NUMERICAL INTEGRATIONS, MODELING CONSIDERATIONS

LECTURE 8

47 MINUTES

LECTURE 8

Evaluation of isoparametric element matrices

Numercial integrations, Gauss, Newton-Cotes formulas

Basic concepts used and actual numerical operations performed

Practical considerations

Required order of integration, simple examples

Calculation of stresses

Recommended elements and integration orders for one-, two-, three-dimensional analysis, and plate and shell structures

Modeling considerations using the elements

TEXTBOOK: Sections: 5.7.1, 5.7.2, 5.7.3, 5.7.4, 5.8.1, 5.8.2, 5.8.3

Examples: 5.28, 5.29, 5.30, 5.31, 5.32, 5.33, 5.34, 5.35, 5.36, 5.37, 5.38, 5.39

NUMERICAL INTEGRATION, SOME MODELING CONSIDERATIONS

- Newton-Cotes formulas
- Gauss integration
- Practical considerations
- Choice of elements

We had

$$\underline{K} = \int_V \underline{B}^T \underline{C} \underline{B} \, dV \qquad (4.29)$$

$$\underline{M} = \int_V \rho \underline{H}^T \underline{H} \, dV \qquad (4.30)$$

$$\underline{R}_B = \int_V \underline{H}^T \underline{f}^B \, dV \qquad (4.31)$$

$$\underline{R}_S = \int_S \underline{H}^{S^T} \underline{f}^S \, dS \qquad (4.32)$$

$$\underline{R}_I = \int_V \underline{B}^T \underline{\tau}^I \, dV \qquad (4.33)$$

In isoparametric finite element analysis we have:

- the displacement interpolation matrix $\underline{H}$ (r,s,t)
- the strain-displacement interpolation matrix $\underline{B}$ (r,s,t)

  Where r,s,t vary from −1 to +1.

Hence we need to use:

$$dV = \det \underline{J}\; dr\, ds\, dt$$

Hence, we now have, for example in two-dimensional analysis:

$$\underline{K} = \int_{-1}^{+1}\int_{-1}^{+1} \underline{B}^T \underline{C}\, \underline{B} \det \underline{J}\; dr\; ds$$

$$\underline{M} = \int_{-1}^{+1}\int_{-1}^{+1} \rho\, \underline{H}^T \underline{H} \det \underline{J}\; dr\; ds$$

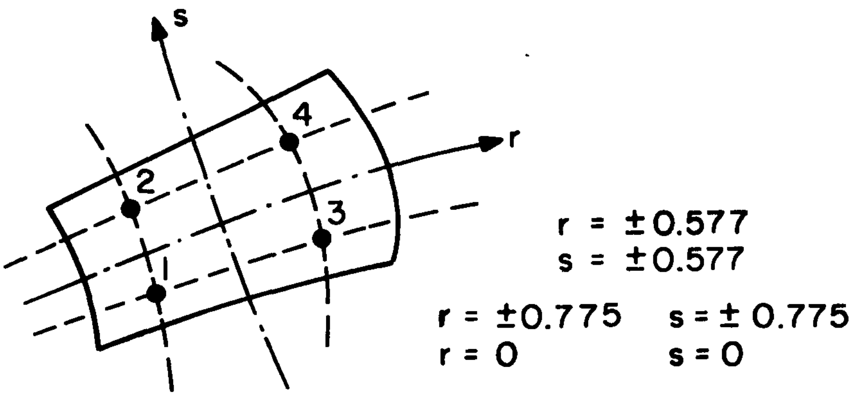
etc...

The evaluation of the integrals is carried out effectively using numerical integration, e.g.:

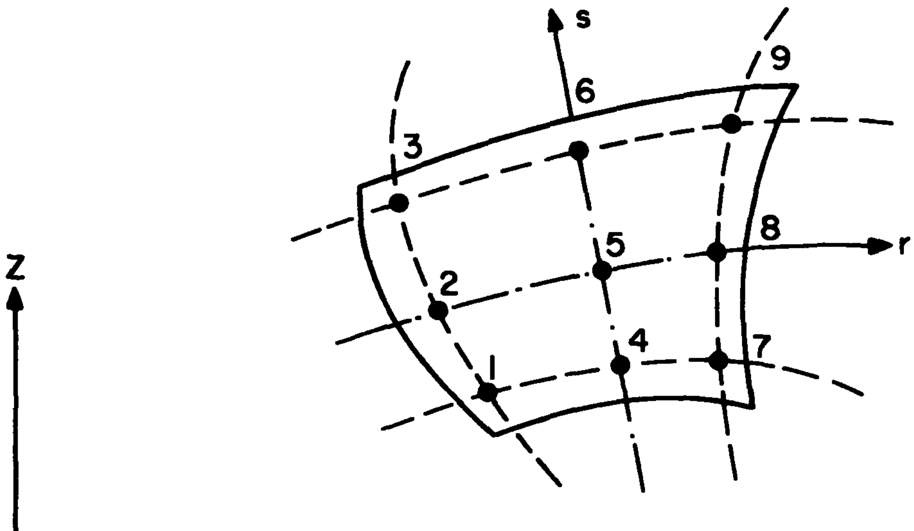
$$\underline{K} = \sum_i \sum_j \alpha_{ij} \, \underline{F}_{ij}$$

where

- i, j denote the integration points
- α_{ij} = weight coefficients
- $\underline{F}_{ij} = \underline{B}_{ij}^T \, \underline{C} \, \underline{B}_{ij} \, \det \underline{J}_{ij}$

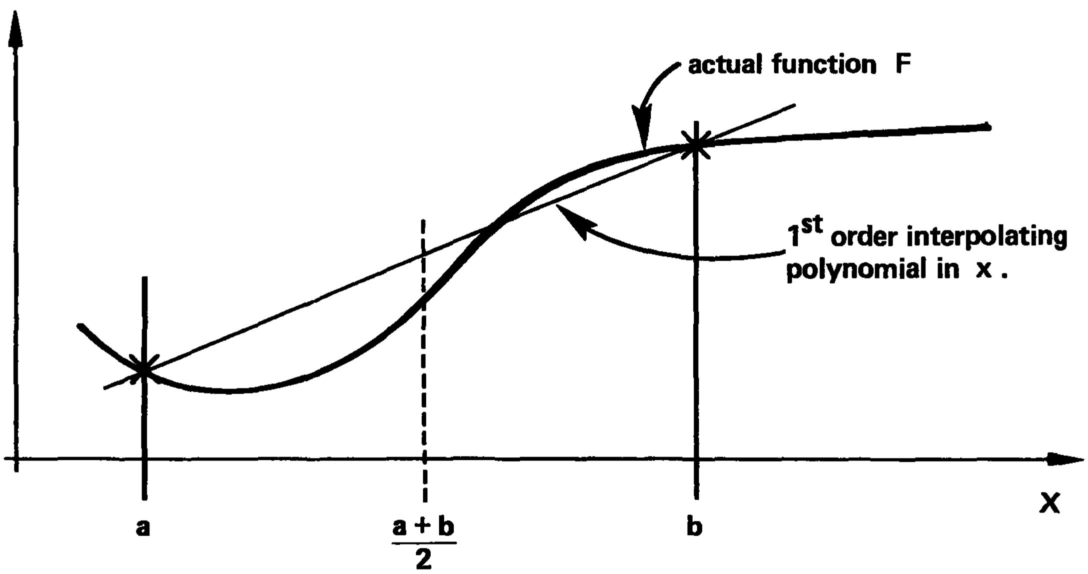


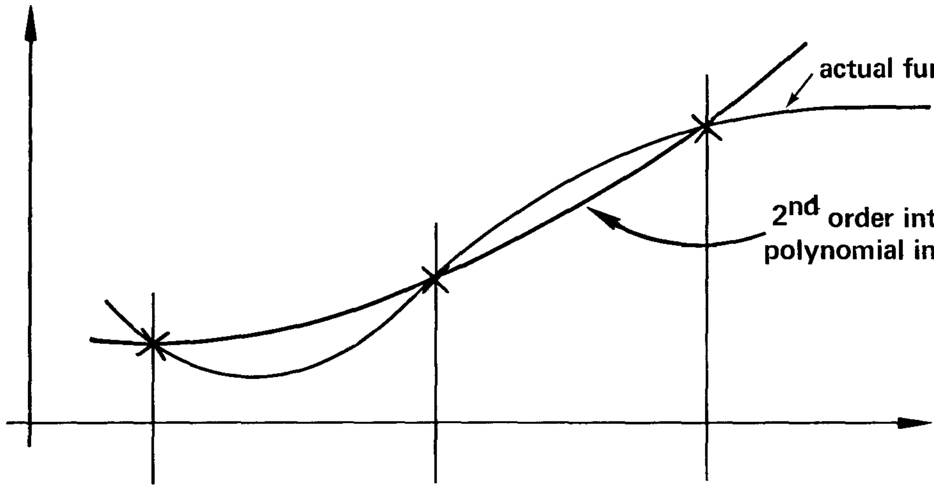
2x2-point integration



3x3 - point integration

Consider one-dimensional integration and the concept of an <u>interpolating polynomial</u>.





In Newton - Cotes integration we use sampling points at equal distances, and

$$\int_a^b F(r)dr = (b-a) \sum_{i=0}^{n} C_i^n F_i + R_n \tag{5.123}$$

n = number of intervals

C_i^n = Newton - Cotes constants

interpolating polynomial is of order n .

Number of Intervals n	C_0^n	C_1^n	C_2^n	C_3^n	C_4^n	C_5^n	C_6^n	Upper Bound on Error R_n as a Function of the Derivative of F
1	$\frac{1}{2}$	$\frac{1}{2}$						$10^{-1}(b-a)^3F^{II}(r)$
2	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$					$10^{-3}(b-a)^5F^{IV}(r)$
3	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$				$10^{-3}(b-a)^5F^{IV}(r)$
4	$\frac{7}{90}$	$\frac{32}{90}$	$\frac{12}{90}$	$\frac{32}{90}$	$\frac{7}{90}$			$10^{-6}(b-a)^7F^{VI}(r)$
5	$\frac{19}{288}$	$\frac{75}{288}$	$\frac{50}{288}$	$\frac{50}{288}$	$\frac{75}{288}$	$\frac{19}{288}$		$10^{-6}(b-a)^7F^{VI}(r)$
6	$\frac{41}{840}$	$\frac{216}{840}$	$\frac{27}{840}$	$\frac{272}{840}$	$\frac{27}{840}$	$\frac{216}{840}$	$\frac{41}{840}$	$10^{-9}(b-a)^9F^{VIII}(r)$

Table 5.1. Newton-Cotes numbers and error estimates.

In Gauss numerical integration we use

$$\int_a^b F(r)dr = \alpha_1 F(r_1) + \alpha_2 F(r_2) + \dots + \alpha_n F(r_n) + R_n \qquad (5.124)$$

where both the weights $\alpha_1, \dots, \alpha_n$ and the sampling points $r_1, \dots, r_n$ are variables.

The interpolating polynomial is now of order $2n-1$.

n	r_i	α_i
1	0. (15 zeros)	2. (15 zeros)
2	±0.57735 02691 89626	1.00000 00000 00000
3	±0.77459 66692 41483	0.55555 55555 55556
	0.00000 00000 00000	0.88888 88888 88889
4	±0.86113 63115 94053	0.34785 48451 37454
	±0.33998 10435 84856	0.65214 51548 62546
5	±0.90617 98459 38664	0.23692 68850 56189
	±0.53846 93101 05683	0.47862 86704 99366
	0.00000 00000 00000	0.56888 88888 88889
6	±0.93246 95142 03152	0.17132 44923 79170
	±0.66120 93864 66265	0.36076 15730 48139
	±0.23861 91860 83197	0.46791 39345 72691

Table 5.2. Sampling points and weights in Gauss-Legendre numerical integration.

Now let,

r_i be a sampling point and

α_i be the *corresponding weight*

for the interval -1 to +1.

Then the actual sampling point and weight for the interval a to b are

$$\frac{a+b}{2} + \frac{b-a}{2} r_i \text{ and } \frac{b-a}{2}\alpha_i$$

and the r_i and α_i can be tabulated as in Table 5.2.

In two- and three-dimensional analysis we use

$$\int_{-1}^{+1}\int_{-1}^{+1} F(r,s)\,dr\,ds = \sum_i \alpha_i \int_{-1}^{+1} F(r_i,s)\,ds \tag{5.131}$$

or

$$\int_{-1}^{+1}\int_{-1}^{+1} F(r,s)\,dr\,ds = \sum_{i,j} \alpha_i \alpha_j F(r_i,s_j) \tag{5.132}$$

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and corresponding to (5.113), $\alpha_{ij} = \alpha_i \alpha_j$, where α_i and α_j are the integration weights for one-dimensional integration. Similarly,

$$\int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} F(r,s,t)\,dr\,ds\,dt$$

$$= \sum_{i,j,k} \alpha_i \alpha_j \alpha_k F(r_i,s_j,t_k) \tag{5.133}$$

and $\alpha_{ijk} = \alpha_i \alpha_j \alpha_k$.

Practical use of numerical integration

- The integration order required to evaluate a specific element matrix exactly can be evaluated by studying the function $\underline{F}$ to be integrated.
- In practice, the integration is frequently not performed exactly, but the integration order must be high enough.

Considering the evaluation of the element matrices, we note the following requirements:

a) stiffness matrix evaluation:

(1) the element matrix does not contain any spurious zero energy modes (i.e., the rank of the element stiffness matrix is not smaller than evaluated exactly); and

(2) the element contains the required constant strain states.

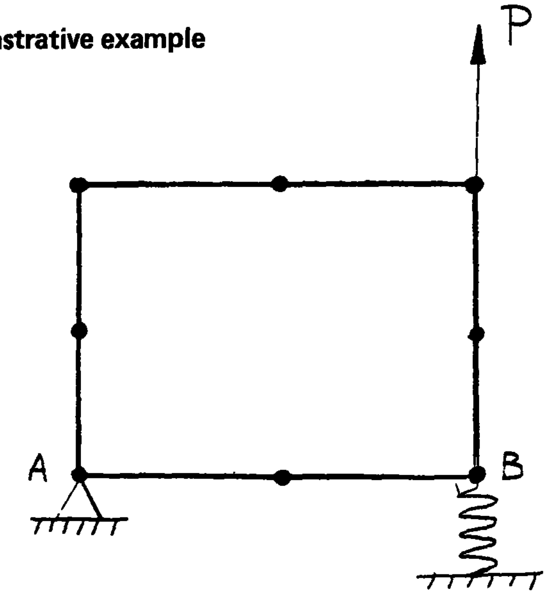
b) mass matrix evaluation:

the total element mass must be included.

c) force vector evaluations:

the total loads must be included.

Demonstrative example



2x2 Gauss integration
"absurd" results

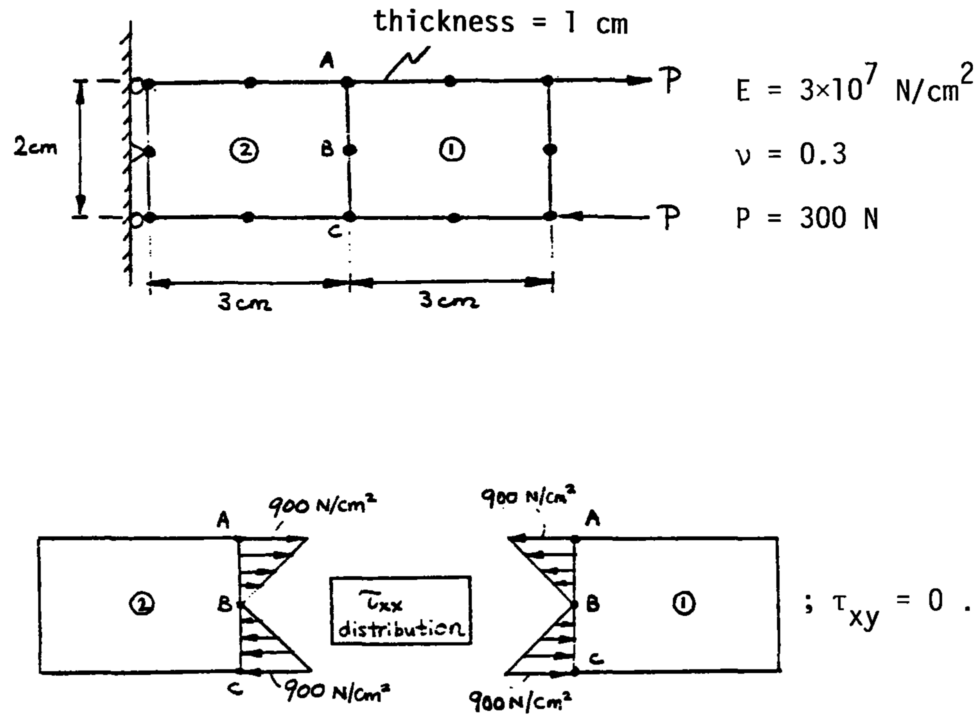
3x3 Gauss integration
correct results

Fig. 5.46. 8 - node plane stress element supported at B by a spring.

Stress calculations

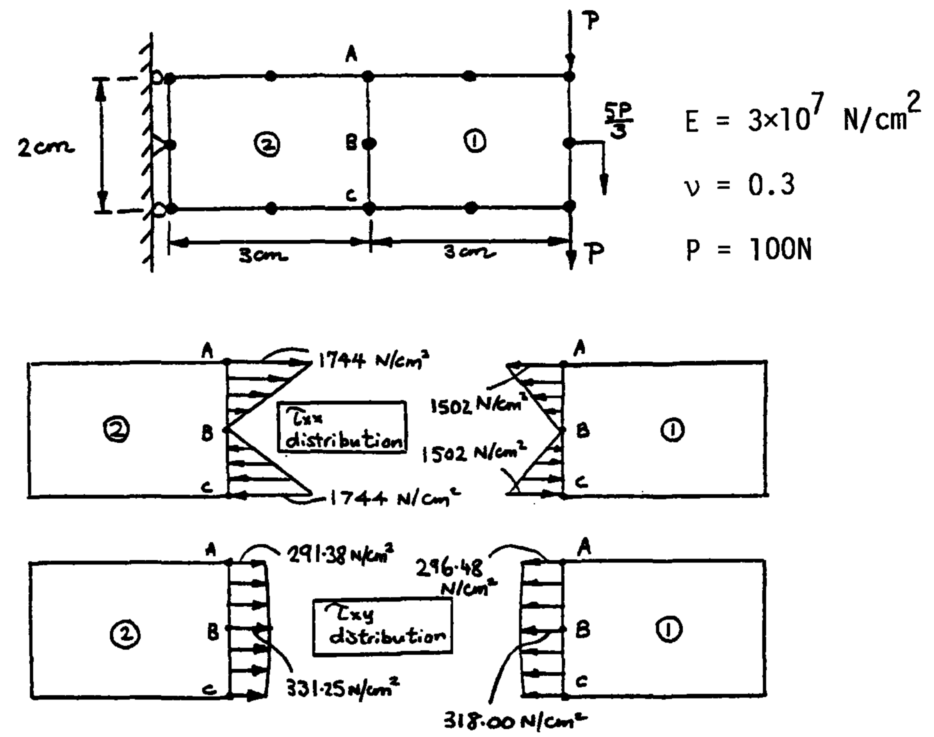
$$\underline{\tau} = \underline{C}\,\underline{B}\,\underline{U} + \underline{\tau}^{I} \qquad (5.136)$$

- stresses can be calculated at any point of the element.
- stresses are, in general, discontinuous across element boundaries.



(a) Cantilever subjected to bending moment and finite element solutions.

Fig. 5.47. Predicted longitudinal stress distributions in analysis of cantilever.



(b) Cantilever subjected to tip-shear force and finite element solutions

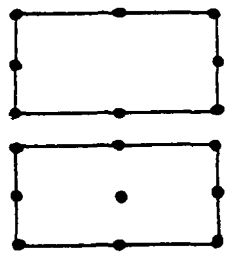
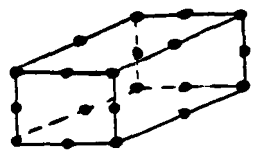
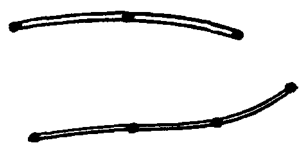
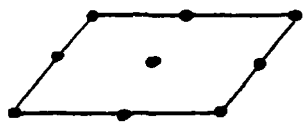
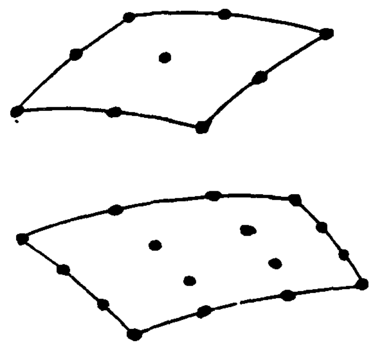
Fig. 5.47. Predicted longitudinal stress distributions in analysis of cantilever.

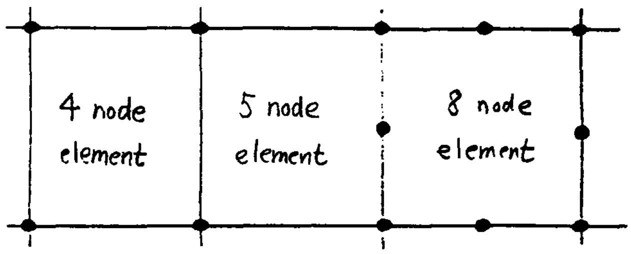
Some modeling considerations

We need

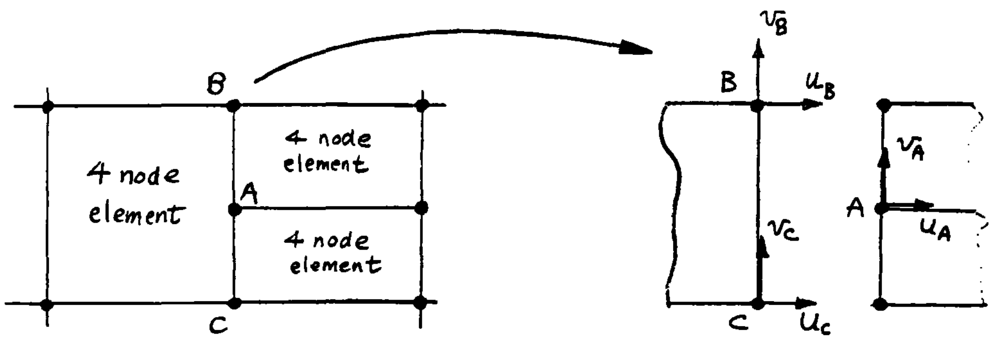
- **a qualitative knowledge of the response to be predicted**
- **a thorough knowledge of the principles of mechanics and the finite element procedures available**
- **parabolic/undistorted elements usually most effective**

Table 5.6 Elements usually effective in analysis.

TYPE OF PROBLEM	ELEMENT	
TRUSS OR CABLE	2-node	
TWO-DIMENSIONAL PLANE STRESS PLANE STRAIN AXISYMMETRIC	8-node or 9-node	
THREE-DIMENSIONAL	20-node	
3-D BEAM	3-node or 4-node	
PLATE	9-node	
SHELL	9-node or 16-node	



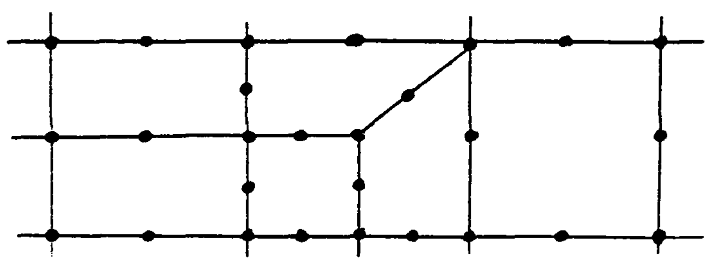
a) 4-node to 8-node element transition region



Constraint equations: $u_A = (u_C + u_B)/2$

$v_A = (v_C + v_B)/2$

b) 4-node to 4-node element transition



c) 8-node to finer 8-node element layout transition region

Fig. 5.49. Some transitions with compatible element layouts