SOME NOTES ON DIFFERENTIAL OPERATORS

A

Introduction

In Part 1 of our course, we introduced the symbol D to denote a function which mapped functions into their derivatives. In other words, the domain of D was the set of all differentiable functions and the image of D was the set of derivatives of these differentiable functions. We then, as usual, introduced the notation $D(f) = f'$.*

Recalling that if f is any function and c is any number, we define the function cf by $[cf](x)$ means $cf(x)$; and that if f and g are any two functions which have the same domain, we define the new function $[f + g]$ by $[f + g](x) = f(x) + g(x)$; we notice that D is a linear mapping. That is, if f and g are both differentiable functions defined on the same domain and if a and b are any (real) numbers then

$$D(af + bg) = aD(f) + bD(g). \tag{1}$$

Notice that (1) is the special case of our notation $L(f)$ or more familiarly, $L(y)$ where $L(y) = y'$.

The key point is that if we restrict the domain of D to the set of all analytic functions**(where by analytic we mean that the function is infinitely differentiable which in turn means that the function possesses derivatives of every order) we can, in a natural way, invent a structure on D that is very analogous to the arithmetic structure of polynomials.

Aside from this structure giving us some very convenient notation and aside from its being interesting in its own right (in fact, the set of analytic functions is a very nice example of a more general vector space which we shall talk about more in Block 3), it is very helpful to us in handling linear differential equations and systems of linear differential equations. These ideas will be discussed in the following sections.

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*It is conventional to write f rather than, say, $f(x)$ because the variable used to denote the "input" is irrelevant. For example, if f is the rule which doubles a number it makes no difference whether we write $f(x) = 2x$ or $f(t) = 2t$ or $f(\$) = 2\$$ etc. This is why we often make remarks like "Define f by $f(t) = 2t$ for all real numbers, t." The point is that it is f which we are studying.

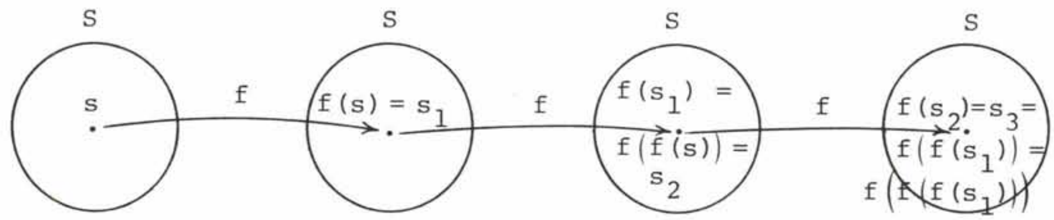
**In many practical cases, one is interested in a specific domain, say the interval $[a,b]$. In these cases, we only require that the domain of D include functions which are analytic on $[a,b]$.

Some Notes on Structure

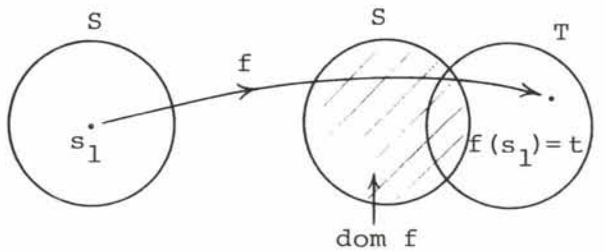
In ordinary multiplication, we accept with little if any difficulty the notation that a^n denotes the product of n factors of a. This notion is extended to any mathematical structure in that we often use the exponent notation to indicate that a certain operation is being carried out successively a certain number of times.*

One place that this notation is used extensively in mathematical analysis is when we refer to composition of functions. For example, suppose that f is a function that maps a set S into itself. Then, for a given $s \in S$, it makes sense to talk about, say, $f(f(f(s)))$. That is, starting with s, we compute f(s). Then, since f(s) is again in S, we may compute the effect of f on f(s); i.e. $f(f(s))$, etc.

Pictorially,



[Notice that it is crucial that the image of f be contained in S since if $f(s_1) = t \notin S$, then $f(f(s_1))$ is not defined because $f(f(s_1)) = f(t)$ and $t \notin$ domain of f. Again, pictorially,



$t \notin S$, dom f = $S \to f(t)$ is not defined

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*An interesting note on notation is that when we use + rather than x to denote the operation, it is conventional to write na rather than a^n. For example, with respect to ordinary addition, to indicate that we want $\underbrace{a + a + \ldots + a}_{n \text{ terms}}$, we write na, not a^n.

In other words, for any real number a and any positive integer n, na denotes the sum $\underbrace{a + \ldots + a}_{n \text{ times}}$.

In terms of specific illustration, let S be the set of integers and define f by $f(s) = \frac{s}{2}$ for each $s \in S$. Then, $6 \in S$ and $f(6) = \frac{6}{2} = 3$. Now, $f(f(6))$ means $f(3)$ and this is undefined if we insist that $f:S \to S$ since $f(3) = \frac{3}{2} \notin S$.]

In any event, as long as $f:S \to S$, we may talk meaningfully about $\underbrace{fo \ldots of}_{n \text{ times}}$ where $\underbrace{fo \ldots of}_{n \text{ times}}$ means $[\underbrace{fo \ldots of}_{n \text{ times}}](s) = \underbrace{f(f(\ldots f}_{n \text{ times}}(s)\ldots))$.

As an illustration, let S denote the set of all integers and define f on S by $f(s) = 2s + 3$ for each $s \in S$. Since $2s + 3$ is an integer if s is an integer, we have that $f:S \to S$. Now, for each $s \in S$, we have

$$f(s) = 2s + 3;$$

hence,

$$\begin{aligned} f(f(s)) = f([f(s)]) &= f(2s + 3) \\ &= 2(2s + 3) + 3^* \\ &= 4s + 9; \end{aligned}$$

hence,

$$\begin{aligned} f(f(f(s))) &= f([f(f(s))]) \\ &= f(4s + 9) \\ &= 2(4s + 9) + 3 \\ &= 8s + 21, \text{ etc.} \end{aligned}$$

For example,

$$f(4) = 2(4) + 3 = 11$$

$$f(f(4)) = f(11) = 2(11) + 3 = 25$$

$$f(f(f(4))) = f(25) = 2(25) + 3 = 53.$$

*Again, keep in mind the irrelevancy of the symbol used to denote the "input." $f(s) = 2s + 3$ means $f([\ ]) = 2[\ ] + 3$ so that when $[\ ] = 2s + 3$, $f([\ ]) = 2[2s + 3] + 3$.

Rather than write, say, $\left(f\ f\left(f\left(f(s)\right)\right)\right)$, it is conventional to abbreviate this by $f^n(s)$, where f^n means $\underbrace{fo \ldots of}_{n \text{ times}}$.

Important Note

This new notation is somewhat unfortunate since it now gives us two entirely different meanings for the same symbol. Namely, we have previously used, say, $f^2(x)$ to denote $[f(x)]^2$ and now we are saying that $f^2(x)$ could also mean $f(f(x))$. Clearly, $[f(x)]^2$ and $f(f(x))$ are entirely different concepts. For example, if $f(x) = 2x + 3$, then $[f(x)]^2 = (2x + 3)^2 = 4x^2 + 12x + 9$ while $f(f(x)) = f(2x + 3) = 2(2x + 3) = 4x + 6$. This same problem occurred as a special case when we first introduced f^{-1} to denote the inverse of f in the sense that $fof^{-1} = f^{-1}of$ = identity function, i.e. $[fof^{-1}](x) = x$ for all x. In this context, we saw that if $f(x) = 2x + 3$, then $[f(x)]^{-1} = \frac{1}{2x + 3}$ while $f^{-1}(x) = \frac{x - 3}{2}$.

Hopefully, whether $f^n(x)$ means $[f(x)]^n$ or whether it means $\underbrace{f(f \ldots f}_{n \text{ times}}(x)) \ldots)$ will be clear from context, but for the remainder of this chapter, $f^n(x)$ shall mean $\underbrace{f(f \ldots f}_{n \text{ times}}(x)) \ldots)$.

With this discussion as background, we are now in a position to define D^2, D^3, D^4, etc. Namely, suppose f is in the domain of D. Then by definition of the domain of D, f possesses derivatives of every order. Since $D(f) = f'$, we see that f' also belongs to the domain of D (i.e. if f has derivatives of all orders, f' must also possess derivatives of all orders since the nth derivative of f' is the (n+1)th derivative of f etc.).

Hence,

$$\begin{aligned} D^2(f) &= D[D(f)] \\ &= D(f') \\ &= f''. \end{aligned}$$

Proceeding inductively, D^n is defined by

$$D^n(f) = f^{(n)},$$

for each $f \in$ dom D. We may also define cD for any constant c. Namely, cD is defined by

$$[cD](f) = cD(f)$$

$$= cf'.$$

We may now invoke the linear properties of D to define what we mean by a polynomial in D. In particular, if $a_o, a_1, \ldots,$ and a_n are constants we define $a_nD^n + a_{n-1}D^{n-1} + \ldots + a_1D + a_o$ to mean

$$[a_nD^n + a_{n-1}D^{n-1} + \ldots + a_1D + a_o](f) =$$

$$= a_nD^n(f) + a_{n-1}D^{n-1}(f) + \ldots + a_1D(f) + a_of =$$

$$= a_nf^{(n)} + a_{n-1}f^{(n-1)} + \ldots + a_1f' + a_of.\text{*} \tag{2}$$

The connection between this definition and our previous study of linear differential equations with constant coefficients should seem rather obvious. Specifically, if we replace f by y in (2), we obtain

$$\left.\begin{aligned} &[a_nD^n + a_{n-1}D^{n-1} + \ldots + a_1D + a_o](y) = \\ &= a_ny^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1y' + a_oy \end{aligned}\right\} \tag{3}$$

Now certainly we would not have gone through all this discussion just to show that we can express linear differential equations in a new notation which involves polynomials in D! What is really important is that in terms of the new polynomial notation, there is a fantastic resemblance between polynomial properties and derivative properties.

Rather than launch into an avalanche of formal proofs, let us begin with a few concrete examples.

Example #1

Consider the expression

*This is why we require that f be analytic. Otherwise $f^{(n)}$ need not exist for each n.

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 3y. \tag{4}$$

Using our new notation, (4) would be expressed as

$$(D^2 + 4D - 3)y.* \tag{5}$$

If we were to look at $D^2 + 4D - 3$ as a "normal" polynomial in D, we would be tempted to "factor" it by writing

$$D^2 + 4D - 3 = (D - 3)(D - 1). \tag{6}$$

The only problem is what do we mean by

$$(D - 3)(D - 1)y, \tag{7}$$

which we get by using (6) in (5).

In terms of multiplication meaning composition of functions, a natural interpretation of (7) would be

$$\begin{aligned}(D - 3)(D - 1)y &= (D - 3)[(D - 1)y] \\ &= (D - 3)[Dy - y] \\ &= (D - 3)\left(\frac{dy}{dx} - y\right) \\ &= D\left(\frac{dy}{dx} - y\right) - 3\left(\frac{dy}{dx} - y\right)^{**} \\ &= \frac{d}{dx}\left(\frac{dy}{dx} - y\right) - 3\left(\frac{dy}{dx} - y\right). \end{aligned} \tag{8}$$

As a check that (4) and (8) are equivalent, notice that (8) may be written as

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*For brevity, it is conventional to write $(D^2 + 4D - 3)y$ rather than $[D^2 + 4D - 3](y)$.

**Notice that if y is analytic, so also is $\frac{dy}{dx} - y$. Thus, we may view $\frac{dy}{dx} - y$ as a simple function (say, u) and apply $D - 3$ to u. That is, $(D - 3)(\frac{dy}{dx} - y) = (D - 3)u = Du - 3u = D(\frac{dy}{dx} - y) - 3(\frac{dy}{dx} - y)$.

$$\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right) - 3\left(\frac{dy}{dx} - y\right) = \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y$$

which is (4).

In other words, "factoring" $(D^2 - 4D + 3)y$ into $(D - 3)(D - 1)$ shows us specifically how the 2nd-order equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$$

may be viewed as the <u>first</u> order linear equation in $(\frac{dy}{dx} - y)$; namely,

$$\frac{d}{dx}\left(\frac{dy}{dx} - y\right) - 3\left(\frac{dy}{dx} - y\right) = 0.$$

<u>Example #2</u>

Consider $(D - 1)(D - 3)y$ which we obtain by permuting the "factors" of $(D - 3)(D - 1)y$. We obtain

$$\begin{aligned}
(D - 1)(D - 3)y &= (D - 1)[(D - 3)y] \\
&= (D - 1)\left(\frac{dy}{dx} - 3y\right) \\
&= D\left(\frac{dy}{dx} - 3y\right) - \left(\frac{dy}{dx} - 3y\right) \\
&= \frac{d}{dx}\left(\frac{dy}{dx} - 3y\right) - \left(\frac{dy}{dx} - 3y\right) \\
&= \frac{d^2y}{dx^2} - 3\frac{dy}{dx} - \frac{dy}{dx} + 3y
\end{aligned}$$

so that

$$(D - 1)(D - 3) = (D - 3)(D - 1)$$

and our multiplication is commutative.

<u>Example #3</u>

The property of commutativity requires that the coefficients of the "powers" of D be constants. For example,

$$(xD + 1)(D - x)y = (xD + 1)(Dx - xy)$$

$$= (xD + 1)(\frac{dy}{dx} - xy)$$

$$= xD(\frac{dy}{dx} - xy) + (\frac{dy}{dx} - xy)$$

$$= x\left(\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y\right) + \frac{dy}{dx} - xy$$

$$= x\frac{d^2y}{dx^2} + (1 - x^2)\frac{dy}{dx} - 2xy$$

$$= [xD^2 + (1 - x^2)D - 2x]y. \tag{9}$$

On the other hand,

$$(D - x)(xD + 1)y = (D - x)(xDy + y)$$

$$= (D - x)(x\frac{dy}{dx} + y)$$

$$= D(x\frac{dy}{dx} + y) - x(x\frac{dy}{dx} + y)$$

$$= \frac{d}{dx}(x\frac{dy}{dx} + y) - x^2\frac{dy}{dx} - xy$$

$$= x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} - x^2\frac{dy}{dx} - xy$$

$$= x\frac{d^2y}{dx^2} + (2 - x^2)\frac{dy}{dx} - xy$$

$$= [xD^2 + (2 - x^2)D - x]y. \tag{10}$$

Comparing (9) and (10), we see that $(xD + 1)(D - x) \neq (D - x)(xD + 1)$, and, moreover, neither $(xD + 1)(D - x)$ nor $(D - x)(xD + 1)$ is equal to the "usual" product $xD^2 + (1 - x^2)D - x$.

In summary, this new arithmetic loses its nice structural appeal if we do not restrict its usage to constant coefficients, and this strongly affects our ability to handle nicely linear equations with non-constant coefficients.

In summary, then, if we introduce the notation of differential operators whereby we rewrite

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y' + a_oy,$$

where $a_o, \dots, a_{n-1}$ are <u>constants</u> as

$$(D^n + a_{n-1}D^{n-1} + \dots + a_1D + a_o)y$$

then

$$D^n + a_{n-1}D^{n-1} + \dots + a_1D + a_o$$

possesses the same structure as does the "usual" polynomial arithmetic. In particular, if $P_1(D)$, $P_2(D)$, and $P_3(D)$ denote three polynomials in D with constant coefficients, it is true (but we omit any formal proofs) that

(i) $P_1(D)P_2(D) = P_2(D)P_1(D)$

(ii) $[P_1(D)P_2(D)]P_3(D) = P_1(D)[P_2(D)P_3(D)]$

and

(iii) $P_1(D)[P_2(D) + P_3(D)] = P_1(D)P_2(D) + P_1(D)P_3(D)$.

Thus, we may manipulate polynomials in D in the "usual" way, as illustrated in the following examples.

<u>Example #4</u>

$$(D^2 + 2D + 2)(D - 1) = D^3 + 2D^2 + 2D - D^2 - 2D - 2$$

$$= D^3 + D^2 - 2.$$

Hence, we may view

$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 2y \tag{11}$$

as

$$(D^3 + D^2 - 2)y,$$

and this in turn is

$$\begin{aligned}
(D^2 + 2D + 2)(D - 1)y &= (D^2 + 2D + 2)(Dy - y) \\
&= (D^2 + 2D + 2)\left(\frac{dy}{dx} - y\right) \\
&= D^2\left(\frac{dy}{dx} - y\right) + 2D\left(\frac{dy}{dx} - y\right) + 2\left(\frac{dy}{dx} - y\right) \\
&= \left(\frac{dy}{dx} - y\right)'' + 2\left(\frac{dy}{dx} - y\right)' + 2\left(\frac{dy}{dx} - y\right) \qquad (12)
\end{aligned}$$

which is second-order and linear in $\left(\frac{dy}{dx} - y\right)$.

By commutativity, we may also rewrite (11) as

$$\begin{aligned}
(D - 1)[(D^2 + 2D + 2)y] &= (D - 1)[D^2y + 2Dy + 2y] \\
&= (D - 1)\left(\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y\right) \\
&= D\left(\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y\right) - \left(\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y\right) \\
&= \left(\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y\right)' - \left(\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y\right) \qquad (13)
\end{aligned}$$

which is first order and linear in $(y'' + 2y' + 2y)$.

The main point is that by (12) and (13), we have that

$$\frac{d^3y}{dx^2} + 2\frac{d^2y}{dx^2} - 2y = 0$$

is equivalent to either of the lower order equations

$$u'' + 2u' + 2u = 0,$$

where $u = y' - y$, or

$v' - v = 0,$

where $v = y'' + 2y' + 2y$.

Hopefully, this analysis supplies some insight as to how we solve linear differential equations, structurally, by reducing them to equivalent lower order equations.

Example #5

Suppose we want to solve the equation

$$(D - 1)(D + 1)(D - 2)y = 0. \tag{14}$$

By our "new" algebra

$$\begin{aligned}(D - 1)(D + 1)(D - 2)y &= (D^2 - 1)(D - 2)y \\ &= (D^3 - 2D^2 - D + 2)y \\ &= y''' - 2y'' - y' + 2y.\end{aligned}$$

Hence, equation (14) is equivalent to

$$y''' - 2y'' - y' + 2y = 0. \tag{15}$$

Letting $y = e^{rx}$ in (15), we obtain

$$r^3e^{rx} - 2r^2e^{rx} - re^{rx} + 2e^{rx} = 0 \tag{16}$$

and since $e^{rx} \neq 0$, (16) implies that

$$r^3 - 2r^2 - r + 2 = 0.$$

Since $r^3 - 2r^2 - r + 2 = (r - 1)(r + 1)(r - 2)$, we see that the general solution of (15) is

$$y = c_1e^x + c_2e^{-x} + c_3e^{2x}. \tag{17}$$

The main observation here is that the "roots" of (14) are also 1, -1, and 2. More generally, using the D-notation, if the nth order homogeneous linear equation with constant coefficients can be represented in the "factored" form

$$(D - a_1)(D - a_2) \ldots (D - a_n)y, \text{ where } a_i \neq a_j \text{ if } i \neq j \tag{18}$$

then the general solution of the equation is simply

$$y = c_1 e^{a_1 x} + \ldots + c_n e^{a_n x}.$$

Example #6

Sometimes the a's in (18) are not all distinct. An interesting question concerns solving, say

$$(D - 1)^3 y = 0. \tag{19}$$

Of course, we have learned to solve (19) by other methods. For example, (19) is equivalent to

$$(D^3 - 3D^2 + 3D - 1)y = 0$$

or

$$y''' - 3y'' + 3y' - y = 0,$$

for which the substitution $y = e^{rx}$ yields

$$r^3 - 3r^2 + 3r - 1 = 0, \text{ or } (r - 1)^3 = 0$$

whereupon

$$y = (c_o + c_1 x + c_2 x^2)e^x$$

is the general solution of (19).

To handle (19) by the method of differential operators, we have the structural property that

$$D^n(e^{mx}y) = e^{mx}(D + m)^n y.^* \tag{20}$$

We may prove (20) by induction, observing that for n = 1, 2, etc., we obtain

*Mechanically, (20) tells us that we may "factor out" e^{mx} and replace D by D + m.

$$D(e^{mx}y) = \frac{d(e^{mx}y)}{dx}$$

$$= e^{mx}\frac{dy}{dx} + me^{mx}y$$

$$= e^{mx}(\frac{dy}{dx} + my)$$

$$= e^{mx}(D + m)y$$

$$D^2(e^{mx}y) = D[D(e^{mx}y)]$$

$$= D[e^{mx}(D + m)y]$$

$$= D[e^{mx}(\frac{dy}{dx} + my)]$$

$$= e^{mx}\frac{d}{dx}(\frac{dy}{dx} + my) + me^{mx}(\frac{dy}{dx} + my)$$

$$= e^{mx}\left[\frac{d^2y}{dx^2} + m\frac{dy}{dx} + m\frac{dy}{dx} + m^2y\right]$$

$$= e^{mx}\left[\frac{d^2y}{dx^2} + 2m\frac{dy}{dx} + m^2y\right]$$

$$= e^{mx}[(D^2 + 2mD + m^2)y]$$

$$= e^{mx}(D + m)^2y, \text{ etc.}$$

and the inductive details are left to the interested reader.
Returning to Example #6, since $D^3(e^{-x}y) = e^{-x}(D - 1)^3y$ [i.e. this is (20) with $n = 3$ and $m = -1$], we have that $(D - 1)^3y = 0$ implies that

$$e^{-x}(D - 1)^3y = e^{-x}0 = 0;$$

that is,

$$e^{-x}(D - 1)^3y = 0.$$

Since $D^3(e^{-x}y) = e^{-x}(D - 1)^3y$, it follows that

$$D^3(e^{-x}y) = 0, \tag{21}$$

whereupon we may integrate (21) successively to obtain

$$e^{-x}y = c_o + c_1x + c_2x^2$$

or

$$y = (c_o + c_1x + c_2x^2)e^x.$$

It is not our purpose here to teach the differential operator method in detail. Our main aim for now is to help you get acclimated to this new language so that later, in Section D, we can show a very nice non-trivial application of this discussion to the fairly sophisticated concept of systems of linear differential equations. For this purpose, Section C may be omitted without any loss of continuity, but the more involved reader may prefer to study Section C if only to gain a little further insight to how the arithmetic of differential operators is further developed.

C

Inverse Differential Operators

Given

$$(D - a)y = f(x), \tag{1}$$

we already know that

$$y = e^{ax}\int e^{-ax}f(x)dx. \tag{2}$$

Namely, (1) may be written as

$$\frac{dy}{dx} - ay = f(x), \tag{3}$$

whereupon an integrating factor of (3) is e^{-ax}. Multiplying both sides of (3) by e^{-ax}, we obtain

$$e^{-ax}\frac{dy}{dx} - ae^{-ax}y = e^{-ax}f(x),$$

or

$$\frac{d(e^{-ax}y)}{dx} = e^{-ax}f(x)$$

Hence,

$$e^{-ax}y = \int e^{-ax}f(x)dx,$$

from which (2) follows.

On the other hand, suppose we wanted to solve (1) "algebraically" treating (D - a) as a "factor," rather than as an operator, in the expression $(D - a)y$. We would "divide" both sides of (1) by $D - a$ to obtain

$$y = \frac{1}{D - a} f(x). \tag{4}$$

What we have really done in obtaining (4) from (1) is to have "composed" both sides of (1) with the inverse of the operator $(D - a)$. In effect, we are writing $(D - a)^{-1}$ as $\frac{1}{D - a}$ where $(D - a)^{-1}$ is defined by the relation

$$(D - a)^{-1}[(D - a)y] = y.$$

The point is that we may now compare (4) [which we would like to mean the value of y] with equation (2) [which we <u>know</u> is the value of y (up to an arbitrary constant)] and conclude that if (4) is going to yield the solution of (1), then we have no choice but to define $\frac{1}{D - a} f(x)$, or, $(D - a)^{-1}f(x)$ by

$$\frac{1}{D - a} f(x) = e^{ax}\int e^{-ax}f(x)dx. \tag{5}$$

<u>Example #1</u>

To evaluate $\frac{1}{D - 3} x^4$, we use (5) with $a = 3$ and $f(x) = x^4$. This yields

$$\frac{1}{D-3}x^4 = e^{3x}\int e^{-3x}x^4 dx$$

$$= e^{3x}\int_c^x e^{-3t}t^4 dt, \tag{6}$$

c an arbitrary, but fixed, constant.

"Inverting" (6), we are saying that

$$(D-3)\left[e^{3x}\int_c^x e^{-3t}t^4 dt\right] = x^4. \tag{7}$$

To show that (7) is correct, we could evaluate $\int_c^x e^{-3t}t^4 dt$ explicitly, but this is not necessary for our purposes. Namely, by the product rule and the fact that $\frac{d}{dx}\int_c^x g(t)dt = g(x)$, we conclude that

$$\frac{d}{dx}\left[e^{3x}\int_c^x e^{-3t}t^4 dt\right] = e^{3x}\frac{d}{dx}\int_c^x e^{-3t}t^4 dt + \frac{d}{dx}(e^{3x})\int_c^x e^{-3t}t^4 dt$$

$$= e^{3x}[e^{-3x}x^4] + 3e^{3x}\int_c^x e^{-3t}t^4 dt$$

$$= x^4 + 3e^{3x}\int_c^x e^{-3t}t^4 dt. \tag{8}$$

Computing the left side of (7) and using (8), we have that

$$(D-3)\left[e^{3x}\int_c^x e^{-3t}t^4 dt\right] = D\left[e^{3x}\int_c^x e^{-3t}t^4 dt\right] - 3\left[e^{3x}\int_c^x e^{-3t}t^4 dt\right]$$

$$= \frac{d}{dx}\left[e^{3x}\int_c^x e^{-3t}t^4 dt\right] - 3e^{3x}\int_c^x e^{-3t}t^4 dt$$

$$= x^4 + 3e^{3x}\int_c^x e^{-3t}t^4 dt - 3e^{3x}\int_c^x e^{-3t}t^4 dt$$

$$= x^4$$

which checks with the right side of (7).

Example #2

Let us use inverse operators to solve

$$\frac{dy}{dx} - 6y = e^{5x}. \tag{9}$$

We have

$$(D - 6)y = e^{5x}.$$

Hence,

$$y = (D - 6)^{-1}e^{5x}. \tag{10}$$

If we now use (5) with $a = 6$ and $f(x) = e^{5x}$, (10) becomes

$$\begin{aligned} y &= e^{6x}\int e^{-6x}e^{5x}dx \\ &= e^{6x}\int e^{-x}dx \\ &= e^{6x}(-e^{-x} + c) \\ &= -e^{5x} + ce^{6x}. \end{aligned} \tag{11}$$

As a check of (11), $y = -e^{5x} + ce^{6x}$ implies

$$\frac{dy}{dx} = -5e^{5x} + 6ce^{6x};$$

whence,

$$\frac{dy}{dx} - 6y = -5e^{5x} + 6ce^{6x} + 6e^{5x} - 6ce^{6x} = e^{5x},$$

which agrees with (9).

In more advanced courses, one pursues the idea of inverse differential operators in more computational detail, but for our purpose of trying to show how the differential operator notation is used, the discussion in this section should be sufficient.

In the next and final section, however, we shall present an application of differential operators in which we really reap the rewards of the structural similarity between differential-operator polynomials and "ordinary" polynomials.

D

Systems of Equations

In the study of numerical algebra, the student begins by practicing on single equations with a single unknown. As he gains experience, he then is exposed to the idea that in many real situations, there is often more than a single variable with which he has to contend.

An analogous situation prevails in the study of differential equations. We first learn to handle equations where we are dealing with a single function of, say, t. We learn to solve such equations as

$$\frac{dx}{dt} + 3x = e^t$$

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = \sin t,$$

etc. Now a natural extension of this problem occurs when we begin to realize that we may have an equation which involves <u>two or more</u> functions of t.

Consider, for example, the equation

$$\frac{dx}{dt} - 3x - 6y = t^2 \tag{1}$$

where x and y are assumed to be functions of t.

What do we mean by a solution of (1)? Obviously, we mean that we want explicit functions $x(t)$ and $y(t)$ which satisfy (1).

Finding such solutions is, in a way, child's play. Namely, we may pick x or y arbitrarily and then solve (1) for the other. By way of illustration, suppose we let

$$y = -\frac{t^2}{6} \tag{2}$$

in (1).

In this case, (1) simplifies into

$$\frac{dx}{dt} - 3x + t^2 = t^2$$

or

$$\frac{dx}{dt} - 3x = 0. \tag{3}$$

e^{-3t} is an integrating factor of (3), so that (3) may be rewritten as

$$\frac{d}{dt}(e^{-3t}x) = 0,$$

whereupon it follows that

$$x = ce^{3t}. \tag{4}$$

As a check, we may use (2) and (4) to show that

$$\begin{aligned}\frac{dx}{dt} - 3x - 6y &= 3ce^{3t} - 3ce^{3t} - 6\left(\frac{-t^2}{6}\right)\\ &= t^2,\end{aligned}$$

so that

$$\left.\begin{aligned} x &= ce^{3t} \\ y &= \frac{-t^2}{6} \end{aligned}\right\}$$

is <u>one</u> family of solutions of (1).

We may generalize our treatment of (1) by allowing $y = y_o(t)$ to be any (integrable) function of t, whereupon (1) would become

$$\frac{dx}{dt} - 3x = t^2 + y_o(t). \tag{5}$$

Once $y_o(t)$ is specified, equation (5) is a linear first order differential equation in x, which may be solved by rewriting (5) as

$$e^{-3t}\frac{dx}{dt} - 3xe^{-3t} = e^{-3t}[t^2 + y_o(t)]$$

or

$$\frac{d}{dt}[xe^{-3t}] = e^{-3t}[t^2 + y_o(t)]$$

or

$$xe^{-3t} = \int e^{-3t}[t^2 + y_o(t)]dt + c,$$

whereupon

$$x = e^{3t}\int [t^2e^{-3t} + e^{-3t}y_o(t)]dt + ce^{3t}. \tag{6}$$

Obviously, the chore of expressing the integral on the right side of (6) depends strongly on the choice of $y_o(t)$, but what should be clear is that (6) shows us how to determine x as a 1-parameter family once $y = y_o(t)$ is given so that we obtain a "general" solution of (1) for each choice of $y_o(t)$.

Before continuing further, notice how this discussion relates to ordinary algebra. When we ask for <u>the</u> solution of $x + y = 7$, we may pick y arbitrarily and then determine x in terms of y. What happened here was that we had an extra degree of freedom to "play with." The same thing happens in our discussion of equation (1). We have two functions, but only one equation. Thus, we expect to have a degree of freedom at our disposal.

In other words, given two functions of t, we expect that we need two differential equations in order to determine the two functions.

Suppose, then, in addition to equation (1), we are told that our functions x(t) and y(t) must also satisfy

$$\frac{dy}{dt} + \frac{dx}{dt} - 3y = e^t. \tag{7}$$

That is, we wish to solve the <u>system</u> of equations

$$\left.\begin{aligned} &\frac{dx}{dt} - 3x - 6y = t^2 \\ &\frac{dy}{dt} + \frac{dx}{dt} - 3y = e^t \end{aligned}\right\}$$

Since the theme of this chapter is differential operators, let us rewrite the given system in this form. Namely,

$$\left.\begin{aligned}\left(\frac{dx}{dt} - 3x\right) - 6y &= t^2\\ \frac{dx}{dt} + \left(\frac{dy}{dt} - 3y\right) &= e^t\end{aligned}\right\}$$

$$\left.\begin{aligned}(D - 3)x - 6y &= t^2\\ Dx + (D - 3)y &= e^t\end{aligned}\right\} . \tag{8}$$

Since differential operators obey the same arithmetic as do polynomials, we suspect that we may be able to solve (8) by treating the coefficients of x and y as polynomials and then eliminating variables just as we do in the algebraic case.

For example, to eliminate y in (8), we "multiply" the first equation by $(D - 3)$, the second equation by 6, and then add the two equations. Thus,

$$\left.\begin{aligned}(D - 3)^2x - 6(D - 3)y &= (D - 3)t^2\\ 6Dx + 6(D - 3)y &= 6e^t\end{aligned}\right\}$$

$$(D - 3)^2x + 6Dx = (D - 3)t^2 + 6e^t. \tag{9}$$

We now invoke the definition of the differential operator as well as its algebraic properties to rewrite (9) as

$$(D^2 - 6D + 9)x + 6Dx = D(t^2) - 3t^2 + 6e^t$$

or

$$D^2x + 9x = D(t^2) - 3t^2 + 6e^t$$

or

$$\frac{d^2x}{dt^2} + 9x = \frac{d}{dt}(t^2) - 3t^2 + 6e^t$$

or

$$\frac{d^2x}{dt^2} + 9x = 2t - 3t^2 + 6e^t. \tag{10}$$

Equation (10) represents a single equation with "one unknown" which can be solved easily by the method of undetermined coefficients. Namely, the solution of the reduced equation is $x_h = c_1 \sin 3t + c_2 \cos 3t$ and we then try for a particular solution of (10) in the form $x_p = Ae^t + Bt^2 + Ct + E$, and we obtain that the general solution of (10) is

$$x = c_1 \sin 3t + c_2 \cos 3t + \frac{3}{5} e^t - \frac{1}{3} t^2 + \frac{2}{9} t + \frac{2}{27}. \tag{11}$$

Similarly, we can eliminate x from (8) by multiplying the first equation by $-D$, the second by $(D - 3)$ and then adding. Thus,

$$-D(D - 3)x + 6Dy = -D(t^2)$$

$$D(D - 3)x + (D - 3)^2 y = (D - 3)e^t$$

$$[(D - 3)^2 + 6D]y = D(-t^2) + D(e^t) - 3e^t$$

or

$$[D^2 + 9]y = \frac{d(-t^2)}{dt^2} + \frac{d(e^t)}{dt} - 3e^t.$$

Hence,

$$\frac{d^2y}{dt^2} + 9y = -2t - 2e^t. \tag{12}$$

Solving (12) by undetermined coefficients yields

$$y = c_3 \sin 3t + c_4 \cos 3t - \frac{1}{5} e^t - \frac{2}{9} t. \tag{13}$$

From (11) and (13) it seems that the system (8) has as its solution

$$\left.\begin{aligned} x &= c_1 \sin 3t + c_2 \cos 3t + \frac{3}{5} e^t - \frac{1}{3} t^2 + \frac{2}{9} t + \frac{2}{27} \\ y &= c_3 \sin 3t + c_4 \cos 3t - \frac{1}{5} e^t - \frac{2}{9} t \end{aligned}\right\} . \tag{14}$$

In (14), notice that we have four arbitrary constants. Namely, for example, the coefficient of sin 3t in (11) does not have to be the same as the coefficient of sin 3t in (13).

Note #1

Just as in the algebraic case, we must check to see whether (14) satisfies (8). The problem can be even more difficult here since "multiplying" by D involves taking a derivative and since the derivative of a constant is zero, we may lose terms in the process, or in different perspective, u and u + c have the property that $Du = D(u + c)$ so that unlike the case of ordinary algebra, $Du = Dv$ does not imply that $u = v$. Thus, it is possible to introduce extraneous solutions and conflicting constants.

While a proof is beyond our present need (as well as scope), there is an interesting theorem which tells us how many arbitrary constants our solution should have in terms of the "determinant of coefficients." More specifically, if our system has the form

$$\left.\begin{aligned} P_1(D)x + P_2(D)y &= f(t) \\ P_3(D)x + P_4(D)y &= g(t) \end{aligned}\right\} \tag{15}$$

where P_1, P_2, P_3, and P_4 are polynomials in D, then the number of arbitrary constants in the solution of (15) is equal to the degree of the polynomial operator obtained from

$$\begin{vmatrix} P_1(D) & P_2(D) \\ P_3(D) & P_4(D) \end{vmatrix} .$$

For example, applying this result to (8), we have

$$\begin{aligned} \begin{vmatrix} D - 3 & -6 \\ D & D - 3 \end{vmatrix} &= (D - 3)^2 - [-6D] \\ &= D^2 - 6D + 9 + 6D \\ &= D^2 + 9. \end{aligned}$$

Since the degree of the polynomial, $D^2 + 9$, is 2, the general solution of (8) should have 2, not 4, arbitrary constants.

Let us take x and y as given by (11) and (13) and see what happens, say, if we want equation (1) to be satisfied.

We have from (11) that

$$\frac{dx}{dt} = 3c_1\cos 3t - 3c_2\sin 3t + \frac{3}{5}e^t - \frac{2}{3}t + \frac{2}{9}.$$

Hence, (1) becomes

$$\left.\begin{aligned}&(3c_1\cos 3t - 3c_2\sin 3t + \tfrac{3}{5}e^t - \tfrac{2}{3}t + \tfrac{2}{9})\\ &-3(c_1\sin 3t + c_2\cos 3t + \tfrac{3}{5}e^t - \tfrac{1}{3}t^2 + \tfrac{2}{9}t + \tfrac{2}{27})\\ &-6(c_3\sin 3t + c_4\cos 3t - \tfrac{1}{5}e^t - \tfrac{2}{9}t)\end{aligned}\right\} = t^2.$$

Hence,

$$(3c_1 - 3c_2 - 6c_4)\cos 3t - (3c_2 + 3c_1 - 6c_3)\sin 3t = 0.$$

Therefore,

$$3c_1 - 3c_2 - 6c_4 = 0 \text{ and } 3c_2 + 3c_1 - 6c_3 = 0$$

or

$$c_4 = \frac{c_1 - c_2}{2} \text{ and } c_3 = \frac{c_1 + c_2}{2}. \tag{16}$$

Putting (16) into (14) yields

$$\left.\begin{aligned}x &= c_1\sin 3t + c_2\cos 3t + \tfrac{3}{5}e^t - \tfrac{1}{3}t^2 + \tfrac{2}{9}t + \tfrac{2}{27}\\ y &= \left(\frac{c_1 + c_2}{2}\right)\sin 3t + \left(\frac{c_1 - c_2}{2}\right)\cos 3t - \tfrac{1}{5}e^t - \tfrac{2}{9}t\end{aligned}\right\} \tag{17}$$

as the general solution of (8).

Technically speaking, we should still check to see whether (17) satisfies (7), however, unless we have made an arithmetic error, our abovementioned theorem guarantees that (17) is the general solution since the general solution must have two arbitrary constants in this example.

Note #2

It should be pointed out that we never had to use operator notation or operator methods for solving system (8). Namely, having been given equations (1) and (7), we might have been ingenious enough to have looked at these two equations and decided that we could eliminate y, we could differentiate equation (1) and then subtract three times equation (1) from this result. [This is precisely what it means to multiply (1) by the differential operator $(D - 3)$.] We could then multiply (7) by 6, and then add these two results to obtain equation (10).

The beauty of the operator method is that since its structure is so similar to that of polynomials, we may work with operator notation pretending that we were dealing with polynomials and thus arrive at correct answers rather easily - even though when this method is translated into the more conventional language of derivatives, the steps do not seem nearly as obvious. It is in this way that the operator technique is more than just a trivial type of shorthand notation.

We shall reinforce this idea in the form of some illustrative examples.

Example #1

Find x and y if

$$\left.\begin{aligned} \frac{dx}{dt} &= x + y \\ \frac{dy}{dt} &= x - y \end{aligned}\right\} . \tag{1}$$

Solution

We rewrite (1) as

$$\left.\begin{aligned} \frac{dx}{dt} - x - y &= 0 \\ -x + \frac{dy}{dt} + y &= 0 \end{aligned}\right\}$$

so that in operator notation, we have

$$\left.\begin{aligned} (D - 1)x - y &= 0 \\ -x + (D + 1)y &= 0 \end{aligned}\right\} . \tag{2}$$

We eliminate y in (2) by "multiplying" the first equation by D + 1. This yields

$$\left.\begin{aligned} &(D^2 - 1)x - (D + 1)y = 0 \\ &-x + (D + 1)y = 0 \end{aligned}\right\},$$

and adding these two equations, we conclude that

$$(D^2 - 2)x = 0.$$

Hence,

$$x = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t}. \tag{3}$$

We can eliminate x from (2) by "multiplying" the second equation by (D - 1) to obtain

$$(D - 1)x - y = 0$$

$$-(D - 1)x + (D^2 - 1)y = 0$$

Hence

$$(D^2 - 2)y = 0,$$

or

$$y = c_3 e^{\sqrt{2}t} + c_4 e^{-\sqrt{2}t}. \tag{4}$$

The determinant of coefficients in (2) is

$$\begin{vmatrix} D - 1 & -1 \\ -1 & D + 1 \end{vmatrix} = (D - 1)(D + 1) - 1$$

$$= D^2 - 2$$

and since $D^2 - 2$ is a polynomial of degree 2, our general solution must have 2 arbitrary constants. In other words, c_1, c_2, c_3, and c_4 in (3) and (4) cannot be independent.

What we can do is use (3) and (4) in (1) to see what conditions are imposed on our constants.

From (3), we have that

$$\frac{dx}{dt} = \sqrt{2}c_1e^{\sqrt{2}t} - \sqrt{2}c_2e^{-\sqrt{2}t}.$$

Hence,

$$\frac{dx}{dt} = x + y$$

implies that

$$\sqrt{2}c_1e^{\sqrt{2}t} - \sqrt{2}c_2e^{-\sqrt{2}t} = c_1e^{\sqrt{2}t} + c_2e^{-\sqrt{2}t} + c_3e^{\sqrt{2}t} + c_4e^{-\sqrt{2}t}$$

or

$$(\sqrt{2}c_1 - c_1 - c_3)e^{\sqrt{2}t} - (\sqrt{2}c_2 + c_2 + c_4)e^{-\sqrt{2}t} = 0. \tag{5}$$

Since $\{e^{\sqrt{2}t}, e^{-\sqrt{2}t}\}$ is linearly independent, (5) implies that

$$\sqrt{2}c_1 - c_1 - c_3 = 0 \text{ or } c_3 = (\sqrt{2} - 1)c_1$$

and

$$\sqrt{2}c_2 + c_2 + c_4 = 0 \text{ or } c_4 = -(\sqrt{2} + 1)c_2.$$

Replacing c_3 and c_4 in (4) in terms of c_1 and c_2, we obtain that the general solution of (1) is

$$\left.\begin{aligned} x &= c_1e^{\sqrt{2}t} + c_2e^{-\sqrt{2}t} \\ y &= (\sqrt{2} - 1)c_1e^{\sqrt{2}t} - (\sqrt{2} + 1)c_2e^{-\sqrt{2}t} \end{aligned}\right\} . \tag{6}$$

If we didn't want to use operator methods, notice that in eliminating y in (2), we were saying

(i) Differentiate $\frac{dx}{dt} = x + y$ with respect to t.

(ii) Add this result to $\frac{dx}{dt} = x + y$.

(iii) Add this result to $\frac{dy}{dt} = x - y$.

This is what "multiplying" by D + 1 means.

In other words,

$$\frac{dx}{dt} = x + y \rightarrow \frac{d^2x}{dt^2} = \frac{dx}{dt} + \frac{dy}{dt} \rightarrow$$

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} = x + y + \frac{dx}{dt} + \frac{dy}{dt} \rightarrow$$

$$\frac{d^2x}{dt^2} = x + y + \frac{dy}{dt} \rightarrow$$

$$\frac{d^2x}{dt^2} = x + y + (x - y) \rightarrow$$

$$\frac{d^2x}{dt^2} = 2x,$$

which agrees with $(D^2 - 2)x = 0$.

Notice, however, how much easier it seemed simply to apply the operator method as we did in deriving (3), at least in comparison with the originality required when the operator method is omitted.

Example #2

Find the solution of the system

$$\left.\begin{aligned} \frac{dx}{dt} &= x + y \\ \frac{dy}{dt} &= x - y \end{aligned}\right\} \tag{1}$$

subject to the initial conditions that $x(0) = 0$ and $y(0) = \sqrt{2}$.

Solution

In the previous example, we saw that the general solution of (1) was given by

$$\left.\begin{aligned} x &= c_1e^{\sqrt{2}t} + c_2e^{-\sqrt{2}t} \\ y &= (\sqrt{2} - 1)c_1e^{\sqrt{2}t} - (\sqrt{2} + 1)c_2e^{-\sqrt{2}t} \end{aligned}\right\} \tag{2}$$

Since (2) contains two arbitrary constants, we may prescribe a pair of initial conditions. [In this exercise, we have done it by prescribing $x(0)$ and $y(0)$. We could also have prescribed, say, $x(0)$ and $x'(0)$, etc.] Letting $t = 0$ in (2), we obtain

$$\left.\begin{aligned} x(0) &= c_1 + c_2 \\ y(0) &= (\sqrt{2} - 1)c_1 - (\sqrt{2} + 1)c_2 \end{aligned}\right\} \tag{3}$$

and since we are given that $x(0) = 0$ and $y(0) = \sqrt{2}$, equation (3) becomes

$$\left.\begin{aligned} &\text{(i)} \quad 0 = c_1 + c_2 \\ &\text{(ii)} \quad \sqrt{2} = (\sqrt{2} - 1)c_1 - (\sqrt{2} + 1)c_2 \end{aligned}\right\} \tag{4}$$

From (i), $c_2 = -c_1$, and making this substitution in (ii) yields

$$\sqrt{2} = (\sqrt{2} - 1)c_1 + (\sqrt{2} + 1)c_1,$$

or

$$\sqrt{2} = 2\sqrt{2}\, c_1.$$

Therefore, $c_1 = -c_2 = \frac{1}{2}$ and our desired solution is

$$\begin{cases} x = \frac{1}{2}\left(e^{\sqrt{2}t} - e^{-\sqrt{2}t}\right) = \sinh \sqrt{2}t \\ y = \frac{\sqrt{2} - 1}{2}\, e^{\sqrt{2}t} + \frac{\sqrt{2} + 1}{2}\, e^{-\sqrt{2}t} \end{cases}$$

Example #3

Find the general solution of

$$\left.\begin{aligned} &\text{(i)} \quad \frac{dx}{dt} + \frac{dy}{dt} + 2x - y = e^t \\ &\text{(ii)} \quad \frac{dx}{dt} + \frac{dy}{dt} - 3x + 2y = e^{2t} \end{aligned}\right\} \qquad (1)$$

Solution

In operator notation, (1) becomes

$$\left.\begin{aligned} &(D + 2)x + (D - 1)y = e^t \\ &(D - 3)x + (D + 2)y = e^{2t} \end{aligned}\right\} \qquad (2)$$

We may eliminate y from (2) by writing

$$\left.\begin{aligned} &(D + 2)[(D + 2)x + (D - 1)y = e^t] \\ &-(D - 1)[(D - 3)x + (D + 2)y = e^{2t}] \end{aligned}\right\}^{*} \qquad (3)$$

or

$$\left.\begin{aligned} &(D + 2)^2x + (D + 2)(D - 1)y = (D + 2)e^t \\ &-(D - 1)(D - 3)x - (D + 2)(D - 1)y = -(D - 1)e^{2t} \end{aligned}\right\} \qquad (4)$$

Hence,

$$(D^2 + 4D + 4 - D^2 + 4D - 3)x = (D + 2)e^t - (D - 1)e^{2t},$$

or

$$(8D + 1)x = D(e^t) + 2e^t - D(e^{2t}) + e^{2t}.$$

Thus,

$$8\frac{dx}{dt} + x = 3e^t - e^{2t},$$

*Without operators, we eliminate y adding twice (i) to the derivative of (i). We then subtract (ii) from the derivative of (ii). Finally, we subtract the latter result from the former.

or

$$\frac{dx}{dt} + \frac{x}{8} = \frac{1}{8}\,(3e^{t} - e^{2t}). \tag{5}$$

An integrating factor for (5) is $e^{\frac{t}{8}}$, so that

$$\frac{d}{dt}\,(xe^{\frac{t}{8}}) = e^{\frac{t}{8}}\left[\frac{3}{8}e^{t} - \frac{1}{8}e^{2t}\right]$$

$$= \frac{3}{8}\,e^{\frac{9t}{8}} - \frac{1}{8}\,e^{\frac{17t}{8}}.$$

Consequently,

$$xe^{\frac{t}{8}} = \frac{1}{3}\,e^{\frac{9t}{8}} - \frac{1}{17}\,e^{\frac{17}{8}t} + c_1,$$

or

$$x = \frac{1}{3}\,e^{t} - \frac{1}{17}\,e^{2t} + c_1e^{-\frac{t}{8}}. \tag{6}$$

Similarly, we may eliminate x from (2) by writing

$$\begin{cases} (D - 3)[(D + 2)x + (D - 1)y = e^{t}] \\ -(D + 2)[(D - 3)x + (D + 2)y = e^{2t}] \end{cases}$$

whereupon,

$$(D - 3)(D - 1)y - (D + 2)^2y = (D - 3)e^{t} - (D + 2)e^{2t}.$$

Therefore,

$$(D^2 - 4D + 3 - D^2 - 4D - 4)y = e^{t} - 3e^{t} - 2e^{2t} - 2e^{2t}$$

or

$$-8\frac{dy}{dt} - y = -2e^{t} - 4e^{2t}$$

or

$$\frac{dy}{dt} + \frac{y}{8} = \frac{1}{4} e^t + \frac{1}{2} e^{2t}. \tag{7}$$

An integrating factor for (7) is again $e^{\frac{t}{8}}$. Hence,

$$\frac{d(ye^{\frac{t}{8}})}{dt} = \frac{1}{4} e^{\frac{9t}{8}} + \frac{1}{2} e^{\frac{17t}{8}},$$

so that

$$ye^{\frac{t}{8}} = \frac{2}{9} e^{\frac{9t}{8}} + \frac{4}{17} e^{\frac{17t}{8}} + c_2.$$

Therefore,

$$y = \frac{2}{9} e^t + \frac{4}{17} e^{2t} + c_2 e^{-\frac{t}{8}}. \tag{8}$$

Using (6) and (7) in (i), we have

$$\left.\begin{aligned} &\frac{1}{3} e^t - \frac{2}{17} e^{2t} - \frac{1}{8} c_1 e^{-\frac{t}{8}} + \frac{2}{9} e^t + \frac{8}{17} e^{2t} - \frac{1}{8} c_2 e^{-\frac{t}{8}} \\ &+ \frac{2}{3} e^t - \frac{2}{17} e^{2t} + 2c_1 e^{-\frac{t}{8}} - \frac{2}{9} e^t - \frac{4}{17} e^{2t} - c_2 e^{-\frac{t}{8}} \end{aligned}\right\} = e^t$$

or

$$e^t + \left(\frac{15}{8} c_1 - \frac{9}{8} c_2\right) e^{-\frac{t}{8}} = e^t,$$

so that

$$\frac{15}{8} c_1 - \frac{9}{8} c_2 = 0.$$

Hence,

$$9c_2 = 15c_1,$$

or

$c_2 = \frac{5}{3} c_1.$

Using this value of c_2 in (8), we obtain that

$$\left.\begin{aligned} x &= \frac{1}{3} e^t - \frac{1}{17} e^{2t} + c_1 e^{-\frac{t}{8}} \\ y &= \frac{2}{9} e^t + \frac{4}{17} e^{2t} + \frac{5}{3} c_1 e^{-\frac{t}{8}} \end{aligned}\right\} \tag{9}$$

is the solution of (1).

As a final check, we need only see whether (9) satisfies (ii). Putting (9) into (ii) yields

$$\begin{array}{rl} & \frac{1}{3} e^t - \frac{2}{17} e^{2t} - \frac{1}{8} c_1 e^{-\frac{t}{8}} \\ + & \frac{2}{9} e^t + \frac{8}{17} e^{2t} - \frac{5}{24} c_1 e^{-\frac{t}{8}} \\ - & e^t + \frac{3}{17} e^{2t} - 3 c_1 e^{-\frac{t}{8}} \\ + & \frac{4}{9} e^t + \frac{8}{17} e^{2t} + \frac{10}{3} c_1 e^{-\frac{t}{8}} \\ \hline & 0\, e^t + e^{2t} + 0\, c_1 e^{-\frac{t}{8}} = e^{2t}. \end{array}$$

The fact that (9) has but one arbitrary constant checks with the fact that the determinant of coefficients in (2) is

$$\begin{aligned} \begin{vmatrix} D+2 & D-1 \\ D-3 & D+2 \end{vmatrix} &= (D+2)^2 - (D-1)(D-3) \\ &= D^2 + 4D + 4 - D^2 + 4D - 3 \\ &= 8D + 1, \end{aligned}$$

which has degree 1.

Thus, as far as initial conditions are concerned, notice that (9) allows us to choose either $x(0)$ or $y(0)$ at random but not both.

Example #4

(a) Show that the general solution of the system

$$\left.\begin{aligned} &\text{(i)} \quad \frac{d^2x}{dt^2} + \frac{dy}{dt} + x = y + \sin t \\ &\text{(ii)} \quad \frac{d^2y}{dt^2} + \frac{dx}{dt} - y = 2t^2 - x \end{aligned}\right\} \qquad (1)$$

has 4 arbitrary constants.

Solution

In operator notation, (1) becomes

$$\left.\begin{aligned} (D^2 + 1)x + (D - 1)y &= \sin t \\ (D + 1)x + (D^2 - 1)y &= 2t^2 \end{aligned}\right\} \qquad (2)$$

The determinant of coefficients is

$$\begin{aligned} \begin{vmatrix} D^2 + 1 & D - 1 \\ D + 1 & D^2 - 1 \end{vmatrix} &= (D^2 + 1)(D^2 - 1) - (D + 1)(D - 1) \\ &= D^4 - 1 - D^2 + 1 \\ &= D^4 - D^2 \end{aligned}$$

which is a polynomial in D of degree 4.

(b) Eliminate y from (2) and then explain what this means in (1) without use of operator notation.

Solution

We have

$$\left.\begin{aligned} &(D^2 - 1)[(D^2 + 1)x + (D - 1)y = \sin t] \\ &(D - 1)[(D + 1)x + (D^2 - 1)y = 2t^2] \end{aligned}\right\} \qquad (3)$$

Hence,

$$[(D^4 - 1) - (D^2 - 1)]x = (D^2 - 1)\sin t - (D - 1)(2t^2)$$

or

$$(D^4 - D^2)x = D^2(\sin t) - \sin t - D(2t^2) + 2t^2.$$

Therefore,

$$\frac{d^4x}{dt^2} - \frac{d^2x}{dt^2} = -2 \sin t - 4t + 2t^2. \tag{4}$$

In terms of (1), (3) tells us to do the following.

(I) Differentiate (i) twice and then subtract (i) from the result. Call this equation (iii).

(II) Differentiate (ii) and subtract (ii) from this result, calling the resulting equation (iv).

(III) Form the equation obtained by subtracting (iv) from (3).

(c) What is the most general function $x = x(t)$ which can satisfy (1)?

Solution

We need only find the general solution of (4). We have

$$D^2(D + 1)(D - 1)x = -2 \sin t - 4t + 2t^2.$$

Thus, the solution of the reduced equation is

$$x = (c_o + c_1 t) + c_2 e^t + c_3 e^{-t}. \tag{5}$$

We may then use undetermined coefficients to find a particular solution of (4) in the form

$$x = At^2 + Bt^3 + Ct^4 + E \sin t + F \cos t. \tag{6}$$

This leads to

$$\frac{dx}{dt} = 2At + 3Bt^2 + 4Ct^3 + E \cos t - F \sin t$$

$$\frac{d^2x}{dt^2} = 2A + 6Bt + 12Ct^2 - E \sin t - F \cos t$$

$$\frac{d^3x}{dt^3} = 6B + 24Ct - E \cos t + F \sin t$$

$$\frac{d^4x}{dt^4} = 24C + E \sin t + F \cos t$$

so that (4) becomes

$$\left.\begin{array}{l} 24C + E \sin t + F \cos t \\ \\ -2A + E \sin t + F \cos t - 6Bt - 12Ct^2 \end{array}\right\} = -2 \sin t - 4t + 2t^2.$$

Hence,

$$\begin{aligned} 24C - 2A = 0 \quad & A = -2 \\ E &= -1 \\ F &= 0 \\ 6B = 4 \rightarrow & B = \frac{2}{3} \\ -12C = 2 \rightarrow & C = -\frac{1}{6} \end{aligned}$$

Therefore, (6) yields

$$x_p = -2t^2 + \frac{2}{3}t^3 - \frac{1}{6}t^4 - \sin t \tag{7}$$

and combining (7) with (5), we have that the general solution of (4) is

$$x = c_o + c_1t + c_2e^t + c_3e^{-t} - 2t^2 + \frac{2}{3}t^2 - \frac{1}{6}t^4 - \sin t. \tag{8}$$