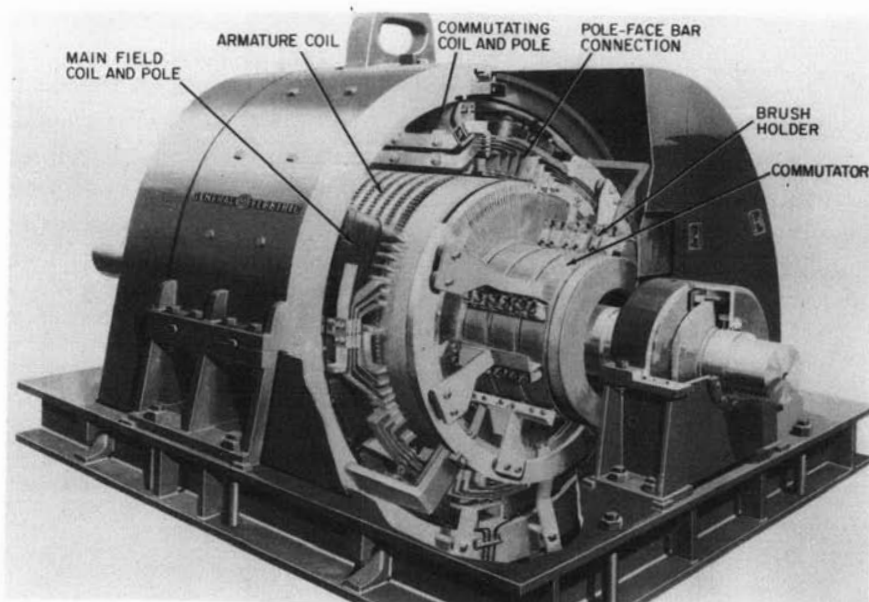


Fig. 6.4.1 Cutaway view of a 2250-hp, 300/600-rpm, 600-V, dc motor. (Courtesy of General Electric Company.)

of a commutator machine is shown in Fig. 6.4.1 with the principal parts labeled. When using the simplified schematic drawings, frequent reference should be made to this practical configuration.

First, consider the schematic end view of a two-pole commutator machine shown in Fig. 6.4.2. This is a salient-pole structure, as defined and discussed in Section 4.2, with the salient poles on the stator. A commutator machine can have any even number of poles (see Sections 4.1.8 and 4.2.4), but we treat a two-pole machine in the interest of simplicity. The rotor is essentially cylindrical with conductors placed in axial slots as indicated. The stator winding is excited directly at its terminals and the rotor conductors are excited through brushes (usually carbon) that make sliding contact with the commutator.

To follow the usual convention, we call the stator winding the *field* winding and denote quantities associated with it by the subscript f ; we call the rotor winding the *armature* winding and denote quantities associated with it by the subscript a .

Currents in the field windings, with directions indicated by dots and crosses in Fig. 6.4.2, produce a flux density distribution that is symmetrical about the field magnetic axis, as indicated in the developed view of the machine in Fig. 6.4.3. Currents in the armature conductors, maintained by the commutator

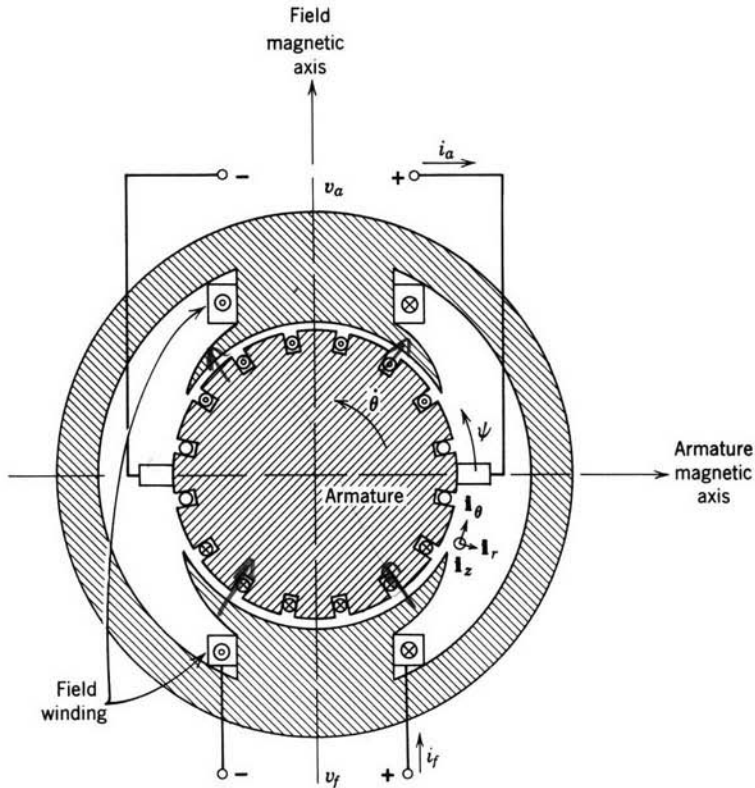


Fig. 6.4.2 Two-pole, commutator machine.

and brushes, have the directions indicated by the dots and crosses in Fig. 6.4.2, independent of armature (rotor) position or speed. Thus armature current produces a flux density distribution that is symmetrical about the armature magnetic axis, as indicated in Fig. 6.4.3.

To indicate qualitatively the shapes of the flux density distributions and to illustrate how armature conductors are connected to the commutator bars, a developed view of the machine is shown in Fig. 6.4.3. The shape of the field flux density distribution is understandable in terms of Ampère's law (1.1.20). For analytical purposes it is often assumed that fringing at the pole edges can be neglected because the air gap is small, that slot and teeth effects are negligible, and that there is no magnetic saturation. Then the flux density distribution has the square shape shown by dashed lines. As developed subsequently, the important quantity is the total magnetic flux per pole; consequently, the idealized curve is a good representation of the actual curve with respect to total flux (area under the curve).

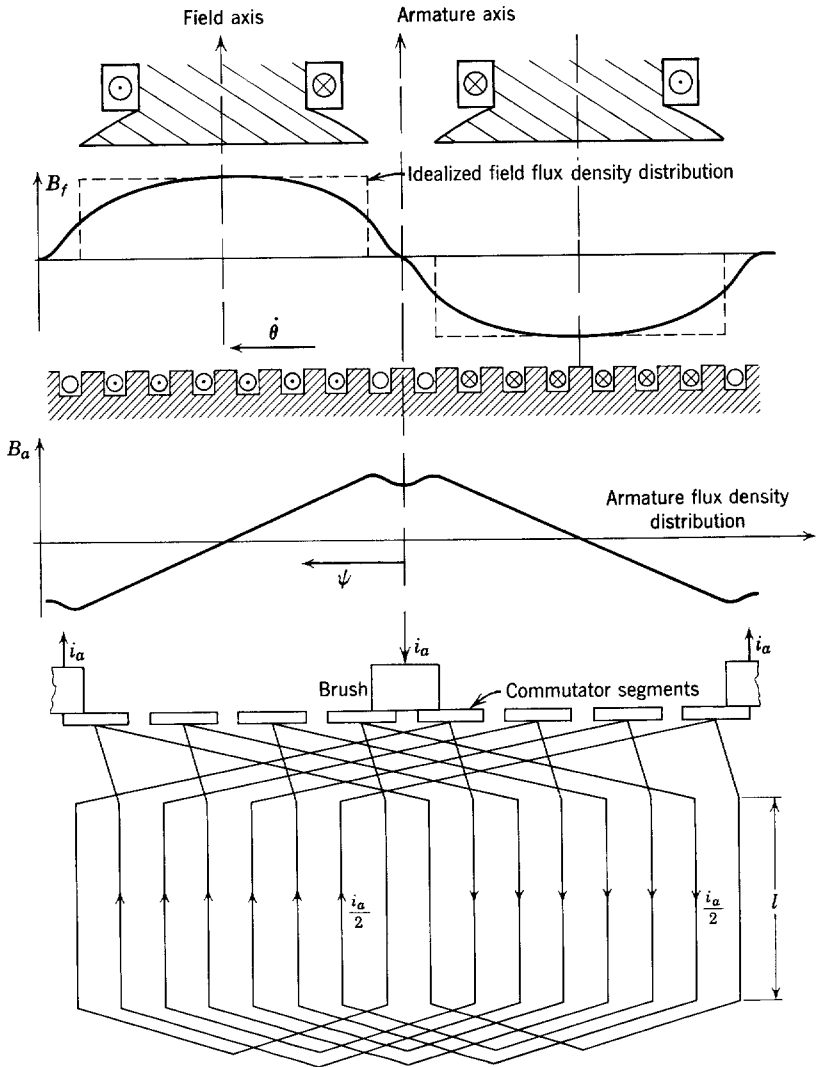


Fig. 6.4.3 Developed views of two-pole commutator machine showing flux distributions and armature connections.

The armature flux density distribution in Fig. 6.4.3 is idealized in two ways: the effects of slots and teeth are neglected and the armature current is assumed to be uniformly distributed along the armature surface. This approximation may seem rather gross with respect to the number and size of slots in Fig. 6.4.3; in practical machines, however, the number of slots is much greater and the slots are smaller; thus the approximate flux density distribution in Fig. 6.4.3 is quite accurate.

The developed view at the bottom of Fig. 6.4.3 shows how armature conductors are connected to commutator bars. A tracing of conducting paths will show that any one-turn coil has its ends connected to adjacent commutator bars and that there are two parallel conducting paths between brushes, each containing three coils in series. Two coils are short-circuited by the brushes. In practical machines with lap windings,* the coils may have one or more turns, with the coil terminals connected as shown in Fig. 6.4.3. Also, in practical machines each slot normally contains two coil sides, one each from two different coils, rather than the one we show for simplicity.

By visualizing what happens to the conductor currents in Fig. 6.4.3 as the armature conductors and commutator bars slide past the brushes, it will become clear that the armature current pattern will shift back and forth by about the distance between two slots. In a practical machine with a large number of armature conductors and commutator bars, this variation will be much less, and it is quite reasonable to assume that the current pattern is fixed at its average position. This switching of current pattern by the commutator can be interpreted as the electrical equivalent of a mechanical ratchet.

As indicated in Fig. 6.4.3 and noted earlier, two coils have no current because the brushes short-circuit them. As the armature turns, armature coils are successively shorted by the brushes. Before a coil is shorted it carries current in one direction, and after the short is removed the current direction is reversed. The process of current reversal is called commutation, and it is complicated by speed voltage and inductive voltages in the shorted coil and by arcing of the contact between brush and commutator. The process of commutation is complex and its practical realization imposes a limitation on the characteristics that can be achieved with commutator machines.† For the purpose of analyzing the terminal behavior of commutator machines we need to know only the geometry of the windings and the fact

* An alternative scheme is called a wave winding. Both schemes, lap and wave windings, essentially yield a continuous armature winding. For a discussion of the two schemes see, for example, A. E. Knowlton, ed., *Standard Handbook for Electrical Engineers*, 9th ed., McGraw-Hill, New York, 1957, Section 8.25.

† For a discussion of commutation and a list of good references see Knowlton, *op. cit.*, Sections 8.33 to 8.55.

that the commutator keeps the armature current pattern fixed in space with respect to the brushes, as illustrated in Figs. 6.4.2 and 6.4.3.

6.4.1b Equations of Motion

To develop the equations of motion for commutator machines we use the field transformations of Section 6.1.1 and the constituent relations of Section 6.3.1. We consider the effects of field flux density and armature flux density separately and superimpose the results. Such a process provides adequate accuracy; when the two flux density distributions in Fig. 6.4.3 are superimposed, however, they add in some regions and subtract in others. In the region in which they add, there may be saturation and resulting distortion of the flux patterns. When this occurs, the armature flux density distribution is skewed and there is a net linkage of armature flux with the field winding. This phenomenon is called armature reaction.* We neglect saturation, and thus the effects of armature reaction, in our analysis.

Consider first the field winding in Fig. 6.4.2. With reference to the flux density distributions of Fig. 6.4.3, it is clear that the armature produces no net flux linkage with the field winding because the axes of symmetry of the flux density distributions are orthogonal. Thus the field winding links only its own flux and we can write the equation (see Section 2.1.1)

$$v_f = R_f i_f + L_f \frac{di_f}{dt}, \quad (6.4.1)$$

where R_f = the field winding resistance,

L_f = the field winding self-inductance.

Effects of armature slots and teeth are neglected in defining the constant field inductance L_f .

Because the armature conductors are in motion with respect to the reference frame in which we are defining fields, we must be careful when writing the voltage equation for the armature circuit. We use Faraday's law in integral form with a *fixed* contour (see Table 1.2):

$$-\oint_C \mathbf{E} \cdot d\mathbf{l} = \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da. \quad (6.4.2)$$

The contour C is fixed so the \mathbf{E} is measured in the fixed reference frame. The contour to be used (shown schematically in Fig. 6.4.4) follows one of the conducting paths through the armature conductors between the brushes (see Fig. 6.4.3). The contour is fixed and the conductors are moving; thus they coincide only instantaneously.

* See, for example, Knowlton, *op. cit.*, Sections 8.27 to 8.32.

To evaluate (6.4.2) we break the contour integration into two parts:

$$-\int_a^b \mathbf{E} \cdot d\mathbf{l} - \int_b^a \mathbf{E} \cdot d\mathbf{l} = \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da. \tag{6.4.3}$$

The first term on the left is the integral between the terminals external to the machine and is

$$-\int_a^b \mathbf{E} \cdot d\mathbf{l} = v_a. \tag{6.4.4}$$

The second integral on the left is taken through the armature circuit of the machine. If the armature conductor material has a conductivity σ , then (6.3.5) indicates that we must write

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{6.4.5}$$

for the armature conductors in which \mathbf{J} is the armature conductor current density and \mathbf{v} is the armature conductor velocity. This expression can be used for all armature circuit conductors, including those from the terminals to the brushes, because $\mathbf{v} = 0$ and $\mathbf{J} = \sigma\mathbf{E}$, as it should be for a conductor at rest.

Solution of (6.4.5) for \mathbf{E} and the use of that result in the second term of (6.4.3) yields

$$-\int_b^a \mathbf{E} \cdot d\mathbf{l} = -\int_b^a \frac{\mathbf{J}}{\sigma} \cdot d\mathbf{l} + \int_b^a (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}. \tag{6.4.6}$$

The first term on the right is just the drop in voltage across the armature resistance and can be written as

$$-\int_b^a \frac{\mathbf{J}}{\sigma} \cdot d\mathbf{l} = -i_a R_a, \tag{6.4.7}$$

where R_a is the armature circuit resistance. To show this for the armature conductors between brushes, assume that the current ($i_a/2$) (see Fig. 6.4.3) is distributed uniformly over the cross section A_w of the wire. Then the magnitude of \mathbf{J} is

$$J = \frac{i_a}{2A_w}. \tag{6.4.8}$$

Specify the total length of wire between brushes as l_w and

$$-\int_b^a \frac{\mathbf{J}}{\sigma} \cdot d\mathbf{l} = -\int_0^{l_w} \frac{i_a}{2A_w\sigma} \, dl = \frac{-i_a l_w}{2A_w\sigma}. \tag{6.4.9}$$

The quantity $(l_w/2A_w\sigma)$ is just the resistance

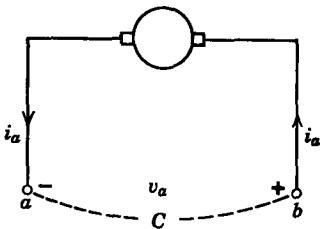


Fig. 6.4.4 Illustration of the contour for finding an armature voltage equation. The contour is completed on the armature, where it follows one of the two conducting paths joining the brushes in Fig. 6.4.3.

of two wires in parallel, each having length l_w , and A_w , and conductivity σ .

The second term on the right of (6.4.6) is a *speed voltage*, which is evaluated as follows. The velocity of a conductor is

$$\mathbf{v} = \mathbf{i}_\theta \dot{\theta} R, \quad (6.4.10)$$

where R = the radius measured from the axis of rotation to the conductor location,

$\dot{\theta}$ = the angular speed of the armature,

\mathbf{i}_θ = the unit vector in the tangential direction taken as positive in the counterclockwise direction in Fig. 6.4.2.

If we assume that appreciable flux density occurs only over the axial length l shown in Fig. 6.4.3 and that this flux density is radial and independent of axial position,* then

$$\mathbf{B} = \mathbf{i}_r B_r(\psi), \quad (6.4.11)$$

where \mathbf{i}_r = the unit vector in the radial direction,

ψ = an angle measured with respect to the fixed reference frame, as indicated in Fig. 6.4.2.

Equations 6.4.10 and 6.4.11 are used to write

$$\mathbf{v} \times \mathbf{B} = -\mathbf{i}_z \dot{\theta} R B_r(\psi), \quad (6.4.12)$$

where \mathbf{i}_z is the unit vector in the axial direction and is positive out of the paper in Fig. 6.4.2. Use of this term in the integral

$$\int_b^a (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l},$$

with the flux density distributions of Fig. 6.4.3 and the contour defined in Fig. 6.4.4, shows that there is no net contribution from armature flux density but there is from field flux density. To evaluate this contribution we can evaluate B_r at the value of ψ for each conductor, multiply by the length, and add up the contributions of each conductor to get the total. In real machines there are many armature conductors such that it is a good approximation to use the average flux density due to the field winding

$$(B_{rf})_{av} = \frac{\int_0^\pi B_{rf}(\psi) d\psi}{\pi} \quad (6.4.13)$$

and write

$$\mathbf{v} \times \mathbf{B} = -\mathbf{i}_z \dot{\theta} R (B_{rf})_{av}. \quad (6.4.14)$$

* This restriction is necessary only for simplicity. The method is general and can include axial variation of radial flux density and effects of the axial component of \mathbf{B} on the radially directed end turns.

We then take twice the axial length ($2l$) for the voltage generated in one turn and the total number of turns N in one path between brushes to write

$$\int_b^a (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = -2lN \dot{\theta} R (B_{rf})_{av}. \quad (6.4.15)$$

Because $(B_{rf})_{av}$ is proportional to field current i_f , it is conventional to write this term as

$$\int_b^a (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = -G\dot{\theta} i_f, \quad (6.4.16)$$

where G is the *speed coefficient* that depends only on geometry and magnetic material properties.

To evaluate the final term, the right side of (6.4.3), we recognize that

$$\int_S \mathbf{B} \cdot \mathbf{n} \, da = \lambda_a \quad (6.4.17)$$

is the flux linking the armature circuit. As indicated earlier, all of this flux linkage is due to armature current, the system is assumed to be electrically linear, and the effects of slots and teeth are neglected; thus

$$\lambda_a = L_a i_a, \quad (6.4.18)$$

where L_a is the constant armature self-inductance and

$$\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da = \frac{d\lambda_a}{dt} = L_a \frac{di_a}{dt}. \quad (6.4.19)$$

The armature voltage equation is now written by combining (6.4.4), (6.4.6), (6.4.7), (6.4.16), and (6.4.19) in (6.4.3); thus

$$v_a = i_a R_a + L_a \frac{di_a}{dt} + G\dot{\theta} i_f. \quad (6.4.20)$$

This is the desired armature circuit equation.

It is clear from this result that the electrical consequence of mechanical motion is represented by the last term on the right of (6.4.20). This term was derived in (6.4.10) to (6.4.20) with a fixed contour and Ohm's law for a moving conductor. It could have been derived with a contour that moves with the armature conductor. In this case we assume a contour from b to a in Fig. 6.4.4 that is moving with the armature conductors and write (6.4.3) in the alternate form

$$-\int_a^b \mathbf{E} \cdot d\mathbf{l} - \int_b^a \mathbf{E}' \cdot d\mathbf{l} = \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da. \quad (6.4.21)$$

We must remember that the contour C is now moving and thus the surface S varies with time. In the frame moving with the conductor

$$\mathbf{J}' = \sigma \mathbf{E}'$$

and the transformation of (6.1.36) gives

$$\mathbf{J}' = \mathbf{J}.$$

Thus the second term on the left of (6.4.21) produces only the resistive voltage drop (6.4.7). The speed voltage (6.4.16) is now generated by the time-varying surface in the term on the right of (6.4.21). This can be verified in a straightforward manner and is not done here. The method corresponds to that of Chapter 3 in which speed voltages were obtained from time rates of change of fluxes. These alternate ways of computing the terminal voltage were also illustrated in Example 6.1.2.

To complete the terminal description of the commutator machine we must evaluate the torque of electric origin. There are several equally valid ways of doing this. The torque is evaluated here by using the Lorentz force density for magnetic field systems.

We shall use the force density (1.1.30)

$$\mathbf{F} = \mathbf{J} \times \mathbf{B}, \quad (6.4.22)$$

with current density in the armature conductors and the radial component of flux density sketched in Fig. 6.4.3. It should be recognized that the teeth shield the conductors in the slots so that the conductors experience only a small fraction of the flux, most of which passes through the teeth. The result is that most of the torque is produced by magnetic forces on the magnetic material of the teeth. Nonetheless, we still get the correct answer by assuming that all the magnetic forces act on the conductors. That it is immaterial whether the force acts on the conductors or teeth but depends only on the magnetic fields in the air gap adjacent to the armature can be verified by using the Maxwell stress tensor to be introduced in Section 8.2.1.*

Making assumptions consistent with those in the derivation of the armature voltage equation, we assume (for mathematical convenience, not necessity) that appreciable flux density exists only along the length l of axial conductors (see Fig. 6.4.3 for definition of l) and that the flux density is radial and does not vary with axial position

$$\mathbf{B} = \mathbf{i}_r B_r(\psi). \quad (6.4.23)$$

* The fact that the force acts on the armature teeth and not on the conductors has the practical advantage that the mechanical forces applied to the conductor insulation are small and the insulation problem is more easily solved.

The current density in the region of interest is uniform across the conductor and axial

$$\mathbf{J} = \mathbf{i}_z J_z. \quad (6.4.24)$$

We use (6.4.23) and (6.4.24) in (6.4.22) to get the force density

$$\mathbf{F} = \mathbf{i}_\theta J_z B_r(\psi). \quad (6.4.25)$$

For uniform current density the magnitude of the current density in one armature conductor (see Fig. 6.4.3) is

$$|J_z| = \frac{i_a}{2A_w}, \quad (6.4.26)$$

where A_w is the cross-sectional area of the conductor. By integrating throughout the volume of one conductor over the active length l , the total force on one conductor is

$$\bar{\mathbf{F}}_1 = \pm \mathbf{i}_\theta \frac{i_a l B_r(\psi)}{2}. \quad (6.4.27)$$

The \pm is determined by the relative directions of current and flux density.

For the system of Fig. 6.4.2 we could simply superimpose the field and armature flux density distributions in Fig. 6.4.3, find the force on each conductor, add the forces, and multiply by the lever arm to get the torque. It is more instructive, however, to consider the two flux densities separately. First, it should be evident from the relative shapes of the armature current distribution and the armature flux density distribution (see Fig. 6.4.3) that no net torque results from their interaction. Furthermore, the relative directions of field flux density and armature current are such that each conductor produces a torque in the $+\theta$ -direction. Thus, with a lever arm (conductor radial position) of R , the torque from one conductor in the $+\theta$ -direction is

$$T_1 = \frac{i_a l R B_{rf}(\psi)}{2}. \quad (6.4.28)$$

We could add the contributions from the 12 active conductors (six coils) in Fig. 6.4.2; there are, however, many conductors, and the practice is to use the average field flux density defined in (6.4.13) and multiply by the number of active conductors, which is four times the number N of coils introduced in (6.4.15) (remember that there are two parallel paths between the brushes); thus the total torque is

$$T^e = 2Nl R(B_{rf})_{av} i_a. \quad (6.4.29)$$

Note that the coefficient of i_a in this expression is the same as the coefficient of $\hat{\theta}$ in (6.4.15), and we use the speed coefficient G defined by (6.4.15) and (6.4.16) to write

$$T^e = G i_f i_a. \quad (6.4.30)$$

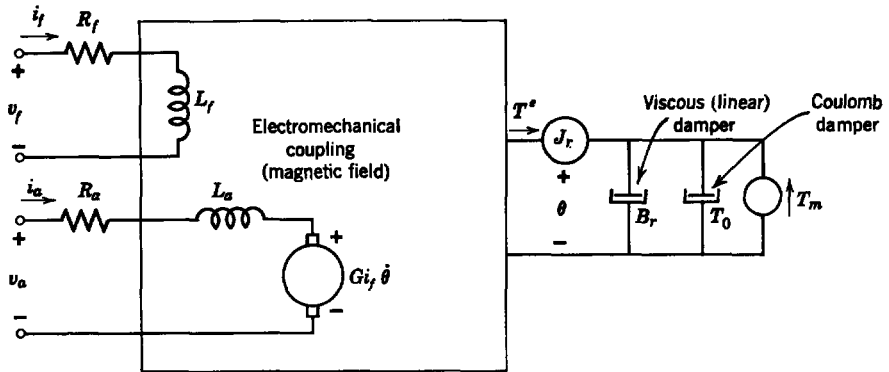


Fig. 6.4.5 Equivalent circuit of the commutator machine in Fig. 6.4.2.

In addition to the Maxwell stress tensor method mentioned earlier, this same torque can be derived by the energy method of Chapter 3, although with the commutator, care must be taken when applying this technique.

We have now completed the description of the electromechanical coupling properties of the commutator machine of Fig. 6.4.2. We complete the terminal description in the nomenclature introduced in Chapters 2 and 3 by drawing the equivalent circuit in Fig. 6.4.5. All losses are taken outside the coupling network which contains only magnetic field energy storage. On the mechanical side the rotor (armature) moment of inertia is J_r , and both viscous (B_r) and coulomb (T_0) damping are included. A commutator machine normally has significant coulomb damping from the brushes sliding on the commutator. The sources T_m , v_a , and v_f are general and can be independent or dependent on some variable.

6.4.1c Machine Properties

We shall now study some properties of the commutator machine by using the equivalent circuit of Fig. 6.4.5. The instantaneous power converted to mechanical form by the coupling network is

$$p_m = T^e \dot{\theta} = G i_f i_a \dot{\theta}, \quad (6.4.31)$$

where (6.4.30) has been used for torque T^e . The instantaneous power absorbed from the armature circuit by the speed voltage $G\dot{\theta}i_f$ is

$$p_e = G\dot{\theta}i_f i_a, \quad (6.4.32)$$

which is equal to the mechanical power output. This leads to the following immediate conclusions:

1. Power conversion between electrical and mechanical form in a commutator machine occurs instantaneously between the mechanical system and the speed voltage source in the armature circuit.

2. The power conversion is proportional to field current, but there is no power conversion between the field circuit and the mechanical system.

Next, assume that the rotor is to be driven by a constant-speed source

$$\dot{\theta} = \omega_m, \quad (6.4.33)$$

open circuit the armature ($i_a = 0$), and apply a battery to the field circuit terminals

$$v_f = V_f. \quad (6.4.34)$$

The open-circuit armature voltage varies with field voltage, as can be seen by considering steady-state conditions and (6.4.1) to find

$$i_f = \frac{V_f}{R_f}. \quad (6.4.35)$$

From (6.4.20) with $i_a = 0$ we find

$$v_a = \frac{G\omega_m}{R_f} V_f. \quad (6.4.36)$$

The armature voltage is proportional to the field voltage. This has the makings of a linear amplifier.

Now with the constraints of (6.4.33) and (6.4.34) applied, put a load resistance R_L across the armature terminals; the armature terminal voltage in the steady-state is

$$v_a = \left(\frac{R_L}{R_L + R_a} \right) \frac{G\omega_m}{R_f} V_f. \quad (6.4.37)$$

The load voltage is linear with field voltage and all the load power comes from the mechanical source. This illustrates the basic mode of operation of a dc generator, but it also indicates that the machine can be operated as an electromechanical amplifier. Direct-current machines are used as power amplifiers in many control applications. The power gain in a single machine is usually in the range of 20 to 30 and the bandwidth over which the amplification factor is constant is limited by field inductance to a few Hertz. Nonetheless, for control applications in which devices and power requirements are large, the required bandwidth is often small. It is difficult to think of a more economical way to make a 100,000-W, linear dc amplifier.

Commutator machines are used widely as electromechanical amplifiers, especially when considerable power is to be handled in control applications. Some amplifiers involve special constraints on a machine like that in Fig. 6.4.2 (Rototrol and Regulex); others have special configurations that involve

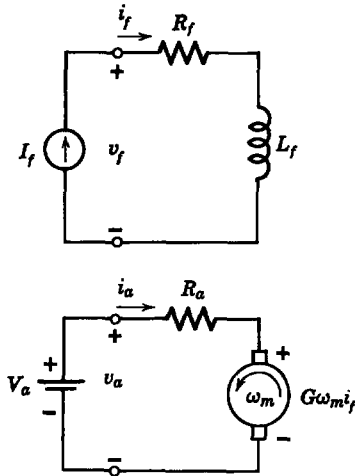


Fig. 6.4.6 Constraints for study of energy conversion properties in a dc machine.

a second set of brushes placed 90° from the original set to form a second armature circuit and a second field circuit placed 90° from the original field circuit (metadyne and amplidyne). All of these amplifiers are analyzed with the same basic techniques we used earlier.*

To learn some of the characteristics of the commutator machine as a dc energy converter we consider a steady-state problem with the constraints indicated in Fig. 6.4.6. The rotor position is constrained by a constant speed source

$$\dot{\theta} = \omega_m;$$

the field is constrained by a direct-current source

$$i_f = I_f;$$

and the armature terminals are constrained by a constant-voltage source

$$v_a = V_a.$$

We have left out armature inductance because we are treating the steady state.

The use of these constraints with the armature circuit equation (6.4.20) yields for the armature current

$$i_a = \frac{V_a - G\omega_m I_f}{R_a}. \quad (6.4.38)$$

* For a good discussion of rotating amplifiers in general and analyses of the specific configurations named see G. J. Thaler and M. L. Wilcox, *Electric Machines*, Wiley, New York, 1966, pp. 135-149.

The electric power fed into the armature terminals is

$$p_a = v_a i_a = V_a \left(\frac{V_a - G\omega_m I_f}{R_a} \right). \quad (6.4.39)$$

The torque of electric origin (6.4.30) is

$$T^e = G I_f \left(\frac{V_a - G\omega_m I_f}{R_a} \right) \quad (6.4.40)$$

and the mechanical power out of the coupling network (6.4.31) is

$$p_m = T^e \omega_m = G I_f \omega_m \left(\frac{V_a - G\omega_m I_f}{R_a} \right). \quad (6.4.41)$$

The quantities given by (6.4.38) to (6.4.41) are sketched as functions of ω_m for constant V_a and I_f in Fig. 6.4.7. As indicated in this figure, there are three regions of operation as defined by energy flow. Generator operation has mechanical power input ($p_m < 0$) and armature power output ($p_a < 0$) and occurs at values of speed at which the speed voltage ($G\omega_m I_f$) is greater than the armature voltage (V_a). The speed voltage then makes current flow to charge the battery on the armature terminals. Motor operation has armature power input ($p_a > 0$) and mechanical power output ($p_m > 0$) and occurs in a range of speed in which the armature terminal voltage V_a is greater than the speed voltage $G\omega_m I_f$ and can feed power into the speed voltage. Brake operation occurs when power is put into the machine both from the armature terminals ($p_a > 0$) and from the mechanical terminals ($p_m < 0$) and all of this power is dissipated in the armature resistance R_a . To see this refer to Fig. 6.4.6 and recognize that with $\omega_m < 0$ the speed voltage $G\omega_m I_f$ and armature battery V_a have aiding polarities and both feed power to R_a .

Figures 6.4.6 and 6.4.7 give the essential features of the operation of a dc machine with separate excitation; that is, the field winding is excited independently from a source separate from the armature excitation. The characteristics of a dc generator with separate excitation can be derived quite easily by using an equivalent circuit like that of Fig. 6.4.6. The transient performance of a separately excited dc generator is limited by field inductance, armature inductance, load inductance, or capacitance, and prime mover characteristics. The study of the characteristics is straightforward and is left to the problems at the end of the chapter.

The principal characteristics of a separately excited dc motor are shown in Fig. 6.4.8 in the sketch of torque-speed curves with armature voltage as parameter and field current constant. For practical machines the relatively small armature resistance leads to the steep slope of the curves. Thus at constant armature voltage there is little variation of speed with torque and the speed can be controlled quite closely by controlling the armature voltage.

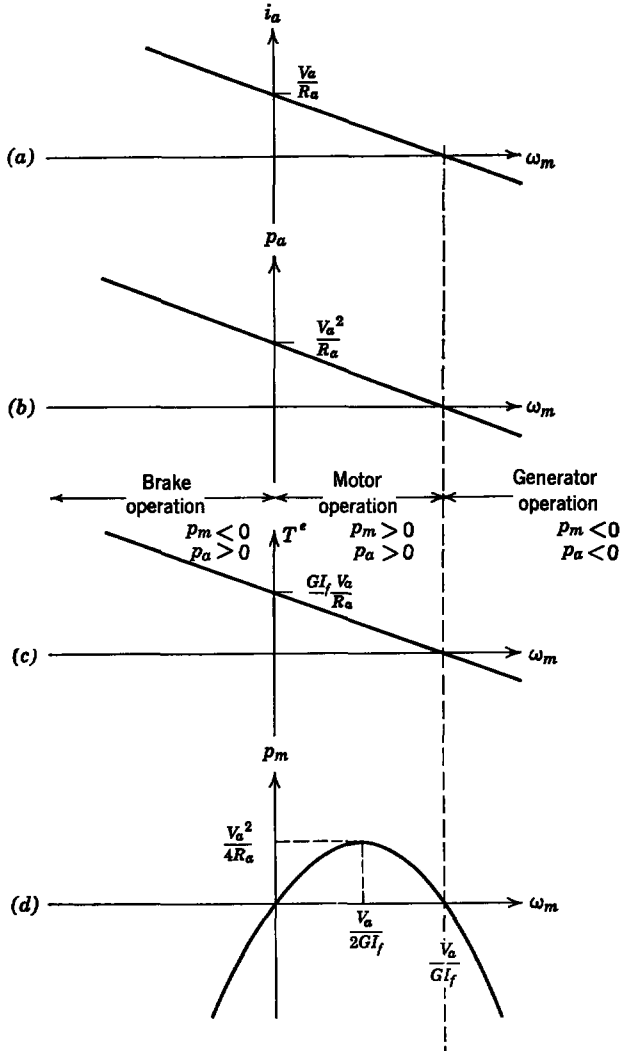


Fig. 6.4.7 Terminal characteristics of commutator machine with constant armature voltage and constant field current: (a) armature current; (b) armature power input; (c) torque of electric origin; (d) mechanical power out of coupling network.

This is the most common mode of operation of a dc motor when speed control is required. More precise speed regulation is obtained by using a feedback control system to sense the speed error and correct the armature voltage accordingly. When a dc generator is used to supply motor armature power in such a system it is called a Ward-Leonard system.*

* See, for example, Thaler and Wilcox, *op. cit.*, p. 291.

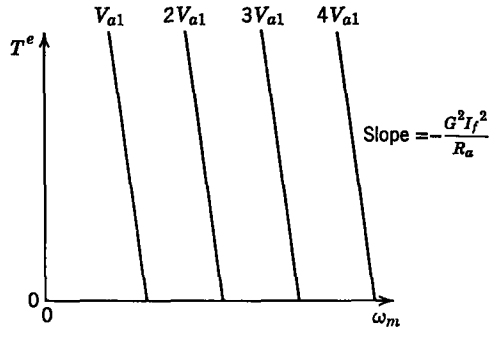


Fig. 6.4.8 Torque speed curves of separately excited dc motor.

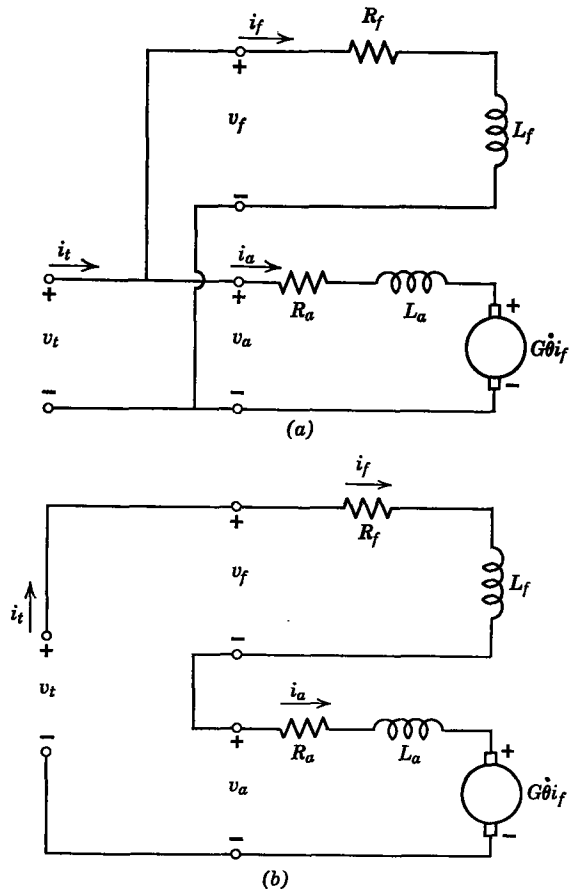


Fig. 6.4.9 Methods of self-exciting a dc motor: (a) shunt excitation; (b) series excitation.

There are two other fundamental ways of providing field excitation in dc motors; shunt excitation, illustrated schematically in Fig. 6.4.9a, and series excitation, illustrated in Fig. 6.4.9b. Considering shunt excitation with a steady state dc problem, we neglect the inductance and write

$$v_f = i_f R_f, \quad (6.4.42)$$

$$v_a = i_a R_a + G\hat{\theta} i_f. \quad (6.4.43)$$

The shunt terminal constraints of Fig. 6.4.9a are

$$v_t = v_a = v_f,$$

$$i_t = i_a + i_f.$$

Using these constraints with (6.4.42) and (6.4.43) yields for the armature current and terminal current

$$i_a = \left(1 - \frac{G\hat{\theta}}{R_f}\right) \frac{v_t}{R_a}, \quad (6.4.44)$$

$$i_t = \left(\frac{1}{R_f} + \frac{1}{R_a} - \frac{G\hat{\theta}}{R_f R_a}\right) v_t. \quad (6.4.45)$$

The torque of electric origin is (6.4.30)

$$T^e = G \left(1 - \frac{G\hat{\theta}}{R_f}\right) \frac{v_t^2}{R_f R_a}. \quad (6.4.46)$$

The torque-speed and terminal-current-speed curves for shunt excitation are sketched in Fig. 6.4.10. Because of the steep slope of the torque-speed curve, speed control is most effectively and efficiently achieved by the control of field resistance R_f because it determines the intercept with the speed axis.

Now consider a steady-state dc problem with series excitation as in Fig. 6.4.9b; (6.4.42) and (6.4.43) still hold but now the series connection imposes the constraints

$$v_t = v_a + v_f,$$

$$i_t = i_a = i_f.$$

Using these constraints with (6.4.42) and (6.4.43), we get for the terminal current

$$i_t = \frac{v_t}{R_f + R_a + G\hat{\theta}}, \quad (6.4.47)$$

and the torque of electric origin is (6.4.30)

$$T^e = G i_t^2 = \frac{G v_t^2}{(R_f + R_a + G\hat{\theta})^2}. \quad (6.4.48)$$

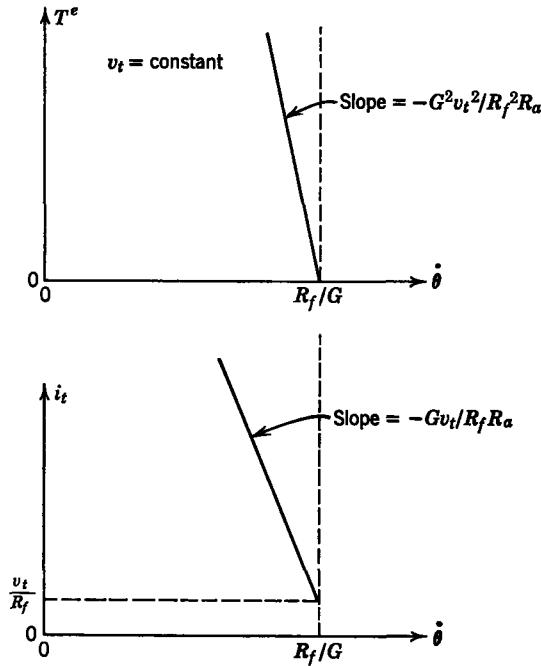


Fig. 6.4.10 Torque and terminal current as functions of speed in a shunt-excited dc motor with constant terminal voltage.

The torque and terminal current are sketched as functions of speed for constant terminal voltage in Fig. 6.4.11. Series-excited motors are used for traction drives and other applications in which high starting torque is required. The starting current is usually limited by variable series resistance, as is the running current.

A commutator machine like that of Fig. 6.4.2 may have more than one field winding on the same field structure. Oftentimes two field windings are used, one for shunt excitation and one for series excitation, in which case the machine is called a *compound* motor. It should be clear that combinations of the characteristics of Fig. 6.4.10 and 6.4.11 in varying amounts can lead to a wide variety of motor characteristics.*

We consider one final example of the possible steady-state characteristics of the commutator machine in Fig. 6.4.2, and that is as a shunt, self-excited, dc generator. By self-excited, we mean that only mechanical input is required to produce an electrical output. The field current is generated by the armature. The successful operation of this machine depends on saturation in the magnetic material, which shows up in the speed coefficient G , thus reducing

* See, for example, Fitzgerald and Kingsley, *op. cit.*, pp. 141–142.

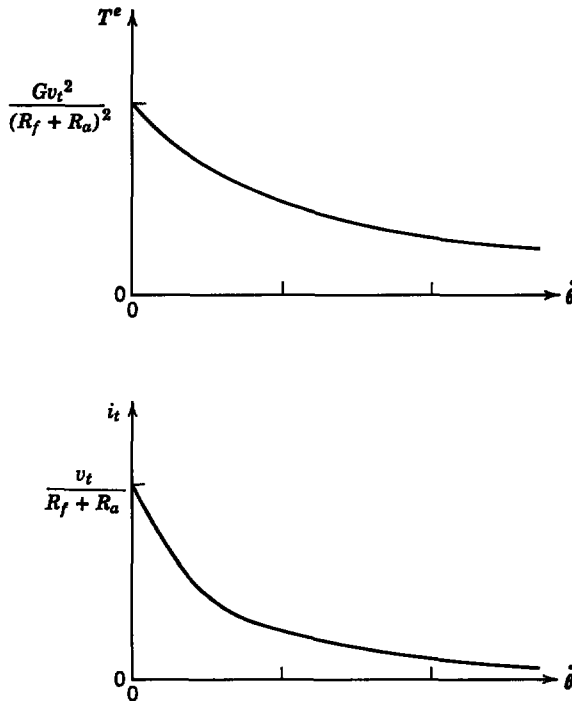


Fig. 6.4.11 Torque and terminal current as functions of speed for series-excited dc motor with constant terminal voltage.

the value of G as the excitation current i_f , hence the speed voltage increase. This saturation is usually represented by a plot of armature open-circuit voltage v_{oc} in Fig. 6.4.12 (the speed voltage $G\omega_m i_f$) as a function of field current with speed held fixed. Such a curve is sketched in Fig. 6.4.12a. Now connect the field winding in parallel with the armature terminals with no load connected, as shown in Fig. 6.4.12b; the field current is determined in the steady state by the field resistance line in Fig. 6.4.12a. The steady-state operating point at which the circuit equations are satisfied in the steady state is shown in Fig. 6.4.12a. If the field current is below the steady-state value, the excess of generated voltage over the iR drop goes into increasing the current as illustrated. Thus, if a small amount of voltage is produced (usually by residual field flux), the terminal voltage will build up automatically to the operating point, the rate of buildup being determined by the relative shapes of the two curves in Fig. 6.4.12a and by the machine inductances. The addition of a load causes a little additional voltage drop in R_a but does not change the essential features of the argument. A moderate range of voltage control is achieved by varying the field resistance R_f , the slope of the field resistance line, hence the intersection of the two curves in Fig. 6.4.12a.

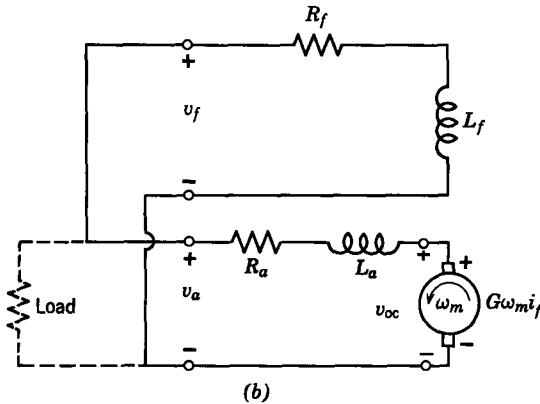
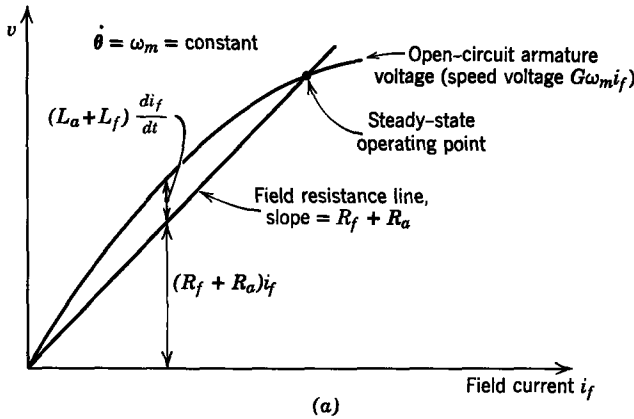


Fig. 6.4.12 Illustrating shunt self-excitation in a dc generator: (a) voltage versus field current curves; (b) shunt-connected generator with no load.

Generators can be series-excited and can use both series and shunt excitation. The techniques of analysis are similar to those used for shunt excitation and a great variety of terminal characteristics can be obtained.*

All of our examples have been based on dc excitation. Commutator machines are also used with alternating current. The techniques of analysis are essentially the same, but the inductances play a vital role in determining steady-state characteristics. Some examples of ac commutator machines are studied in the problems at the end of this chapter.

6.4.2 Homopolar Machines

A physically different type of dc rotating device is the homopolar machine, but, as we shall see, it has terminal behavior much like that of a commutator

* Fitzgerald and Kingsley, *op. cit.*, pp. 139-141.

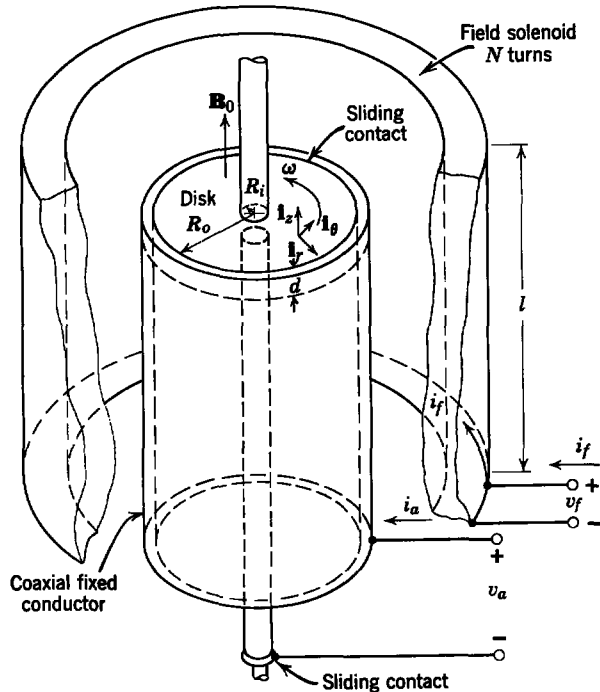


Fig. 6.4.13 Configuration for analysis of homopolar machine.

machine. One version of a homopolar machine is introduced in Example 6.3.1 to illustrate the application of Ohm's law for a moving conductor. A cutaway view of a practical homopolar generator appears in Fig. 6.3.2.

In Example 6.3.1 a limited analysis is made of the armature characteristics under steady-state conditions. In this section that analysis is generalized to include transients in the armature circuit and extended to find the terminal equations for the field circuit and the mechanical system.

For the analysis we use the configuration of Fig. 6.4.13 which is representative of a homopolar machine that would use a superconducting field coil. The system consists of a disk of the dimensions shown and connected to a shaft that is rotating with angular speed ω . A coaxial fixed conductor makes sliding electrical contact (conventionally through a liquid metal) with the outside surface of the disk. Another sliding contact is made with the shaft as shown and these two connections to the disk form the armature terminals. The whole assembly is surrounded by a coaxial solenoid that produces an axial flux density at the disk. The terminals of the solenoid are the field circuit terminals. The reason for the carefully specified cylindrical symmetry is mathematical simplicity and is not necessary for the analytical techniques to

be correct. In many practical cases, however, symmetry is desirable to cut down on harmful electrical and electromechanical effects like circulating currents, thrust on bearings, and forces on the solenoid.

Current to and from the disk in the shaft and in the coaxial conductor is axial and has cylindrical symmetry. (We assume that the armature terminals are far enough removed from the disk and solenoid that the end effects can be ignored.) Current in the disk is radial and distributed uniformly at any radius. Consequently, the flux density produced by armature current is tangential and none of it will link the field solenoid. Therefore, we can write the field circuit equation immediately as

$$v_f = R_f i_f + L_f \frac{di_f}{dt}, \quad (6.4.49)$$

where R_f = field circuit resistance,

L_f = field circuit inductance.

This is exactly the same form as (6.4.1) for the commutator machine.

The electromagnetic behavior of the disk was analyzed for steady-state conditions in Example 6.3.1. When we assume that any transients occur slowly enough not to disturb appreciably the uniformity of current density in the disk, (we pursue this point in Chapter 7), the results of Example 6.3.1 are still valid, provided we add a self-inductance term. (There is no net field-produced flux linkage with the armature.) Thus the armature voltage equation is

$$v_a = R_a i_a + L_a \frac{di_a}{dt} + G \omega i_f, \quad (6.4.50)$$

where R_a is the armature resistance found in Example 6.3.1 (where it is called R_{int})

$$R_a = \frac{\ln(R_o/R_i)}{2\pi\sigma d}; \quad (6.4.51)$$

L_a is the armature self-inductance that can be calculated from the geometry and G is a speed coefficient found from equating the open-circuit voltage in Example 6.3.1 to the speed voltage.

$$G \omega i_f = \frac{\omega B_0}{2} (R_o^2 - R_i^2). \quad (6.4.52)$$

As for the commutator machine, this speed coefficient G depends only on geometry and material properties.

We obtain an approximate value for G by assuming that the solenoid has N total turns and is axially long with small radial buildup. With an axial

length l , the value of flux density near the center is (from Ampère's law)

$$\mathbf{B}_0 = i_z B_0 = i_z \frac{\mu_0 N i_f}{l}. \quad (6.4.53)$$

The use of this result in (6.4.52) and the solution for G yields

$$G = \frac{\mu_0 N}{2l} (R_o^2 - R_i^2). \quad (6.4.54)$$

Equation 6.4.50, which is the armature voltage equation for the homopolar machine, is the same form as the armature equation for the commutator machine (6.4.20). We could have derived this equation in a manner analogous to that used for the commutator machine; for example, if we had chosen a contour C , illustrated schematically in Fig. 6.4.14, then, with the contour fixed in the laboratory frame, we could write Faraday's law as

$$-\int_a^b \mathbf{E} \cdot d\mathbf{l} - \int_b^a \mathbf{E}' \cdot d\mathbf{l} = \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da. \quad (6.4.55)$$

The first term on the left is the terminal voltage, the second term on the left contains the resistive voltage drop and the speed voltage, and that on the right is the self-inductance term. To complete such an analysis we must simply follow the steps used in going from (6.4.3) to (6.4.20) for the commutator machine.

Alternatively, the contour C can be fixed to the disk in Fig. 6.4.14, and Faraday's law is written as

$$-\int_a^b \mathbf{E} \cdot d\mathbf{l} - \int_b^a \mathbf{E}' \cdot d\mathbf{l} = \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da. \quad (6.4.56)$$

In this case the first term is still the terminal voltage, but now the second term on the left contains only the resistance voltage drop, and because of the time-varying surface S the term on the right includes both the speed voltage and voltage of self-inductance.

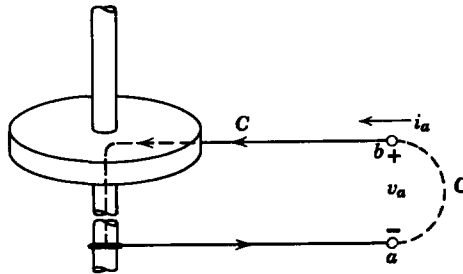


Fig. 6.4.14 Contour of integration for Faraday's law.

To complete the description of the homopolar machine, the torque of electric origin must be obtained. For this we use the force density of electric origin for magnetic field systems (1.1.30).

$$\mathbf{F} = \mathbf{J} \times \mathbf{B}. \quad (6.4.57)$$

Using the unit vectors for a cylindrical coordinate system shown in Fig. 6.4.13, the current density in the disk at a radius r is

$$\mathbf{J} = \mathbf{i}_r J_r = -\mathbf{i}_r \frac{i_a}{2\pi r d}. \quad (6.4.58)$$

[See (b) of Example 6.3.1 and note that the assumed positive direction of armature current has been reversed.] The only flux density that interacts with this current density to produce a torque about the axis of rotation is the field flux density given by (6.4.53). The use of these two quantities in (6.4.57) yields

$$\mathbf{F} = \mathbf{i}_\theta \frac{\mu_0 N i_f i_a}{2\pi l dr}. \quad (6.4.59)$$

The force is tangential; thus we multiply by the lever arm r and integrate throughout the volume of the disk to find the torque T^e in the direction of positive rotation ω

$$T^e = \int_{R_i}^{R_o} \int_0^{2\pi} \left(\frac{\mu_0 N i_f i_a}{2\pi l} \right) r dr d\theta. \quad (6.4.60)$$

Evaluation of the integral yields

$$T^e = \frac{\mu_0 N}{2l} (R_o^2 - R_i^2) i_a i_f = G i_a i_f, \quad (6.4.61)$$

where the speed coefficient was defined in (6.4.54).

The results of (6.4.61) can be derived also by the energy methods of Chapter 3, provided great care is exercised in defining the moving circuit. It is much more reasonable and straightforward to evaluate the torque in the manner we did.

We now use the results of (6.4.49), (6.4.50), and (6.4.61) to redraw the equivalent circuit of Fig. 6.4.5. We have included the usual mechanical elements, and it would be well to remark that with liquid-metal brushes there is little coulomb friction (T_0) in a homopolar machine.

The equivalent circuit of Fig. 6.4.5 was originally drawn for the commutator machine discussed in Section 6.4.1. This emphasizes the similarity of commutator and homopolar machines. All of the discussion of the properties of commutator machines holds equally well for homopolar machines, with the qualification that relative parameter values in the two types of machine

are different; for example, the homopolar machine has essentially a one-turn armature and is always a low-voltage, high-current device. Thus matching a wire-wound coil to this low impedance is difficult and few homopolar machines can be self-excited, either with series field windings or with shunt field windings. Consequently, most homopolar machines have separately excited field windings.

Because of the similarity between homopolar and commutator machine characteristics we terminate the discussion here and treat homopolar machines further in the problems at the end of this chapter.

6.5 DISCUSSION

In this chapter we have made the necessary generalizations of electromagnetic theory that are needed for analyzing quasistatic systems with materials in relative motion. This has involved transformations for source and field quantities between inertial reference frames, boundary conditions for moving boundaries, and constituent relations for moving materials. In addition to some simple examples, we have made an extensive analysis of dc rotating machines because they are devices that are particularly amenable to analysis by the generalized field theory.

Having completed the generalization of field theory with illustrative examples of lumped-parameter systems, we are now prepared to proceed to continuum electromechanical problems. In Chapter 7 we consider systems with specified mechanical motion and in which electromagnetic phenomena must be described with a continuum viewpoint.