CALCULUS REVISITED
PART 3

A Self-Study Course

SUPPLEMENTARY NOTES

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CONTENTS

1

INVENTED NUMBER SYSTEMS

A

Introduction

There are many mathematical quotations to the effect that God gave man the whole numbers, but that the rest of the number system was invented by man. The meaning of this notion is that it was rather natural for man to be aware of the need to count. Indeed, even in the most modern mathematical vocabulary these "counting" numbers are called the natural numbers.

Negative numbers were unnatural in this context in the sense that it was not possible to have a collection with fewer than no members. In fact, even zero was considered an unnatural number because one would not need to use a symbol to indicate that nothing was present. This idea is carried through in a trivial way in the usual place-value notation. We write 107 to indicate the number one hundred seven with the zero serving as a place holder telling us that no tens are present. We need the zero only because we have no way of keeping track of the various denominations in place value notation other than by the position of a digit. For example, the Romans would write one hundred seven as CVII and the fact that no tens were present was apparent from the absence of the concrete (visible) symbol X. In this same sense, when we write the number one hundred thirty seven in place value, we write 137 - not 0137 to indicate that there are no thousands. In other words, zero was invented when place value was invented, but man was dealing with arithmetic long before the invention of place value.

In any case, when the natural numbers are augmented by zero, the resulting system of numbers is referred to as the whole numbers, and when we allow negative numbers as well as positive, the resulting system is called the integers.

The aim of this chapter is to create the mood for understanding why man invents new numbers. In this sense, we shall revisit the integers, the rational numbers, and the irrational numbers. The union of the rational numbers and the irrational numbers is called the real numbers, and what we would like to do is to make it seem natural that the word "real" in this context is not very appropriate. Once this is done, we shall devote the next chapter of these notes to the discussion of the complex numbers, a number system which somehow seems unreal to the student, yet is as real as any of the previously studied number systems.

B

A Note About Number Versus Numeral

In any artificial (man-made) language there is a difference between a concept and the words used to denote the concept. In mathematics this problem usually is first encountered in the study of elementary arithmetic. We learn about the concept of number, but we denote numbers by symbols called numerals. Thus, for example, X, 10, 7 + 3, 5 x 2, and IIIIIIIII are each numerals which denote the number ten.

Now while the natural numbers might have been "God given", the numerals to denote these numbers were invented by man. It is fair to assume that man found the simplest symbols that would suffice and only invented more complicated symbols when the simpler ones proved to be inadequate or cumbersome. For example, tally marks are a very visual system of numerals. That is, somehow or other it is easier to visualize the concept of "three-ness" looking at the numeral III than at the numeral 3, but by the same token tally marks would be extremely awkward as a numeral system if we wished to denote the number one billion (a number rather easy to denote in ordinary place value numerals, namely, 1,000,000,000).

Looking at tally marks, it is not difficult to see that the concepts of addition and multiplication lent themselves very nicely to this system of numerals. For example, to add two natural numbers* we had only to "amalgamate" the tally marks that represented each of the two (natural) numbers.

By way of illustration, to denote that the sum of three and four is seven, we write 3 + 4 = 7. In the tally system we would not write that III + IIII = IIIIIII. Rather all we would do is write III IIII, the sum obviously being the total number of written tally marks. Notice that this explains very vividly why the natural numbers are commutative with respect to addition. That is, while it may not seem self evident that 3 + 4 = 4 + 3**, it surely seems

*We say "natural" since tally marks would not be used in any other context.

** In fact if 3 + 4 = 4 + 3 seems natural because all we did was change the order, notice that 3÷ 4 and 4÷3 represent different numbers even though "all" we did was change the order.

clear that III IIII and IIII III represent the same number of tally marks.

In a similar way, one could view multiplication of natural numbers very nicely in terms of tallies, and this idea is reflected in the fact that we read 3 x 4 as 3 <u>times</u> 4 even though the word "times" does not appear in the artificial expression 3 x 4. That is, we view 3 x 4 as three, four times (or as four, three times). Thus, in terms of tallies we have

III III III III

or

IIII IIII IIII

We could pursue the advantages of tally systems to great extremes and while such a discussion is highly informative and very interesting, it is not necessary to make the point we are hitting at in this chapter. Rather, with the preceeding remarks as an introduction, notice that neither subraction nor division lent themselves too well to a tally-mark interpretation. For example, in terms of tallies we could take three from five but we could not take five from three. That is, in terms of our modern notation 5 - 3 = 2 would be written very nicely in tally notation (and notice here that we actually seem to capture the feeling of what it means when we say "take away"; that is, in the tally system we actual take away [delete] some tallies from the rest). On the other hand, 3 - 5 could not be interpreted in terms of tallies.

As for division, certainly 6÷3 = 2 could be viewed pictorially as III III indicating that if six tallies are divided into groups of three we get three such groups. Yet we cannot view 5÷3 in the same way, since we cannot "break up" five into a whole number of groups of three. To be sure, we would get one such group and two of the necessary three to form another, but the fact remains that if all we have is the tally system, we must <u>invent</u> new numbers (not just numerals) to represent the type of number named by 3 - 5 or 5÷3.

In the remainder of this chapter we shall emphasize number concepts and introduce systems of numerals only for the purpose of highlighting certain remarks. We may also tend to get a bit careless in our colloquial use of "number" when we mean "numeral", etc., but we hope that if this should occur it will be clear from context what it meant.

C

The Rational Numbers

Why did man invent numbers other than whole numbers? Often one answers this question by giving excuses rather than reasons. Among the various excuses are that we must often take fractional parts of the whole (a frightening statement to beginning students). Yet it is more common to refer to <u>one</u> ounce than to <u>one-sixteenth</u> of a pound, or to <u>one</u> dime rather than to <u>one-tenth</u> of a dollar.

Even the practical side is often misunderstood. For example, the existence of the fraction $\frac{3}{2}$ would hardly help the father figure out how to bequeath three race horses to two sons! Most likely the father would give each son <u>one</u> horse and distribute the profit from the sale of the third horse.

In essence, the concept of division was not always meaningful in problems that lent themselves to a tally-mark interpretation. However, there arose applications of numerical problems in which only natural numbers were mentioned but the solution required more than natural numbers. Such problems could not be viewed effectively in terms of tally marks. For example, suppose we have a 5 inch length and we want to cut it into three parts of equal length. Notice that we either know how to trisect a line segment or we don't, and that once we know how to divide a length into three parts of equal length it really makes no difference what the length of the origianal segment is (except in terms of the actual answer).

In a similar way, one could encounter the same kind of problem in the sense that it is no more meaningful for a particle to travel six feet in two seconds than to travel five feet in two seconds, and that in each case we want to compute the average speed of the particle. Notice, in other words, that there are meaningful problems which involve knowing only the natural numbers, but whose solution can not be obtained within the framework of the natural numbers.

To rephrase this idea from a more algebraic point of view, notice that one does need any number system more extensive than the natural numbers for the equation

$$2x = 3$$

to be meaningful. Namely the equation asks us to find a number such that when it is multiplied by 2 (a natural number) the result is 3 (also a natural number). However, since twice any natural number is an even natural number and since 3 is odd, we see that the meaningful equation $2x = 3$ cannot have a solution which is a natural number.

In any event, we now arrive at a crossroads that is the very crux of scientific investigation. We have the choice of saying that we don't care whether the equation $2x = 3$ has a solution or saying it is imperative that $2x = 3$ have a solution. Should we decide that we want $2x = 3$ to have a solution, we must invent new numbers, and until they are invented the equation has no solutions, or, if we wish to give into the fact that we are comfortable only with that which we believe exists, we may say that the solutions of the equation are imaginary. Hopefully, it is clear to you that the word "imaginary" is used in this context only to denote a concept that does not appear to be meaningful to the use. That is, it is in this sense that realness is in the eyes of the beholder.

Moreover, to make sure that the rational number system (recall that system implies structure) was a "legitimate" extension of the natural number system, one had to make sure that whatever rules were invented for combining rational numbers they had to be consistent with the rules for how these numbers would be combined had they been natural numbers. For example, one might have like to add common fractions (by the way, common fractions are but one form of numeral for denoting rational numbers; another well-known numeral system for denoting the rational numbers is in terms of decimals and this will be discussed in more detail in the next section) by adding numerators and adding denominators. Had we done this, then the sum of two and three would depend on how we wrote the numbers. For example, we would have

$$2 + 3 = 5$$

$$\frac{2}{1} + \frac{3}{1} = \frac{5}{2}$$

etc.

At any rate, since the ancient Greeks were primarily interested in geometry and since the concept of length and distance were so crucial to this study, it should not seem too strange that the ancient Greeks were the first to develop the rational numbers in the form of a logical science.

Of course, the tally system became inadequate as a model for the rational numbers and for this reason the number line replaced the tally system as a model for the "new" real number system, that is, the system of rational numbers.

D

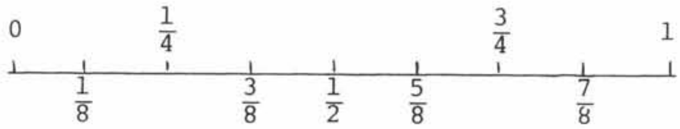
The Irrational Numbers

Once the rational numbers were invented, the ancient Greek believed that no other numbers would ever have to be invented. He believed, in other words, that every (real) number was expressable as the quotient of two whole numbers and accordingly he never thought much about the need to invent other (extended) number systems.

Now, in order for us to appreciate the fact that it seemed natural that the rational numbers completed the number system, let us review the type of reasoning the ancient Greek used in arriving at his decision.

In terms of the number line, let us look at a segment of unit length. If we bisect this segment, the midpoint is a rational number (quite in general, the average of two rational numbers is again a rational number). If we then bisect each of the two new segments the new points of division also denote rational numbers. If we continue in this way it seems that we eventually fill in the entire segment. Of course, we never do really fill in the entire segment philosophically-speaking but it is in reality filled in because a physical line has thickness and eventually the distance between two points of subdivision is less than the thicknessof the line drawn with our pencil. Nevertheless, we get the feeling that we can make our lines thinner and thinner and that in this way we can neglect the thickness of our lines; and eventually the entire segment does get filled in.

Well, what we can say for sure is that even if the line is not filled in completely, any errors can be made arbitrarily small; but the fact that we do not fill in the line completely can be seen very dramatically from the following observation. If we assume that our segment extends from 0 to 1, then the midpoint of the segment is at the point $\frac{1}{2}$. If we now bisect each of the two new segments we obtain the points $\frac{1}{4}$ and $\frac{3}{4}$; and if we then bisect each of these new segments we obtain the additional points $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, and $\frac{7}{8}$. Pictorially,



Notice then that the only points we can obtain in this way are those which when represented as common fractions have denominators equal to 2, 4, 8, 16; and in general, 2^n. In other words, not only don't we fill in the entire line in this way, but we even miss <u>most</u> of the rational numbers - in particular those with denominators 3,5,6,7,9, 10,11,12,13,14,15, etc.

Certainly, this result does not prove the existence of numbers which are non-rational, but it does demonstrate that the given proof is inadequate.

Perhaps the easiest way to see why there should be irrational numbers is in terms of decimals. It should be noted that when represented as a decimal every rational number either terminates or else repeats the same cycle of digits endlessly (in this sense, all rational numbers are represented by repeating decimals since a terminating decimal such as 0.5 may be viewed as the repeating endless decimal 0.50000000.....). The easiest way to see this is that if we try to convert $\frac{m}{n}$ into a decimal, we either eventually obtain a remainder of 0 in which case the decimal perminates or else we have the same remainder occur twice in which case the entire cycle between the repeated remainders also repeats. For example if we write $\frac{1}{7}$ as a decimal we obtain

```
   0.142857
7 )1.0000000
   7
   30
   28
    20        The remainder 1 repeats.
    14
     60
     56
      40
      35
       50
       49
        1     so we again have 1 followed by endlessly many 0's,
              just as when we began.
```

[Notice, of course, that the first <u>repeated</u> remainder need not be the first remainder. For example in writing $\frac{8}{15}$ as a decimal we have

```
    0.533...
15 )8.0000     First remainder
    7 5
     50        First repeated remainder
     45
      50                               ].
```

The fact that a remainder must repeat when $\frac{m}{n}$ is written in decreased from follows from the profound, but simple, observation that since there can be but n remainders (0 through n - 1) when a number is divided by n any set of more than n numbers must have the property that at least two* have the same remainder when divided by n. Hence no later (but it may happen sooner) than after the $(n + 1)^{th}$ 0 a remainder must repeat.

Therefore, if we admit that any decimal should name a real number and if we agree that we can invent decimals that are neither terminating nor repeating, then we have admitted that there must be numbers other than rational numbers. That is, if every rational number is represented by a terminating or a repeating decimal and there are decimals that neither terminate nor repeat, then there must be numbers which are not rational.

As an example of a non-terminating, non-repeating decimal consider

$0.494994999499\hat{9}....$

*This apparently simple observation is known under the ominous name of Dedekind's "Chest of Drawers Principles". For example, given 400 people at least two must celebrate their birthday on the same day of the year (but we can't be sure which day (or days) it will be, since there are only 365 (or at most 366) days in a year.

where ^ means that each time another 9 is added to the cycle. Certainly the above decimal cannot repeat the same cycle endlessly since each cycle contains one more 9 than the previous cycle.

Notice of course that we can approximate an irrational number by a suitable chosen rational number to as great a degree of accuracy as we desire. For example, with respect to our above irrational number notice that the rational number 0.49499 (which is rational because it terminates) is accurate to five decimal places as an approximation to the given irrational.

If this seems abstract, recall that we were often told in high school that $\pi = \frac{22}{7}$. Yet without proving the point here, the fact remains that is an irrational number (in fact it is transcendental*) while $\frac{22}{7}$ is rational.

What is meant in this context was that while π and $\frac{22}{7}$ were not equal, as decimals they were represented as

$$\left.\begin{array}{l} \pi = 3.14 \,|\, 158\ldots \\ \frac{22}{7} = 3.14 \,|\, 2857 \quad \overline{142857} \end{array}\right\}$$

If we can't measure beyond the nearest hundreth we cannot distinguish between π and $\frac{22}{7}$.

The fact that we can approximate any irrational number by a rational number to as close a degree of accuracy we desire means that from a practical point of view we never need irrational numbers (since they can always be replaced by "sufficiently close" rational numbers). Nevertheless, we hope that our discussion has shown that there are certain "real" numbers which seem unnatural.

What may be even more alarming is that we cannot even say that there are relatively few of such disturbing numbers as the irrationals. While we do not want to enter into a discussion of different orders of infinity at this time, the fact is that there are <u>more</u> irrational numbers than rational numbers. As an intuitive device to help sense what this means immagine a device that allows us to pick a digit from

*See Note #3 at the end of this chapter for a definition of transcendental.

0 through 9 at random.

Suppose we then agree to construct an endless decimal by using our random device to pick a number which we shall use for our first decimal place, a second number for our second decimal place, etc. It would seem that the chances of this device yielding an endless chain in which the same cycle of digits was ultimately always repeated is very small (in fact, from this point of view, it seems like a miracle that there should even be one rational number). Yet unless the same sequence of digits is ultimately repeated, the decimal cannot represent a rational number.

At any rate, we shall continue with the more quantitative aspects of irrational numbers in the notes at the end of this chapter. But for now we would like to conclude with the observation that once he discovered that not all numbers were rational, the ancient Greek invented the term irrational to mean all numbers that were not rational, and in this way, he apparently completed the entire number system.

Yet he had not reckoned with the negative numbers, if only because he had not thought about directed lengths. The idea of directed lengths was due mainly to Descartes in the sixteenth century, and once this was accomplished the real number system (at least as we define "real" today) was completed. Geometrically the real numbers were the set of all points on the number line.

Algebraically, they were the set of all numbers whose squares were non-negative, i.e., $\{x : x^2 \geq 0\}$.

Are there numbers which are not "real". Well, in the same way that the solution of $2x = 3$ is "non-real" if the only "real" numbers are the integers, the solutions of $x^2 + 1 = 0$ must be "non-real" if the only "real" numbers are those whose squares are non-negative; for clearly from the rules of algebra, if x is a number such that $x^2 + 1 = 0$ then $x^2 = -1$. Since a real number cannot have a negative square, if we want x to be a number it must be a "non-real" number.

It is this topic that we shall discuss in the next chapter.

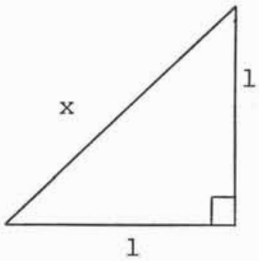
Note #1

The Ancient Greek Proof That $\sqrt{2}$ is Irrational

Obviously the ancient Greek didn't discover the irrational numbers by his knowledge of decimals if only because he knew about the irrationals by 600 B.C., and decimals were not invented as such until many hundreds of years later.

The ancient Greek's approach was that he identified numbers with length. In particular, therefore, he felt that any length he could construct (at least by the principles of plane geometry) denoted a real number (and indeed so it was since the length existed).

Now consider the isosceles right triangle each of whose legs is 1 unit of length. Then by the Pythagorean Theorem the length of the hypotenuse must be $\sqrt{2}$. That is, the length must satisfy $1^2 + 1^2 = x^2$, or $x^2 = 2$



Now, if x is rational it has the form $x = \frac{m}{n}$ where m and n are whole numbers, and we may further assume that we have chosen $\frac{m}{n}$ to be in lowest terms (i.e., m and n show no natural number except 1 as common factor).

We then have upon squaring that

$$\frac{m^2}{n^2} = 2$$

or

$$m^2 = 2n^2. \qquad (1)$$

Since $2n^2$ is divisible by 2 and $m^2 = 2n^2$ it follows that m^2 is also divisible by 2. Now since 2 is a prime number, it follows that m must itself be divisible by 2.*

*A natural number n is called prime if (1) $n > 1$ and (2) n has no natural numbers as factors except for itself and 1.

[Note: If a,b, and p are natural numbers and ab is divisible by p, we cannot validly conclude that either a or b is divisible by p. For example 4 x 9 is divisible by 6 yet neither 4 nor 9 is divisible by 6. What happened was that since 6 = 3 x 2 we "used up" the 3 as a factor of 9 and the 2 as a factor of 4. But a prime can't be broken up into factors which can be used piecemeal. In the present context notice that whole 6 is not divisible by 4, 6^2 is. The reason again is that $\frac{6^2}{4} = \frac{6}{2} \times \frac{6}{2}$. For a prime number p, however, it is true that ab is divisible by p if and only if at least one of the numbers. a or b, is divisible by p]

Now since m is divisible by 2 we can write it as $m = 2k_1$ where k_1 is a natural number, whereupon

$$m^2 = 4k_1^2 \ . \qquad (2)$$

Substituting the value of m^2 in (2) into equation (1) we obtain

$$4k_1^2 = 2n^2$$

or

$$n^2 = 2k_1^2. \qquad (3)$$

From equation (3), by reasoning as we did above, it follows that n^2, hence n,is divisible by 2.

But the fact that both m and n are divisible by 2 contradicts the given facts since we chose m and n so that $\frac{m}{n}$ was in lowest terms, hence m and n cannot have 2 as a common factor. Where did this contradiction come from? Well, all of our mathematical arguments were valid, so the fact that we have a false conclusion means that we must have begun with at least one false assumption. Yet the only assumption we made was that $\sqrt{2}$ is rational (otherwise, we could not assume that it could be written in the form $\frac{m}{n}$ where m and n were natural numbers). Hence it must be that our assumption that $\sqrt{2}$ is rational is false since it leads validly to a false conclusion; but if it is false that $\sqrt{2}$ is rational, then by definition it is irrational.

As a passing remark, the type of proof in which we show that something is true by showing that the assumption is false leads to a valid but false conclusion is known as the Indirect Proof and also under the name Reduction ad Absurdem.

Note #2

The Unique Factorization Theorem

A well-known theorem of arithmetic is that any natural number can be written uniquely as a product of powers of primes, except for the order of the factors. (It is for this reason that 1 is not considered a prime since we can multiply as many factors of 1 as we wish without changing the value of the product.)

For example while 24 can be factored in several ways into the product of two or more natural numbers, such as 4 x 6, 12 x 2, and 2 x 2 x 6; there is only one way in which we can break it down completely into prime factors and that is as 2 x 2 x 2 x 3, or $2^3 \times 3$. We could have written it in the order 3×2^3, but other than for this rather trivial switch in order, we see that 24 can be factored uniquely (i.e., in one and only one way) as a product of powers of primes.

The unique factorization theorem also supplies us with an indirect proof that $\sqrt{2}$ is irrational. Namely the assumption that $\sqrt{2}$ is rational leads to $m^2 = 2n^2$ and this is impossible since m^2 can have only an even number of factors of 2 while $2n^2$ has an odd number of factors of 2.

That is m^2 has double the number of factors of 2 than does m and since the double of a whole number (including 0) is even, m^2 has an even number of factors of 2 (or for that matter m^2 has an even number of any prime factor). In a similar way we see that n^2 has an even number of factors of 2 and since $2n^2$ has one more factor of 2 than does n^2, we see that $2n^2$ has an odd number of factors of 2.

Thus if $m^2 = 2n^2$ we have that the same number (i.e., the one denoted by m^2 and $2n^2$) has both an even number of factors of 2 and an odd number of factors of 2 and since no number is both even and odd, this assumption that $m^2 = 2n^2$ contradicts the unique factorization theorem.

Note #3

The Fundamental Theorem of Polynomial Factorization

Suppose we have a polynomial whose coefficients are integers and that we wish to investigate the solutions of the equation obtained when the polynomial is equated to 0. For example suppose the polynomial equation is given by

$$a_n x^n + \dots + a_1 x + a_o = 0 \tag{1}$$

where $a_o, \dots,$ and a_n are integers. Then the amazing result is that if there is a rational root of (1), say, $x = \frac{p}{q}$ then it follows that q must be a divisor of a_n and that p must be a divisor of a_o.

For example if

$$3x^8 + 9x^5 - 7x^3 + 5 = 0 \tag{2}$$

has a rational root, say $\frac{p}{q}$ then q must be a divisor of 3 (that is, q is either 1, -1, 3, or -3) and p must be a divisor of 5 (so that p = 1, -1, 5, or -5). In other words unless $\frac{p}{q}$ is one of the numbers ± 1, ± 5, $\pm \frac{1}{3}$, or $\pm \frac{5}{3}$ then $\frac{p}{q}$ cannot be a rational root of (2).

[Notice that we are not saying that ± 1, ± 5, $\pm \frac{1}{3}$, and $\pm \frac{5}{3}$ are roots of (2) - only that they are the only possibilities]

To prove this result (known as the Fundamental Theorem of Polynomial Factorization) we replace x in (1) by $\frac{p}{q}$, where $\frac{p}{q}$ is assumed to be in lowest terms. We obtain

$$a_n \left(\frac{p}{q}\right)^n + \dots + a_1 \left(\frac{p}{q}\right) + a_o = 0$$

or

$$a_n \frac{p^n}{q^n} + \dots + \frac{a_1 p}{q} + a_o = 0 \tag{3}$$

and if we now multiply both sides of (3)by g^n we obtain

$$a_n p^n + a_{n-1} p^{n-1} q + \dots + a_1 p q^{n-1} + a_o q^n = 0. \tag{4}$$

Rewriting this as

$$a_n p^n + \dots + a_1 p q^{n-1} = -a_o q^n \tag{5}$$

and observing that the left side of (5) is divisible by p (since each terms is) we see that the right side is also divisible by p. Since p and q are assumed to have no factors (other than $\pm$) in common, the only way that $-a_o q^n$ can be divisible by p is if a_o is divisible by p.

Similarly if we rewrite (4) as

$$a_n p^n = -(a_{n-1}p^{n-1}q + \ldots + a_o q^n)$$

we can conclude that $a_n p^n$ must be divisible by q. Hence, since p and q have no factors in common, a_n is divisible by q.

The interesting case occurs when $a_n = 1$ (in which case the polynomial is called monic. In other words, a monic polynomial is one whose leading coefficient is 1). In this case, since q must be a divisor of a_n, q is either 1 or -1 and consequently $\frac{p}{q} = \pm p$ which is a whole number.

This tells us a great deal about how to establish what types of numbers can be irrational. Namely, any rational root of a monic, integral (integral means that the coefficients are integers), polynomial equation must itself be an integer since its denominator is a factor of a_n which is 1.

Thus, any root of a monic, integral, polynomial equation which is not an integer must be an irrational number.

For example $\sqrt{2}$ satisfies the integral, monic polynomial equation

$$x^2 - 2 = 0.$$

Since this equation cannot have an integer as a root (i.e., 1 x 1 is too small to be 2 and 2 x 2 is too big, etc.), any root must be irrational, but $\sqrt{2}$ is a root of this equation, so $\sqrt{2}$ is irrational.

[Another way of obtaining this result is to observe that if $\frac{p}{q}$ is a root of $x^2 - 2 = 0$ then q must be a divisor of 1 and p a divisor of -2. Hence $q = \pm 1$ while $p = \pm 1$ or ± 2. Thus, the only possible rational roots of $x^2 - 2 = 0$ are ± 1 and ± 2; and none of these numbers equals $\sqrt{2}$. Hence $\sqrt{2}$ must be irrational since it is a root of the equation but can't be a rational root.]

We should point out that "most" irrational numbers (believe it or not) cannot be roots of integral polynomial equations. An irrational number which is the root of such an integral equation is called an algebraic irrational, while any irrational number which cannot be the root of such an equation is called a transcendental number. For example (and we do not prove these facts since the proofs are very difficult as well not necessary for our purposes) among the transcendental numbers are π and e.

Notice that any rational number is automatically the root of some integral polynomial equation. Namely, the rational number m/n satisfies the equation $x = m/n$, or $nx - m = 0$, and this equation, since m and n are integers, is an integral equation which has m/n as a root.

Notice also our emphasis on integral polynomial. For example π is trivially a root of the linear polynomial equation $x - \pi = 0$ but this equation is not integral, since no matter what else π might be, it is not an integer.

It turns out that any irrational number which involves nothing worse than arithmetic combinations of roots of integers are algebraic numbers. In particular $\sqrt{2}$ is algebraic since it satisfies $x^2 - 2 = 0$.

As a not so trivial example notice that $\sqrt{2} + \sqrt{3}$ is also algebraically irrational. Namely, $\sqrt{2} + \sqrt{3}$ satisfies the non-integral equation

$$x = \sqrt{2} + \sqrt{3}$$

or

$$x - \sqrt{2} = \sqrt{3}.$$

Squaring both sides yields

$$x^2 - 2\sqrt{2} + 2 = 3$$

or

$$x^2 - 1 = 2\sqrt{2}\ .$$

Again squaring we obtain

$x^4 - 2x^2 + 1 \ (2\sqrt{2})^2 = 8$

or

$$x^4 - 2x^2 - 7 = 0. \qquad (6)$$

Thus, $\sqrt{2} + \sqrt{3}$ is algebraic because it is a root of the integral polynomial equation (6). Moreover, by the Fundamental Theorem, any rational root, $\frac{p}{q}$, of (6) must have the property that q is a divisor of 1 and p a divisor of -7.

Thus, no numbers other than $\pm$ 1 or $\pm$ 7 are even eligible to be candidates for roots of (6). Since $\sqrt{2} + \sqrt{3}$ is not equal to $\pm$ 1 or to $\pm$ 7 it cannot be a rational root of (6). Thus, since $\sqrt{2} + \sqrt{3}$ is a root of (6), it is irrational.