PROFESSOR: Ladies and gentlemen, welcome to lecture number seven. In this lecture, I would like to present to you the formulation of structural elements. We will be discussing beam, plate, and shell elements, and I would like to introduce to you the isoparametric approach for interpolations.

There are two approaches in the formulation that we can follow. The first one is a strength of materials approach, in which case we look at a straight beam element, we use a beam theory including shear effects. If you look at a plate element, we use a plate theory including shear effects, also. The names associated with these theories that we're using are the names of Reissner and Mindlin.

In the second approach, we have the continuum mechanics approach, in which we use the general principle of virtual displacement, but we exclude the stress components not applicable. For example, in a plate, we set the stress through the thickness of the plate equal to zero. In addition, also, we have to impose in the use of the principle of virtual displacement the kinematic constraints for particles on sections or originally normal to the mid-surface. Namely, we have to put the constraint into the structure that the particles remain on a straight line during deformation.

Well, as examples, I've plotted here, I've shown here two structures, a beam and a shell. Let's look first at a beam. In this case, we have that the original particles normal to the mid-surface, or the neutral axis of the beam, are on this orange line. I've shown here a large number of particles. The kinematic constraint that we're talking about is that during deformations, these particles remain on a straight line. They move over to the yellow line here. In other words, point A goes to point a prime. Another particle here goes over to this particle here. This particles goes over to that particle, and so on, and these particles remain on a straight line. That is the basic kinematic assumption. However,

We should also notice that there's a right angle between the mid-surface, or neutral axis off the beam, and this line of particles initially. This right angle is not preserved during deformation. In other words, this angle here is not a right angle after deformations anymore. In the case of the shell, the kinematic constraint is quite similar. Here, we have now the mid-surface of the shell shown as a dashed line. The particles on a line normal to that mid-surface are shown here again in orange. This is the initial line.

And during deformation, these particles remain on a straight line. Now they have come to be the yellow line here, and we notice that, again, there's a right angle initially here, but that right angle is not preserved during deformations because, in each case, we are including shear effects. Here, we include shear effects, and similarly here, we also include shear effects.

Well, I've prepared some view graphs to show these facts a little bit more distinct. Here, we have a first view graph on which I show the assumptions of the basic Bernoulli-Euler beam theory that is used in the development of conventional beam elements. We have here the original beam element with its neutral axis in a dashed line. And that beam element, during deformation, becomes this. It goes into this shape here. We notice-- and this is important-- that this section here, which I now mark in blue, goes over into that section, and that the displacement is w at the mid-surface, and that the slope here at right angles to that section is nothing else than dw dx. In other words, this angle here is really nothing else than that angle, dw dx. This is the Bernoulli-Euler beam theory excluding shear deformations.

The important point is that when we use this beam theory, we have to match between two elements, w, in other words, the displacement at the mid-surface has to be the same for element one and element two. And in addition, this slope has to be the same for both elements, dw dx for element one on the left-hand side must be equal to dw dx on the right-hand side. This is the conventional beam theory that is used to develop the Hermitian beam elements. And what I'd like to introduce to you now is the beam theory that we're using in the isoparametric formulation, namely in the development of modern beam elements, pipe elements, shell elements, and plate elements.

I should say at this point that the conventional Hermitian beam element that you're probably familiar with is more effective in engineering analysis than the beam element that I'm talking about here when we just look at a straight beam. However, I want to look at the straight beam here as an example to introduce to you the formulation of the structural elements that I'm talking about. The formulation is very well displayed, very well demonstrated. The basic features are very well demonstrated looking at the beam element, although the application of this formulation to a straight beam element is not as effective as just the usage of a Hermitian beam element.

However, if we talk about curved beam elements, pipe elements, then this formulation is indeed very effective. And, of course, for plates and shells, it is really very effective. Let me show to you, then, the basic points-- the basic important points-- that are being used in the formulation. Here we have, again, our original beam element. The neutral access is shown here, and this beam element now moves over into this piece, into that shape, during deformations. We have a section here, and as I pointed out earlier, that section has to remain straight during deformations. In fact, it moves right to that section here.

Now notice that what we're talking about is a slope, dw dx here, which is this angle here, plus a shear deformation angle, gamma, and dw dx minus gamma is this angle. And that is the angle beta. In fact, this is the rotation of this line of particles. In other words, we are not talking just about one state variable, w, as we do in the Hermitian formulation, but we talk about two state variables now, beta and w. Beta and w both are independent, and we will see that, later on, we will interpolate them as independent quantities.

The important point, then, that if you look at two elements, if you do interpolate w and beta independently, then we need, between two elements, continuity in w and continuity in beta. We do not talk about continuity in dw dx. We do not talk about that, and that is the important point, particularly when we talk about the formulation of plate elements and shell elements. The fact that we'll talk about independent interpolations of w and beta, including shear effects, of course, in an approximate way, but including shear affects, that fact alleviates us of many difficulties that we encounter otherwise. In other words, that we encounter if we use a classical plate theory excluding shear deformations.

A good starting point for the development of the elements is the use of the total potential, pi, of an element. That total potential I've written down here, and we have set pi is equal to a contribution from the bending part plus a contribution from the shearing part, or the shearing deformations, and, of course, there is the external work due to distributed pressure, p, on a beam element, and moments, externally applied moments, m, onto the beam element.

Now, notice the quantities that I'm using here. The bending part is given by dw dx squared, of course with the flex or rigidity in front, and this part here is given only in terms of the section rotation beta, which is independent of the translation of the neutral axes, w. Here, we have the shearing part, dw dx, minus beta As, the shear strength. And they have been written down here once more. If we look at this equation, here we get dw dx minus beta is equal to gamma.

The other quantities, of course, that I used in the derivation of this-- pi, the stress being equal to v over as, v being the shear force on the section, As is the shear area. Notice that we are assuming the shear strength through the thickness of the beam element to be constant. They are constant because of this equation here, basically. w, of course, varies along the length of the beam. Beta varies along the length of the beam, but that means gamma is constant through to the thickness of the beam.

Now, since gamma is constant through the thickness of the beam, we have to say, of course, that also our shear stress is constant through the thickness of the beam, and we have to introduce a shear correction factor, k, which is equal to As over A, where As is an equivalent shear area. Using these quantities we obtain this pi functional, where, once again, we simply add the bending contribution, bending strain energy to the shear strain energy, and we subtract the total potential of the external loads. If we invoke the stationarity of this functional, in other words, we invoke that delta pi is equal to zero, we obtain the principle of virtual work, or principle of virtual displacement, which I have discussed with you in an earlier lecture.

The result of invoking that del pi is equal to zero, or invoking the stationarity of pi is this equation here. Now notice that in this equation we have, if we want to interpret it once physically here, basically the real stress part, here, the virtual strain part. Similarly here, the real stress part and the virtual strain part, and, of course, the virtual work of the external loads. The important point is that we only integrate along the length of the beam, and not through the thickness anymore, because we talk about quantities stress resultants over the thickness. Well, once we have arrived at this equation, we can proceed in much the same way as we have been proceeding in the development of continuum isoparametric elements.

Here, we look at a particular case. Let us say we have a beam elements such as shown here. Of course, this is the loading applied, P. The bending moment that they're talking about here is shown here. It is a distributed bending moment over any part of the beam. Similarly, P is only applied over a certain part of the beam. The depth of the beam is b, the width of the beam is a. As a shear factor for a rectangular beam, the sheer fact k that I introduced to you briefly is 5/6. Of course, I, the moment of inertia, is ab cubed divided by 12.

The interpolations that we would use for such a beam element are one dimensional interpolations. We only integrate along the length, x. And the one dimension interpolations we discussed already earlier. We simply use the same that we have been using already in the formulation of truss elements. Two point interpolation would be this element here. Here, we use a three point interpolation. Notice-- and this is important-- that we're talking about w and beta, the section rotations, as independent quantities. So for a three point, or three nodal point beam, we would have, in just a plane analysis, we would have six degrees of freedom. For a cubic element, we have eight degrees of freedom.

Of course, for a Hermitian beam element, we would have only these degrees of freedom and those degrees of freedom, w and dw dx here, and w, dw dx here. So, that is the reason why the Hermitian beam element is more effective. However, we can use these interpolations directly to develop curved beam elements, pipe elements, and then, of course, this approach is-- even for beam elements-- more effective, as I mentioned earlier.

Well, let us then write down the basic interpolations that we are using. Here, we have w being the sum of hi wi. hi, we discussed earlier already. And these, of course, are nodal point transverse displacement. These are the nodal point rotations of the sections, beta. In other words, I could have used here, as a notation, beta i, but I chose to use theta i. If you write these equations in matrix form, we directly obtain this relation here, where hw simply list hi, u lists the displacements and section rotations, and similar here, beta is given in terms of an h beta matrix. We take the differentiations of hw and h beta, and we get dw dx equal to Bw times u d beta dx equal beta times u.

Once we have the principle virtual work established for the element that we're considering and have chosen our interpolations, the approach of developing the stiffness matrices, the load vectors, is exactly the same as in the development of continuum elements.

Well here, I have written down the various quantities. U transposed lists, as I said earlier, the displacement vector nodal points and the rotation vector nodal points. Hw simply gives the interpolation functions, q, of course, being equal to the number of nodal points we are using. And here, we have H beta. The Bw is given right here. Notice our J inverse comes in there because we have to transform from r to x, x being the actual physical coordinate along the length. Similarly, B beta being given here. Again, the J inverse here to transform from r to x coordinates.

Once we have written down these, we can directly substitute into the principal of virtual work, and we come up with the stiffness matrix, given here, and the load vector given here. Let me point out a few important things here. Of course, this B beta matrix here-- just to remind you-- comes from d beta dx. It's in fact, really, d beta dr, but because we're integrating from -1 to plus 1, however, we have our determinant J there to take into account the volume transformation. Here, of course, we have d beta dx transposed. This comes from the virtual strains, and that comes from the real strains. Of course, we have the stresses. Here, so these strains times the stress strain law, E, being the Young's modulus, gives us the stresses.

Here, we talk about shearing deformations. Notice that we have here dw dx, that is, the Bw minus the beta. So we have the derivative in here of w, but no derivative in H beta because we are simply interpolating here the beta values. Again, of course, a transformation to x, and therefore we have a determine J in there. The load vector looks just the same way as in the development of continuum elements. This is the transverse loading applied P. This is, of course, interpolating-- this [UNINTELLIGIBLE] interpolates the virtual transverse displacements.

Here, we have the moment loads, the real moment loads, and this interpolates here the section rotations, beta, along the lengths of the beam. Of course, again, the volume transformation from r to x. This is really a straightforward application of what we discussed earlier.

There is one important point, however, now, that I have to point out to you. If we consider the functional pi, as I mentioned earlier, there's a bending part here and there's a shearing part here. Notice that in this development I have divided through by EI over 2, so that I introduce a value alpha there, and that alpha is really GAK divided by EI. Now, if we look at that alpha value, and let's look at the value for a rectangular section, we would see that the a value, of course, is a times b, and the I value gives us really an ab cubed, if this is, here, b, and that is a. An ab cubed over 12, of course. And we have, also, the GK that gives us these, and of course an E in front here.

But the important point that I want to now concentrate on is really this b over b cubed. We can see that as the element gets thinner and thinner, as the element gets thinner and thinner, that alpha gets larger and larger. If alpha gets larger and larger, this term here will be predominant. This term will be predominant. Now this means, however, that if we want to finally converge to a beam in which the shear strengths are negligible, in other words, in which the shear strengths are extremely small, what we would have to be able to represent in the formulation is that this value here goes to zero. And what happens in the formulation, really, is that as this value gets larger and larger, any error introduced in the formulation, due to the fact that this is not exactly zero in the finite element interpolation, that error is largely magnified. Is magnified and, in fact, can introduce a very large error if this value is not zero due to the fact that alpha becomes larger and larger if the element becomes thinner and thinner.

In other words, in summary once more, if we are talking about a beam element that gets thinner and thinner for which we know the shear strengths should becomes smaller and smaller, our finite element interpolation must be able to represent this fact. Now, if we look at the shear strengths, and this is, of course here, nothing else then basically the shear strength squared, we now identify that dw dx minus beta, when interpolated using our interpolation functions, must be able to be very, very, very small. And that is a restriction on the formulation. So what we have to do, really, is use high enough order interpolations so that dw dx minus beta can be smaller for thin elements.

Of course for thick beam elements, that is not really a constraint because we know that there are shearing deformations, and the shear deformations can be quite significant. However, for thin elements, we must be able to represent the fact that gamma is small, and therefore, we have to use high order interpolations. In fact, the parabolic interpolation is really the lowest interpolation that one can recommend. It would be better to use cubic interpolation. In fact, we use the cubic interpolation in practice. In that case, this gamma value can be small, and we run into no difficulties and the element can be very, very, very thin.

Another approach would be to use a discrete Kirchhoff theory or reduced numerical integration. These approaches have been developed for low order elements. The discrete Kirchhoff theory approach is very effective. The reduced numerical integration can also be effective, but has to be used with care. In particular, as I will point out in the next lecture, we have to be careful that we do not introduce spurious rigid body modes into the system.

The development that I just talked about really is applicable to an element that has a rectangular section and the element was also lying in a plane. We looked at a straight element. Let us now see how we can generalize these concepts directly to the formulation of general curved beam elements. And for that purpose I've shown here-- I'm showing here-- a more general beam element that lies in a three-dimensional space. It's still rectangular, however, we could also have a circular section instead. In fact, when we look at a pipe element, we do talk about-- we have, of course, a circular section.

Well, in this particular beam element, notice I'm looking at node one here, node two there, and generally node three here and node four here, because we want to pick up the curvature of the element. I have this element lying in a three dimensional space, x, y, z. Notice that that element has local coordinates r, s, t. These are the isoparametric coordinates. And psi, eta, zeta, these are the actual continuous physical coordinates in the beam element.

We define normals at each nodal point. The normal in the t direction here is 0 Vt 1. The normal into the s direction is 0 Vs 1. And similarly, we have two normals at each nodal point. Notice that for this particular rectangular beam element, I have the thickness, a1, here and b1, there, and a2 here, b2 there. These thicknesses can be different. Now what I want to use are the same basic assumptions that we have familiarize ourselves already with when we looked at the special case of a straight beam element in planar deformations. I want to use those basic assumption now in the development of this more general beam element. And I want to use a continuum approach.

Well, what we are doing, then, is the following. We interpolate the coordinates, x, y, and z, along the beam element in terms the nodal point coordinates of the nodes that lie on the neutral axes of the beam plus an effect that comes in due to the thickness of the beam. Now let us go and look in detail at the x interpolations. The L denotes 0 or 1, 0 being the initial configuration, 1 being the final configuration. So let's put simply a 0 in there, think in terms of a 0 there, and let's look at the initial configuration first.

Well, here we have the initial x-coordinates of the nodal points, and there are q of them. These are the one dimensional interpolation functions, just the same that we use for a truss element, for example. Here, we have the t-axis. This is the t-axis into the direction of the normal into the t direction. In other words, this is a t-axis here. Notice that that t-axis here corresponds to this normal here. That s axis here corresponds to this normal here. So here, we have t/2, and ak being the total thickness of the beam corresponding to that t direction, hk being the one dimensional interpolation functions again, and these are the direction cosines of the normal in the t direction. Here, I should really say, this is a direction cosine corresponding to the x-axis, corresponding to the x-axis. When we talk later about the y and z-axis, then we use the y direction cosine and the z direction cosine.

This part comes in because the beam basically has a thickness into the t direction. Now, we have also to introduce the s direction part, and here we s/2, the thickness into s direction, the one dimensional interpolation functions, and the direction cosine in the x direction of the normal in the s direction. Well, if we want to find, in other words, the coordinates of any point in the beam element, and let us look, once more, back at the beam element. If I want to find the coordinate of a point, p, lying in that beam element here-- there's, say, point p-- what I have to do is I have to identify the r, s, and t coordinates of that point, p, and then substitute these r, s, and t-coordinates into this part here. And I would get the corresponding x-coordinate.

I proceed similarly with the y and z-coordinates. Notice, as I pointed out earlier, we are talking still about the thickness ak hk here, here, and here, but we're using the x direction cosines, the y direction cosines, and the z direction cosines here. And similarly for the s direction, we are using similar quantities.

So this is the interpolation of the beam element, and using these three formerly, we can directly obtain-- we can directly obtain-- the x, y, and z-coordinates in this system of axes of any point in the beam element. This is the most important fact. The interpolations that I've listed here are the starting point of the development of the strain displacement matrices and displacement interpolation matrices.

Now, let us identify that if these are the original coordinates for L being 0, then we can also apply, of course, after the deformation, the same interpolation, and we put L equal to 1. If we subtract the x1 minus zero x, we should get the displacements, u. Notice that the displacements, u, are in the directions of the x-axis. Well, this is exactly how we proceed. We use these interpolations for before deformation and after deformation. And we subtract these interpolations as shown here and directly obtain the u, v, and w displacements, of course, as a functional of r, s, and t.

Notice that if I proceed this way-- notice that if I proceed this way, I have used the basic assumption that plane sections remain plane during deformation. This was the assumption that I pointed out to your earlier. And we are using it in this general information just in the same way as in the special application that I showed you earlier. Well, having then-- just to refresh your memory, the u, v, and w, in terms of r, s, and t, of course, via these subtractions, we obtain directly these equations here. Notice that here we have, now, the nodal point displacements, uk, vk, wk, and we have the change in the direction cosines. These are the changes in the direction cosines.

Well, these changes in the direction cosines we want to express in terms nodal points rotations. And that is achieved as shown on this view graph. We can express these changes in the direction cosines directly by taking the cross product of a vector of nodal points rotations, and I've listed here this vector, this is a nodal point rotation about the x, y, and z-axis at nodal point, k. You're taking the cross product of this vector times the original normal. Time the original normal. And we get, then, the change in the normal, of course, for the t direction and for the s direction.

If we substitute this relation here, of course, remember that these quantities are known. They are given. So the unknowns now are theta k. If you substitute from here into our relation here that I developed earlier, we directly obtain the displacements of any point, p, in the beam in terms of nodal point displacements and nodal points rotations. Because these quantities here now have been eliminated and have been expressed in terms of nodal point rotations.

Well, now we have all the quantities that we need to develop our strain displacement matrix for the beam element. Remember, all we need are the coordinate interpolations, which I have developed already, and the displacement interpolations. With those two quantities, we can immediately calculate, via the procedures that I discussed with you earlier, the strain displacement transformation matrix. And here it is given. Notice that we are now talking about strain into the eta directions. In other words, the eta, psy, and zeta directions. These are the directions that I pointed out to you earlier, which are the physical coordinate directions along the beam.

Let me show it to you once more, the picture. The r, s, and t-coordinates are the isoparametric coordinates. The eta, psy, and zeta coordinates are the physical coordinates along the beam. In other words, for the one dimensional beam that we looked at, this eta axis was, in fact, the x-axis. Of course, our x, y, and z-axes are now global Cartesian axes.

Well then, with that information, we can directly calculate the strain displacement matrix here. This is done effectively using numerical integration, as I will be discussing in the next lecture. The transformations that are necessary are also done on the integration point level. Notice that the uk here lists the nodal point displacements and the nodal point rotations, and, of course, we have to also remember one important fact, that for the beam we are talking about stresses into the eta, psy, and zeta directions, normal stresses, shear stresses here, that are related via this stress strain law to the normal strains and shear strains.

Notice that there's again the shear correction factor, k, which we want to also include in the formulation because we have assumed constant shearing strains through the thickness of the beam, whereas we know that for a rectangular beam, for example, we have parabolic shear strain distributions if the beam is straight.

I'd like to now go on with the development of plate elements. Here, we are talking basically about the same approach. As I mentioned already once, the beam element that I'm talking about here-- that I have been talking about-- is really only an effective formulation when we talk about, and we want to develop, a curved beam element. For pipe elements also, in the case of pipe elements, I should briefly mention that, of course, we have to introduce, also, an ovalization degree of freedom. That ovalization degree of freedom interpolates basically the ovalization along the curved pipe. That is an additional degree of freedom that has to be introduced in the curved beam element formulation.

So, the beam element formulation is effective for curved beams, pipes. However, it does show the basic procedure that we are following also in the development of plate and shell elements. And here, we have a typical plate element. In other words, a flat shell. The u, v, and w displacements are now interpolated in this way. Notice u be the displacement into the x direction, v the displacement into the y direction, w being the transverse displacement, and again, we're talking about section rotations. Beta x being the section rotation about the y-axis. That is, the beta x section rotation. Beta y is a section rotation about the x-axis. So that our v, measuring z positive upwards. Notice, our [? v4 ?] point here is negative, and that's why we have a negative sign there.

Well, with that then given, we can immediately develop our strains by using the strength of material equations that tell us epsilon xx is del u del x, epsilon yy is del v del y, et cetera. And, of course, we're getting also our shear strengths. Having developed the strains, we also recognize that our stresses are given in terms of these formulae, where we have now the stress strain law for planed stress analysis, because we are looking at the plate as an assemblage of thin elements, plane stress elements, lying on top of each other. The stress through the plate, of course, is 0, and this is, therefore, the plane stress [? material ?] that we have been putting in here. And the z times this vector here gives us the strains. The shearing stresses are given here, and our functional pi that I used also for the beam element already is given here.

Notice that we now have to integrate through this thickness of the plate element. Here's our shear correction factor again, which is introduced just the same way as in the beam element. Notice here we have the work, or the total potential of the external loads, I should say. It is convenient now to integrate through the thickness because we can integrate prior to interpolating the quantities, and that then yields this value for pi, where our Cb parts and Cs part here, these two matrices, embody the fact that we have integrated through the thickness, so we have the following definitions here.

Kappa simply lists basically the bending strains, or I should say the rotations of the sections. Of course here, we have the shearing strains. Gamma lists the shearing strains. Cb now is a function of h cubed, just like in classic plate theory. Of course, we also have an h cubed entering into the formulation. And this Cb matrix embodies the fact that we have integrated through the thickness. Here is our Cs. Of course, in the Cs, we also have the k part. Notice that, again, we have an h cubed here and an h there. Therefore, to use our interpolation for plate and shell elements, we will have to use high enough shell interpolations to be able to represent the fact that the shear strains go to 0 for thin plates if we want to use this formulation for thin plates and shells.

Well, invoking now the fact that pi shall be stationary, we directly obtain this equation here. And this, of course, is nothing else than the principal of virtual displacement for the plate element. Notice that from this point onwards, we simply need to substitute only our interpolations. The interpolations that we are using are now interpolations for w, beta x, and beta y. And, of course, we also interpolate x and y. These interpolations, beta x and beta y, are independent from the interpolations of w, and that is the important points, as I mentioned in the development of the beam element. The fact that we are dealing, here, with three interpolations of course means that an each nodal point, we have three unknowns, w's, and section rotations.

Let us now look very briefly at shell elements. The same concept that we used to develop the general beam element after having discussed the special beam element is also employed now in the development of what you might call a general shell element, versus the special plate element that I just discussed, or the special shell element that I just discussed, because a flat shell is of course nothing else than a plate, if we also don't have membrane forces.

Well here, I'm showing a shell element, a nine-noded shell element. And notice that in this case, now, we are talking about this normal only. In the beam, we had two normals, vt and vs. Now we only have one normal. At a nodal point, we are defining the membrane displacements, uk, vk, wk, being a transverse displacement, and the rotations, alpha k and beta k. These rotations are defined about the axis v1 k and v2 k. Now notice that these two rotations, alpha k and beta k, will give us the change in vn during deformations. And this is really how we use alpha k and beta k. We express a change in vn in terms of alpha k and beta k. The procedure is the same as in the case of the beam element.

First, we express our x, y, and z-coordinates. And we're using here the original normal, vn, the x, y and z direction cosines. These are, here, the x, y and z-coordinates of the nodal point, k. We have our two dimensional interpolation functions, hk, now here, because we talk about a two dimensional surface, the mid-surface of the shell. Of course, at each nodal point, the shell can have a different thickness, and that is denoted by using a different ak value at each nodal point.

Applying this interpolation here to the initial configuration and the final configuration and subtracting 0x from 1x, and similar for y and z, we directly obtain the displacements u, v, and w. Notice that the displacements now are involving the nodal point displacements and the change in the direction cosines of the normal, denoted here. These changes in the direction cosines of the normal can directly be expressed in terms of the rotations, alpha k and beta k.

Now, notice here that once we have done this, of course here we involve now, as I mentioned earlier, the v1 and v2 directions. And these v1 and v2 directions are arbitrarily selected. In fact, they are-- for our shell element here-- selected as shown here. But once we have selected v1 and v2 at each nodal point, and they can vary from nodal point to nodal point, then we can use this relation to attain directly the change in the direction cosines of the normal during deformations, when the deformations are alpha k and beta k.

So with this equation, then, and the earlier equation that I've given to you, we can directly obtain the displacement interpolation matrix and the strain interpolation matrix. One important point that I should briefly mention is that, of course, for the shell, we have 0 stresses through the thickness. So, we have to use this stress strain law here. Notice there are 0's in this row and column. And this is, here, the plane stress part for the bending, and that is the shear part here. This is the stress strain law defined in a local convected coordinate system where we are talking about the stresses through the thickness being this direction here. And these other stresses are aligned with the coordinate system. We have to transform this one here to the global coordinate system in order to be able to use it directly in our formulation. And that transformation is achieved via these transformation matrices.

Now, this element has been effectively implemented in the ADINA computer program, and we want to use it using high order interpolation, as I mentioned earlier, as the basic element that therefore is very useful, which can be used as a flat element, a curved element, all curved element, or it can have curvature in both directions. Also, this is the basic element that is being used. We can also collapse nodes and derive other elements. As I pointed out earlier, the low order elements should only be used in very special cases. I would not recommend these elements. Although they can be used in principle, the element that is really useful is this one and that one. Both, of course, can be used as curved elements as I pointed out.

Another feature, and this is the final view graph that I wanted to show in this lecture, is that we can use these elements also in transition regions. Namely, here we have the shell element, now being flat-- that I discussed-- and we can directly couple this element into another element, which we call a transition element, which has the shell degrees of freedom at these nodes, but translational degrees of freedom only here. In other words, a continuum element degrees of freedom right here. Notice, three degrees of freedom at this node, only translations, whereas here, we would have five degrees of freedom-- three translations, two rotations.

Similarly, here we have a curved shell going into a solid, and again here, we have a transition element. Here, we show the five degrees of freedom at a shell node and the three degrees of freedom at a continuum element node. This is an effective approach to be able to couple director shell elements into solid elements.

I have not talked, of course, about the actual derivation of the matrices used in the formulation of the transition element. That is a little bit beyond what I wanted to present in this lecture. But the basic concepts are those that we discussed already in the earlier lecture for continuum element and in this lecture for structural elements. Thank you very much for your attention.