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Continuum Electromechanics

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For Section 5.3:

Prob. 5.3.1 For flow and field that are two-dimensional and represented in Cartesian coordinates using the definitions suggested by Table 2.18.1, show that lines along which the charge density is constant are represented by Eq. 5.3.13a.

Prob. 5.3.2 For flow and field that are axisymmetric in cylindrical coordinates, as represented by that case in Table 2.18.1, show that lines along which the charge density is constant are given by Eq. 5.3.13b.

For Section 5.4:

Prob. 5.4.1 Gas passes through the planar channel shown in Fig. P5.4.1 with the velocity $4U(x/d)[1 - x/d]\vec{i}_y$. An electric field is imposed by placing the lower plane at potential V relative to the upper one. Between $x = 0$ and $x = a$ on this lower plane, positively charged particles having mobility b are injected through a metallic grid. A goal is to determine the current i collected by an electrode imbedded opposite the injection grid. It is presumed that the potential of this electrode remains essentially zero.
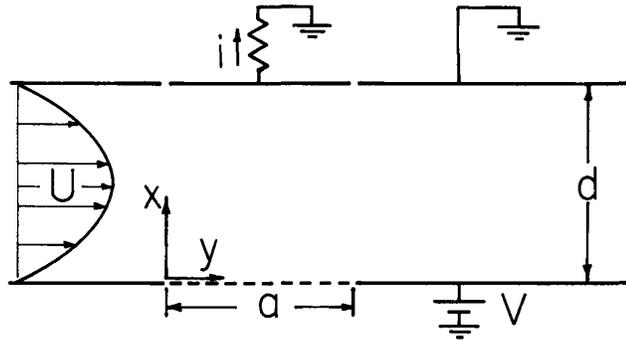
(a) Use the result of Prob. 5.3.1 to show that the injected particles follow the characteristic lines

$$-2 \frac{U}{d} x^2 (1 - \frac{2x}{3d}) + \frac{bV}{d} y = \text{constant}$$

(b) Show that the current-voltage relation is

$$
i =
\begin{cases}
\frac{bV}{d} nq[a - \frac{2}{3} \frac{Ud}{(bV/d)}] \, , & V > \frac{2}{3}Ud^2/ba \\[2em]
0, & V < \frac{2}{3}Ud^2/ba
\end{cases}
$$

Fig. P5.4.1

Prob. 5.4.2 The potential of a spherical particle having radius R is constrained to be

$$\Phi(r=R) = V\cos\theta$$

(This could be accomplished by making the surface from electrode segments, properly constrained in potential.) The sphere is surrounded by fluid generally moving in the z direction. The flow is solenoidal and irrotational, consistent with its being inviscid and entering at $z \rightarrow -\infty$ without rotation. (See Fig. P5.4.1. Such flows are taken up in Chap. 7.) The fluid flow velocity is given as

$$\vec{v} = -\nabla\Phi_v \quad ; \quad \Phi_v = -UR[\frac{r}{R} + \frac{1}{2} \frac{R^2}{r^2}] \cos\theta$$

There are no other sources of field than those on the sphere itself. The following steps establish the electrical current on the sphere created by ions entering uniformly with the fluid at $z \rightarrow -\infty$.
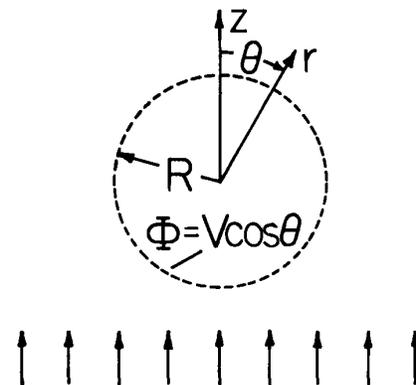
(a) Assume that the contribution of the ion space charge to the field is negligible, and represent $\vec{E}$ and $\vec{v}$ in terms of Λ_E and Λ_v.

Fig. P5.4.2

Prob. 5.4.2 (continued)

(b) Find the expression for the particle trajectories in the form

$$f(\frac{r}{R}, \quad \theta, \quad \frac{Vb}{UR}) = \text{constant}$$

(c) Assume that $V > 0$ and that the ions are positive. Find the critical points in the region outside the sphere.

(d) Plot the characteristic lines in two cases: for $bV/RU < \frac{3}{2}$ and for $bV/RU > \frac{3}{2}$. Identify the critical points in the case where they exist in the region outside the sphere.

(e) Find the current i to the particle as a function of bV/RU. (Be sure to identify any "break points" in this V-i relation.

Prob. 5.4.3 A circular cylindrical conductor having radius a has the potential V relative to a surrounding coaxial cage having radius R_0 (Fig. P5.4.3). Hence it imposes an electric field $\vec{E} = (V/r)/\ln(R_0/a)$ on the air in the region $a < r < R_0$. The wind passing perpendicular to this conductor has the velocity

$$\vec{v} = -U(1 - \frac{a^2}{r^2}) \cos \theta \ \vec{i}_r + U(1 + \frac{a^2}{r^2}) \sin \theta \ \vec{i}_\theta$$

consistent with an inviscid model. (Thus, there is a finite tangential wind velocity at the surface of the conductor.) Charged particles enter uniformly at the appropriate "infinity." This might be a model for the contamination of a high-voltage d-c conductor by naturally charged dust.
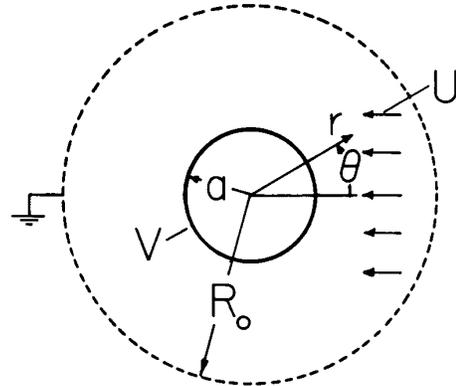
(a) Consider two cases: (i) conductor and particles of the same polarity and, (ii) conductor and particles of opposite polarity. This is equivalent to taking the particles as positive and V as positive or negative. Find the critical points (lines).

Fig. P5.4.3

(b) Find the characteristic lines and sketch them for the two cases.

(c) Determine the electrical current to the conductor as a function of V.

Prob. 5.4.4 Fluid enters the region between the electrodes shown in Fig. P5.4.4 through a slit at the top (where $x = c$). The system extends a length ℓ into the paper and the volume rate of flow through the slit is Q_v m^3/sec. The electrodes to left and right respectively are located at $xy = -a^2$ and $xy = a^2$ and have the constant potentials $-V_0$ and V_0. The electrodes in the plane $x = 0$ are essentially grounded, with the one between $x = -a$ and $x = a$ used to collect the current i. Entrained in the gas as it enters at $x = c$ is a charge density that is uniform over the cross section at that location. The charge density is ρ_0. The fluid velocity is

$$\vec{v} = 2C(x\vec{i}_x - y\vec{i}_y)$$

(a) What is the constant C?

(b) Find the critical lines, if any.

(c) Given a certain volume rate of flow Q_v, find the current i to the center electrode as a function of bV_0, where b is the mobility of the charged particles. Present $i(bV_0)$ as a dimensioned sketch. (Assume that Q_v and V_0, as well as the charge density ρ_0, are positive.)
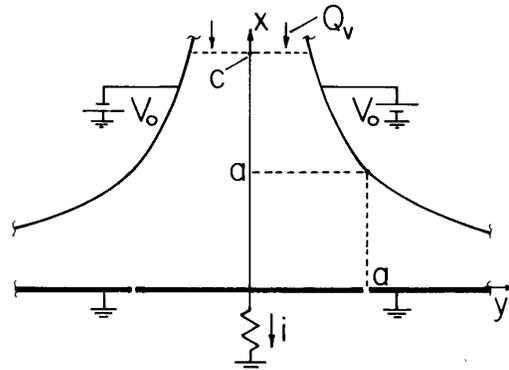
Fig. P5.4.4

For Section 5.5:

Prob. 5.5.1 For a "drop" in an ambient electric field and flow as discussed in this section, both positive and negative "ions" are present simultaneously. The objective here is to make a charging diagram patterned after those of Figs. 5.5.3 and 5.5.4. Because there are now two different

mobilities, b_+ and b_-, it is best to make the abscissa the imposed electric field $\vec{E}$. Construct the charging diagram, including charging trajectories, showing final values of charge. (With bipolar charging, the final charge can be less than q_c in magnitude. Expressions should be derived for these limiting values of charge.)

Prob. 5.5.2 The objective is to determine the charging diagrams, Figs. 5.5.3 and 5.5.4, with the low Reynolds number flow represented by Eq. 5.5.5 replaced by an inviscid flow. (See Sec. 7.8 for discussion of this class of flows.) Important here is the fact that such a flow can have a finite tangential velocity on a rigid boundary. The fluid velocity is given here as

$$\vec{v} = -U[1 - (\frac{R}{r})^3] \cos\theta\vec{i}_r + U[1 + \frac{R^3}{2r^3}] \sin\theta\vec{i}_\theta$$

(a) Find Λ_v and the general characteristic equation that replaces Eq. 5.5.6.

(b) Because both tangential and normal velocity are zero on the surface of the "drop" for the low Reynolds number flow, the points on the surface described by Eq. 5.5.10 are critical points. With an inviscid flow, matters are not so simple. Show that, as before, there are now two types of critical points, one type lying on the z axis and the other not. Find analytical expressions for the (r,θ) locations of these latter critical lines.

(c) Construct the charging diagrams for positive and negative "ions."

For Section 5.6:

Prob. 5.6.1 Unless some of an initial charge distribution reaches a boundary, self-precipitating charge of one polarity must conserve its total value. With the charge density given as a function of time by Eq. 5.6.6 and the volume filled by this density described by Eqs. 5.6.9 and 5.6.10, show that for the example of Fig. 5.6.3 this is indeed the case.

Prob. 5.6.2 Fig. P5.6.2 shows a one-dimensional configuration involving a unipolar conduction transient. Gas flows through a duct with the uniform velocity $U\vec{i}_z$. Screen electrodes at $z = 0$ and $z = \ell$ have the constant potential difference v. When $t = 0$, there is a uniform distribution of charged particles having charge density ρ_o and mobility b in the region between $z = z_B$ and $z = z_F$. The regions in front of this layer and behind it have no initial charge density. Assume that the charge is positive. In the following the evolution of the layer is to be described during the time that it has not encountered the screen electrodes.
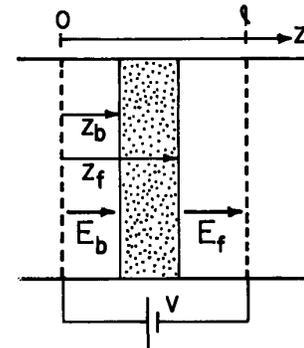

Fig. P5.6.2

(a) Show that the charge density within the layer remains uniform and find its dependence on time.

(b) Use Gauss' law to deduce that $(z_f - z_b) = (1 + t/\tau)(z_F - z_B)$; $\tau \equiv \epsilon_o/\rho_o b$.

(c) Use Gauss' law and the potential constraint to relate $E_b(t)$, $E_f(t)$, $z_b(t)$ and $z_f(t)$.

(d) Use the second characteristic equations to also relate these four quantities.

(e) Find $z_f(t)$ and $z_b(t)$ and sketch the charge evolution in the z-t plane (as in Fig. 5.6.3).

For Section 5.7:

Prob. 5.7.1 The steady-state charge distribution of Eq. 5.7.3 is time-varying from the particle frame of reference. Hence, in accordance with Eqs. 5.6.2 and 5.6.3, the charge density decays from the frame of reference of a given particle. Start with these characteristic equations and deduce Eq. 5.7.3.

For Section 5.9:

Prob. 5.9.1 When $t = 0$, a region of fluid described by the bipolar laws, Eqs. 5.8.9 and 5.8.10, has uniform neutral density n_o and species charge densities $\rho_+ = \rho_- = 0$. A self-consistent picture of the ensuing dynamics has these densities evolving uniformly. This is possible because there is no applied

electric field and because $\rho_+ = \rho_-$, so there is no self-field either.

(a) Use the conservation laws to show that $\rho_+ = \rho_-$ is consistent with $E = 0$.

(b) Write an ordinary differential equation for $n(t)$ and one for $\rho_\pm (t)$.

(c) Argue that the stationary equilibrium state is one having $\beta n = \frac{\alpha}{q}\rho_+\rho_-$.

(d) Show that the time characterizing the early stages of the system's approach to this equilibrium is $\tau_{th} = q/\beta$.

For Section 5.10:

Prob. 5.10.1 (conductivity model) In the region $0 < x < d$, the fluid velocity is $\vec{v} = U(x/d)\vec{i}_z$. When $t = 0$, the volume charge density is zero for $z < 0$ and is a constant ρ_o for $0 < z$. Describe $\rho_f(x,z,t)$ for $t > 0$. Represent the distribution in the (x-z) plane, giving analytical expressions for wavefronts and decay rates.

Prob. 5.10.2 (conductivity model) The fluid velocity is as in Prob. 5.10.1. When $t = 0$, $\rho_f(x,z)$ $= 0$ for $z > 0$. A source of charge is used to constrain the charge density to be a step function in the $z = 0$ plane. That is, $\rho_f(x,0,t) = \rho_s u_{-1}(t)$. Describe the charge evolution, including sketches in the x-z plane and analytical expressions for wavefronts and decay rates. What is the steady state condition and at a given position (x,z) when is it established?

Prob. 5.10.3 A particle initially has a net charge $q = q_0$ and is immersed in an electrolyte that has uniform conductivity and permittivity. Write integral statements of Gauss' law and the conservation of charge for a volume enclosing the particle. Show that $q(t) = q_o\exp(-t/\tau)$, where τ is the charge relaxation time ε/σ.

For Section 5.12:

Prob. 5.12.1 The planar layer of Table 2.16.1 is composed of a material having uniform permittivity ε and uniform anisotropic conductivity σ_{ij}, such that

$$\vec{J} = \sigma_x E_x \vec{i}_x + \sigma_y E_y \vec{i}_y + \sigma_z E_z \vec{i}_z$$

(a) Show that for variables taking the form $\Phi = \mathrm{Re}\hat{\Phi}(x)\,\exp j(\omega t - k_y y - k_z z)$, the current density $\mathscr{J} \equiv (j\omega\varepsilon + \sigma_x)\hat{E}_x$ (the sum of the displacement and conduction currents needed to write the conservation of charge boundary condition at an interface) evaluated at the (α,β) surfaces is related to the potentials there by

$$
\begin{bmatrix} \mathscr{J}^\alpha_x \\ \\ \mathscr{J}^\beta_x \end{bmatrix}
= (j\omega\varepsilon + \sigma_x)\gamma
\begin{bmatrix} -\coth\gamma\Delta & \dfrac{1}{\sinh\gamma\Delta} \\ \\ \dfrac{-1}{\sinh\gamma\Delta} & \coth\gamma\Delta \end{bmatrix}
\begin{bmatrix} \hat{\Phi}^\alpha \\ \\ \hat{\Phi}^\beta \end{bmatrix}
$$

where $\gamma^2 \equiv [k_y^2(\sigma_y + j\omega\varepsilon) + k_z^2(\sigma_z + j\omega\varepsilon)]/(\sigma_x + j\omega\varepsilon)$

(b) Consider as a special case $\sigma_y = \sigma_z = 0$, so that conduction is only in the x direction. Discuss implications of γ for penetration of the field in the x direction as function of frequency and of $k^2 \equiv k_y^2 + k_z^2$. In particular, what is the nature of field distribution in the limit $\omega \to 0$?

(c) Consider $\sigma_x = 0$ and $\sigma_y = \sigma_z = \sigma_0$, so that conduction is confined to y-z planes. Discuss the field distribution as in (b) and draw contrasts.

Prob. 5.13.1   A circular analogue of the case study considered in this section is shown in Fig. P5.13.1. A rotating shell has radius R and angular velocity Ω. A traveling wave of potential is applied to electrodes around the shell at a radius a, while an equipotential electrode is at the center with radius b.

(a) Find the surface potential of the rotating shell.

(b) Determine the electrical torque acting on the shell.

Prob. 5.13.2   As a continuation of Prob. 5.13.1, a tachometer is constructed as shown in Fig. 5.13.4. Determine the output current in forms analogous to Eqs. 5.13.15 and 5.13.16.
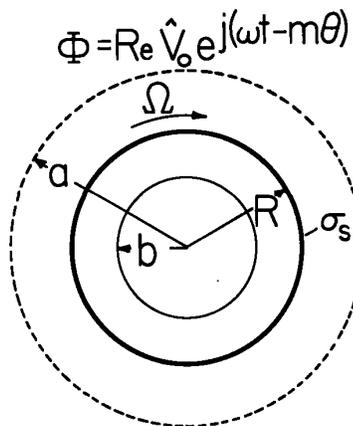
For Section 5.14:

Prob. 5.14.1   The circular analogue of the planar configuration considered in this section is shown in Fig. 5.14.2. The following steps are intended to parallel those of the text for this configuration. Define the angular velocity of the rotor as $\Omega = U/R$.

(a) Write the electrical torque in a form analogous to Eq. 5.14.6.

(b) Find the surface potential of the rotor in a form analogous to that of Eq. 5.14.8.

(c) Write the electrical torque in a form like that of Eq. 5.14.11, identifying S_e and τ_E.

$$\Phi = Re \, \hat{V}_0 e^{j(\omega t - m\theta)}$$

Fig. P5.13.1

Prob. 5.14.2   Motions of Von Quincke's rotor, shown in Fig. 5.14.4c, can be of far greater complexity than the steady rotations considered here. To study these motions, it is appropriate to develop a "lumped parameter" model which exploits the fact that the dynamics enter only through the boundary conditions at the rotor interface. Plane parallel electrodes are used to impose an electric field $-E(t)\vec{i}_x$ perpendicular to the cylinder. The region surrounding the rotor is electrically taken as extending to "infinity," where the electric field is this imposed field. In the region immediately surrounding the rotor, the potential takes the form

$$\Phi = E(t)r\cos\theta + P_x(t)\frac{\cos\theta}{r} + P_y(t)\frac{\sin\theta}{r}$$

Permittivities of the surrounding fluid and the cylinder are respectively ε_a and ε_b. The cylinder is insulating while the fluid has conductivity σ. The rotor has radius b, moment of inertia per unit axial length I and a viscous damping torque per unit length $-B\Omega$, where $\Omega(t)$ is the rotor angular velocity.

(a) Show that motions of the rotor are in general described by the nonlinear equations

$$P_e^{-1}\overset{o}{\underline{\Omega}} + \underline{\Omega} = \underline{E}\,\underline{P}_y$$

$$\overset{o}{\underline{P}}_x + \underline{\Omega P}_y + \underline{P}_x = H_e^2(-f\overset{o}{\underline{E}} + \underline{E})$$

$$\overset{o}{\underline{P}}_y - \underline{\Omega P}_x + \underline{P}_y = fH_e^2\underline{E}\,\underline{\Omega}$$

where variables have been normalized such that

$$t = \underline{t}\tau_e \quad ; \quad \tau_e \equiv (\varepsilon_a + \varepsilon_b)/\sigma$$

$$\underline{\Omega} = \underline{\Omega}\tau_e \quad ; \quad \underline{E}(t) = E(t)/\mathscr{E}$$

$$\underline{P}_{\binom{x}{y}} = \frac{2\varepsilon_a\pi\tau_e}{B}P_{\binom{x}{y}}$$

so that $\mathscr{E}$ is a typical electric field intensity. For example, if $E(t)$ is a constant, $\mathscr{E}$ is that constant and $E = 1$. Other dimensionless parameters are the electric Hartmann number H_e (given in

Sec. 8.7 as the square root of the ratio of the charge relaxation time to the electro-viscous time τ_{EV}) and the electric Prandtl number p_e (the ratio of the charge relaxation time to the viscous diffusion time). Thus

$$H_e \equiv \sqrt{\frac{2\varepsilon_a \pi R^2 \tau_e}{B}} \quad ; \quad p_e = \tau_e / I/B \quad ; \quad f = \frac{\varepsilon_b - \varepsilon_a}{\varepsilon_b + \varepsilon_a}$$

If I is the moment of inertia of the rotor alone (ignoring inertial effects of the fluid), $I = \pi b^4 \rho / 2$. If viscous diffusion in the liquid is complete, $B = 4\pi b^2 \eta$, where η is the fluid viscosity and ρ is the rotor mass density. (See Sec. 9.3). Then $H_e^2 = \tau_e / \tau_{EV}$; $\tau_{EV} \equiv 2\eta / \varepsilon_a \mathscr{E}^2$ and $p_e \equiv \tau_e / \tau_V$; $\tau_V \equiv \rho b^2 / 8\eta$.

(b) The imposed field is raised very slowly. Use the results of (a) to deduce the threshold value of H_e at which the static equilibrium of the rotor is unstable. What steady values of $\underline{\Omega}$ result from raising H_e beyond this critical value?[1]

For Section 5.15:

Prob. 5.15.1   Identify the temporal modes for the rotor of Prob. 5.13.1.

Prob. 5.15.2   Identify the temporal modes for the rotor of Prob. 5.14.1.

Prob. 5.15.3   An insulating spherical particle having radius R and permittivity ε_b has angular velocity Ω about the z axis. It is surrounded by insulating material of infinite extent having permittivity ε_a. On its surface is a conducting coating having surface conductivity σ_s. Find the natural modes of decay for charge distributed on the surface. Modes included should represent the ϕ dependence exp $(jm\phi)$ by the mode number m, and the θ dependence by the mode number n of the function P_n^m. From these modes, pick the one that represents the rate of decay of a spherical particle initially in a uniform electric field, which is then suddenly turned on or off. Your result should be $\tau = R(2\varepsilon_a + \varepsilon_b)/2\sigma_s$.

Prob. 5.15.4   A particle has the properties given in Prob. 5.15.3. In addition, it has a bulk conductivity σ_b and the surrounding material has a bulk conductivity σ_a. Show that the relaxation time of the nth mode is

$$\tau_n = \frac{\varepsilon_a (n+1) + \varepsilon_b n}{\sigma_a (n+1) + \sigma_b n + \dfrac{\sigma_s}{R} n(n+1)}$$

Prob. 5.15.5   The planar layer described in terms of transfer relations in Prob. 5.12.1 is bounded in the planes $x = \Delta$ and $x = 0$ by equipotentials.

(a) Find an expression for the eigenfrequencies of the temporal modes.

(b) Show that as the material becomes isotropic in conductivity, so that $\sigma_x = \sigma_y = \sigma_z$, the infinite set of temporal modes all degenerate to the same eigenfrequency.

(c) Identify the eigenfrequencies for conduction confined to the x direction ($\sigma_y = \sigma_z = 0$) and plot as a function of $k \equiv \sqrt{k_y^2 + k_z^2}$ with the mode number n as a parameter.

(d) Proceed as in (c) for the case $\sigma_x = 0$, $\sigma_y = \sigma_z = \sigma_o$.

For Section 5.17:

Prob. 5.17.1   For the same configuration as developed in this section, define the sheet position as being at $x = 0$. Find the potential distribution for the regions above ($0 < x < d$) and below ($-d < x < 0$) the sheet. The expressions should reduce to Eqs. 5.17.17, 5.17.18 and 5.17.19 on the sheet surface ($x = 0$).

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1. Aperiodic motions such as these have been studied in connection with mathematically analogous models for thermal convection. See W.V.R. Malkus, "Nonperiodic Convection at High and Low Prandtl Number," Memoires Societe Royale des Sciences de Liege, 6 serie, tome IV, (1972), pp. 125-128.

Prob. 5.17.2   The system shown in Fig. P5.17.2 is the same
as considered in Sec. 5.14, except that the excitation on the
upper boundary starts at $z = 0$ and ends at $z = \ell$. The poten-
tial upstream and downstream on this surface is zero. Also,
the interface is midway between the transverse boundaries,
so a and b from Sec. 5.14 are equal to d.

(a) The potential at the interface in the sinusoidal steady
state is $\Phi^b(z,t) = \mathrm{Re}\,\hat{\Phi}^b(z,\omega)e^{j\omega t}$. Show that

$$\hat{\Phi}^b = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\hat{V}_o[(\omega-kU)\epsilon_a - j\sigma_a][e^{j(\ell-z)k}\,e^{-j\beta\ell} - e^{-jkz}]dk}{(k-\beta)\,D(\omega,k)}$$

where

$$D(\omega,k) = \cosh kd[(\sigma_a+\sigma_b) + j(\omega-kU)(\epsilon_a+\epsilon_b)]$$

(b) Show that the wavenumbers of the spatial modes are

$$k_n = \begin{cases} \dfrac{\omega}{U} - j\dfrac{(\sigma_a+\sigma_b)}{U(\epsilon_a+\epsilon_b)}\,, & n = 0 \\[3mm] \dfrac{\mp(|2n|-1)}{d}j\dfrac{\pi}{2}\,, & n = \pm\infty \ldots \pm 1 \end{cases}$$

Sketch the transverse and longitudinal dependences of these modes. Why do modes $n \neq 0$ have no depend-
ence on material properties, ω, or U?

(c) Use the Cauchy integral theorem to find $\Phi^b(z,t)$ from the result of part (a) and the modes of part (b).

(d) Find the total time average electrical force exerted in the z direction on the material. The
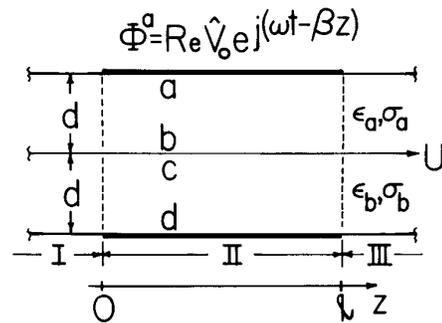expression can be left as an integral on k.

$$\Phi^a = \mathrm{Re}\,\hat{V}_o e^{j(\omega t - \beta z)}$$

Fig. P5.17.2

Problems for Chap. 5