CALCULUS REVISITED
PART 3
A Self-Study Course

STUDY GUIDE

Block 1
An Introduction to Functions
of a Complex Variable

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PREFACE

The material covered in Parts 1 and 2 of this course serves as the foundation of virtually any other mathematics course, pure or applied, in the curriculum. Specifically, the notion of the "game" of mathematics which served as our central theme for Part 2 is indeed the backbone of any course concerned with the concept of mathematical structure; and the importance of differentiation, integration, and their related subtopics should be obvious from our previous study.

While the previous material is the basis of all of advanced calculus, there are three topics which traditionally act as buffers between elementary* calculus and advanced calculus. These three topics are often included as part of a one year course which begins with the equivalent of our Part 2 and which concludes with these three topics as special applications.

The three topics are

(1) An Introduction to Complex Variables
(2) An Introduction to Ordinary Differential Equations
(3) An Introduction to Linear Systems (Linear Algebra)

We stress the word "introduction" not to be humble, but rather to indicate that the three topics are sufficiently profound to justify spending a minimum of an entire term on each topic. The aim of elementary calculus is to introduce these topics only to the depth needed by the student to gain an overview in the hope that when he encounters these topics in more depth later, he is better prepared to grasp their significance.

Now there are probably dozens of different ways to construct a central theme in which these three topics are presented as an integrated whole. The trouble seems to be that there are so many different applications of these topics that, depending on the background of the particular student, it is quite possible that no matter which theme is chosen by the instructor any of the others may have seemed better.

*The word "elementary" is not being used in a condescending way here. You are cautioned not to confuse "elementary" with "simple." Indeed, when Sherlock Holmes says: "Elementary, my dear Watson!," the result is anything but simple or obvious to poor Dr. Watson. We identify "elementary" with "basic" in this course, and basic things may well be very difficult. In summary, in the present context, we define elementary calculus to consist of those building blocks from which the more advanced courses are built.

For this reason, we have chosen a middle-of-the-road policy, and have presented these topics as more-or-less independent subjects, each important and self-contained in its own right. The order of the topics, by and large, was arrived at by a reason no more profound than that this is the order in which the topics are covered in our accompanying textbook.

While it is not important for you to have studied our Parts 1 and 2 in order to understand Part 3 (i.e. Part 3 stands as a self-contained course whose prerequisites are the usual topics involving partial derivatives, multiple and line integrals, and a smattering of matrix algebra), we have planned Part 3 as a natural continuation of Part 2 so that the student who has studied Part 2 as a 1-semester course can augment this study by Part 3 to form a rather natural full year course.

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INTRODUCTION

This self-study course consists of several elements which supplement one another. As in most courses, the central building block is the textbook. The remaining parts augment the text. First, there are the lectures. These are designed to give an overview of the material covered in the text and to supply motivation and insight in those areas where the oral word is more helpful than the written word.

Because the lectures are on film (or tape) it is assumed that you will be able to view a lecture more than once. You may use the lecture as an introductory overview and then review the unit by watching the lecture again when the rest of the assignment for that unit has been completed.

Yet, the fact remains that most students will not, for one reason or another, watch the lecture as often as might be advisable. For this reason, photographs of the blackboards, exactly as they appeared at the end of the lecture, have been made and reproduced as part of the study guide. Consequently, as you proceed through an assignment, there is always a rather convenient reminder and summary of the lecture. In fact, it might very well happen that once you have seen the lecture and done the assignment, the photographs of the blackboards will be sufficient to supply you with an "instant replay" whenever needed.

After the lecture, there are times when additional material is needed to bridge the gap between where the lecture ends and the text begins. Certain topics in the text are particularly difficult and as a result the student requires additional points of view or even a rehash of what's in the text. Frequently, a choice of approaches to a difficult topic is the best psychological boost to the student. Certain important topics are sometimes presented in the text in order to solve a specific problem, but it turns out that these topics have applications far beyond the specific problem in question; consequently the student would benefit from a more detailed explanation. Finally, there are certain topics that form a "twilight zone" for the student. Roughly speaking, these are the topics that the college professor assumes the student learned in high school and the high school teacher thinks he will learn in college. In such cases a student may need more explanation than is offered in the text. For these reasons, the course includes a volume entitled "Supplementary Notes."

Finally, it often happens that even under the assumptions that one has given the best possible lectures, has chosen the best possible textbook, and has written the best supplementary notes, there are still things the student does not learn unless he sees them occur in solutions to exercises. For this reason a major portion of the course consists of exercises together with in-depth solutions.

Some of the exercises serve as a vehicle (indeed, almost as an excuse) for introducing difficult but important topics in the guise of solutions to important exercises. When this occurs the exercise is labeled (L) to indicate to the student that it is a learning exercise, an exercise for which (even if he can solve the problem) he should read the solution because one or more important asides will be made there.

Other exercises are more routine and are supplied simply so that the student can test whether he can handle the material. These exercises appear without special designation.

The final type of exercise is called the optional exercise. It often happens that in the learning of a new concept one has to see beyond where he is in order to appreciate better his present position. For example, it often happens that a student does not begin to master algebra, geometry, or trigonometry until he studies calculus where these topics are used as a means toward an end rather than as an end in themselves, or he learns calculus better while he is studying differential equations, etc. At any rate, for this reason we have interjected optional problems for the purpose of giving new ideas additional exposure. An optional exercise together with its solution forms a step-by-step supplement to the material already presented. Whenever an optional problem is given, there is an explanation as to why it is assigned and what it hopes to accomplish.

Actually, it is the hope of the author that the student will not distinguish between optional and "regular" exercises (to an author nothing is optional and everything is important), but we recognize that certain students may not have either the desire or the time to delve in depth into certain topics. Accordingly, wherever we feel that there is no serious damage done by the omission of the exercise, we have called it optional.

Two other features round out our course. Since many people who will be taking this course may have studied some of the topics previously, we have included a pretest at the beginning of each new block of material. The nature of this pretest is to let the student know whether he needs the material contained in the block. That is, if he can pass the pretest comfortably, he may elect to by-pass the block. It may be of interest to know that each pretest problem occurs as a learning exercise within the block, so that the student who has trouble with such an exercise in the pretest can rest assured he will see a detailed solution later.

As important as the pretest is the post-test or what is more colloquially known as the quiz. Somehow or other, there is no substitute for a comprehensive test to see what the student has retained. For this reason there is a "final examination" at the end of each block. The correct answers together with rather detailed solutions are supplied so that the student can better analyze his difficulties.

In summary, then, our typical format for a lesson unit is:

1. See a lecture.
2. Read some supplementary notes.
3. Read a portion of the text.
4. Do the exercises.

When assigned, these four steps almost always occur in the given order, but there are some assignments in which (1) and/or (2) are omitted, and there are a few places where the supplementary notes form the only reading assignment, especially in the treatment of topics not covered in the text.

Finally, I would like to acknowledge the very able assistance I have received from several people in the preparation of this self-study course. First and foremost, I am deeply indebted to John T. Fitch, the manager of our self-study development program. He discussed and helped me plan the lectures and written material -- unit by unit. He made suggestions, offered improvements, and, in many cases, put himself in the role of the student to help me "tone down" certain topics to the extent that they became (hopefully) more understandable to the student. In addition to all this, he was a friend and colleague, and this went a long way towards making a very difficult undertaking more palatable for me.

I would also like to thank Harold S. Mickley, the first director of CAES, whose idea it was to make "Calculus Revisited" available as a self-study course. Most of this present course reflects his ideas as to what constitutes a meaningful continuing education, calculus course. He, too, during his stay at the Center was a constant source of inspiration to me, and, in a certain sense, this course belongs more to him than to anyone else.

If you think that having to read all this material is difficult, imagine what it would have been like to have had to type the entire manuscript. Yet this job was accomplished, in an efficient and good-

natured manner, by our able staff of secretaries - in addition to maintaining all the other responsibilities of their office. In particular, I am grateful to Miss Elise Pelletier who worked on the manuscript from its very inception and to Miss Nina Ness for their help in the preparation of the manuscript.

I hope that your studying this course will be as rewarding and enjoyable as preparing the course has been to me. Good luck.

Study Guide

BLOCK 1:
AN INTRODUCTION TO FUNCTIONS OF A COMPLEX VARIABLE

Pretest

1. Express $\frac{3 + 5i}{7 + 9i}$ in the form a + bi, where a and b are real.

2. In terms of the Argand diagram, describe the locus of points defined by

$$\begin{cases} |z - (1 + i)| < 2 \\ |z - 2i| > \frac{3}{2} \end{cases}$$

3. (a) In terms of the Argand diagram, describe the set

   $S = \{z : z = \cos t + i \sin t,\ 0 \leqslant t \leqslant \pi\}$.

   (b) Describe f(S) if f is defined by $f(z) = z^2$.

4. If $f(z) = z^3$, write f in the form u(x,y) + iv(x,y) and show that u and v satisfy the Cauchy-Riemann conditions.

5. If u(x,y) = 3x - 2y + 5, how must v(x,y) be defined if u(x,y) + iv(x,y) is to be conformal?

6. Let f(z) be defined by $f(z) = \sum_{n=0}^{\infty} (-1)^n (n + 1) z^n$.

   (a) Find the radius of convergence for f(z).

   (b) Compute $f(\frac{i}{12})$ to three decimal places.

7. Under what condition is the integral

$$\int_{-1}^{2} \frac{dz}{z^2}$$

   well-defined, and what is its value in this case?

8. Suppose C is any simple closed curve which contains the points z = i and z = -i as interior points. Evaluate

$$\oint_C \frac{z\ dz}{z^2 + 1}.$$

Unit 1: A Prelude to the Complex Number System

1. Overview

The study of complex variables is not an easy one. Quite apart from other very valid reasons, the fact remains that the idea of an imaginary or non-real number causes most students to look at complex numbers as a rather abstract and completely non-practical system of arithmetic. Obviously, therefore, one aim of this block must be to change the image of the complex numbers in the eyes of the typical student.

What we have elected to do is to show that the concept of real versus non-real has occurred many times in the mathematics curriculum, at levels far more elementary than in the study of complex numbers. In particular, there were times when any numbers other than whole numbers were considered to be imaginary, or at least, unnatural. The ancient Greeks referred to negative numbers as being imaginary since they viewed numbers as lengths and it was impossible to visualize a length so short that even when three inches was added onto it, it was still invisible!

In this unit (which for students who already feel that they understand the subtlety that "non-real" numbers are indeed quite real, may be viewed as an optional unit), we shall try to develop the irrational numbers in a way that enhances all of the usual problems which are involved in attempts to make the complex numbers seem more real. In this way, it is hoped that our treatment of the complex numbers themselves, which begins in the next unit, will be that much easier for you to grasp.

2. Read Supplementary Notes, Chapter 1.

3. Exercises:

1.1.1(L)

Use the indirect proof, first in terms of the unique factorization theorem and then in terms of mimicking the ancient Greek's proof that $\sqrt{2}$ was irrational, to prove $\sqrt{5}$ is irrational.

1.1.2(L)

a. Assuming the facts that the sum (difference) of two rational numbers is a rational number and that $\sqrt{5}$ is irrational (from the previous exercise), use the indirect proof to show that $3 + \sqrt{5}$ is also an irrational number.

b. Find a polynomial equation with integer coefficients which has $\sqrt{5} + \sqrt{2}$ as a root. Then use the fundamental theorem of factorization of polynomials to prove that $\sqrt{5} + \sqrt{2}$ is irrational.

1.1.3

a. Use the unique factorization theorem together with the indirect proof to prove that $\sqrt[3]{7}$ is irrational.

b. Find an integral polynomial equation which has $\sqrt[3]{7} + \sqrt{5}$ as a root.

c. Use the result of (b) to conclude that $\sqrt[3]{7} + \sqrt{5}$ is irrational.

1.1.4

a. Find the common fraction (in lowest terms) which names the number represented by the decimal $0.324324\overline{324}\ldots$ (where the bar over 324 means that the cycle 324 repeats endlessly).

b. Suppose $n = 0.37377377\hat{7}\ldots$ (where ^ means that each time another 7 is added to the cycle). Explain why n cannot represent a rational number.

c. With n as in part (b), find another irrational number m such that $n + m$ is rational and is between $\frac{4}{5}$ and $\frac{9}{10}$.

1.1.5

With n as in Exercise 1.1.4, find two rational numbers L and U such that

(i) $L < n < U$

and

(ii) $U - L = 10^{-10}$.

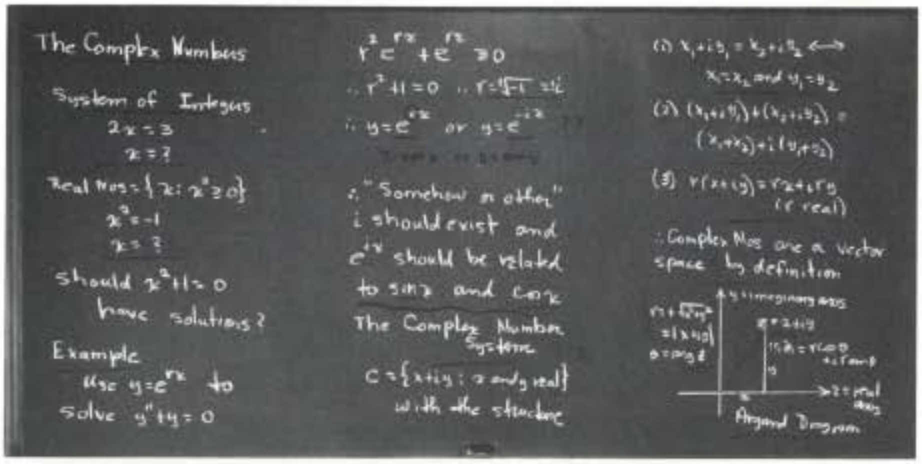
Unit 2: Complex Numbers from an Algebraic Point of View

1. Overview

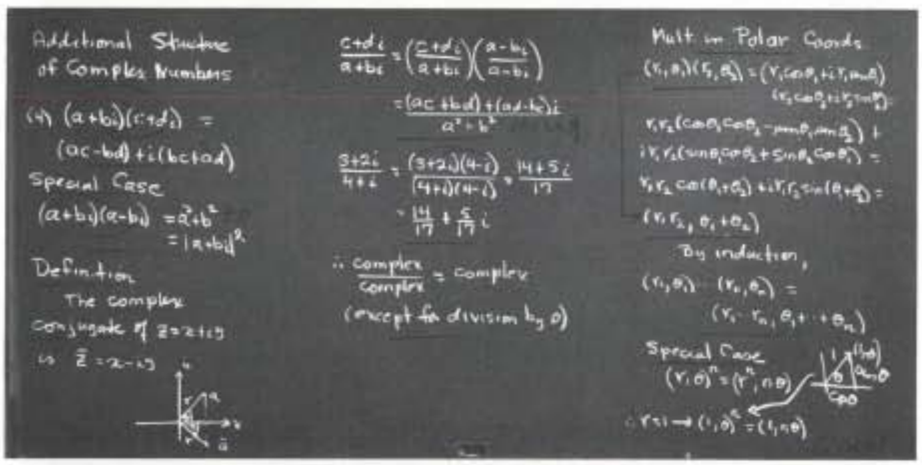
In the next two units we shall develop the complex numbers as being a natural outgrowth of the real numbers. In this unit we shall stress the algebraic development of the complex numbers, while in the next unit we shall study the same problem but from a geometric point of view. The student who is fairly well versed in the concept of complex numbers may find that it is feasible to study these two units as one long unit, but in general it is probably better to study the two units separately so that both the algebraic flavor and the geometric flavor of the complex numbers becomes as second-nature to you as possible.

The same lecture (1.010) covers both units and you may well wish to see this lecture more than once. The textbook material covered in these two units consists of Sections 19.1 and 19.2 of Thomas. Unit 2 supplements Section 19.1 and Unit 3 supplements Section 19.2.

2. Lecture 1.010



a.



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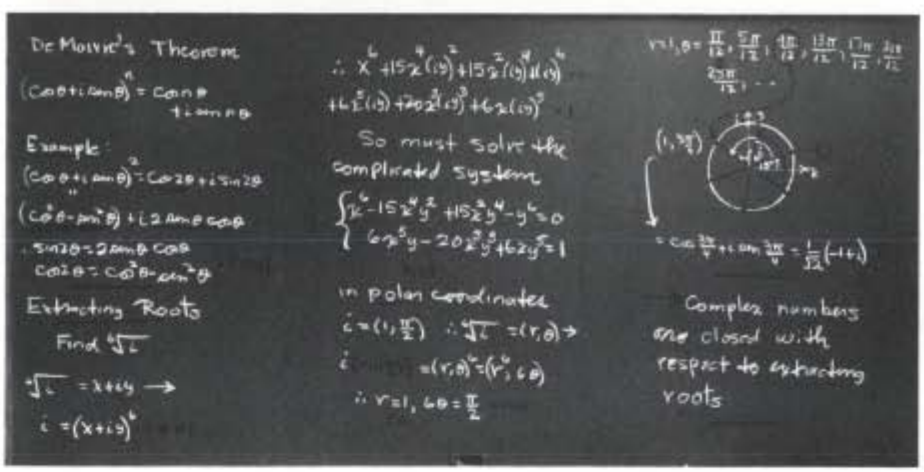

c.

3. Read Supplementary Notes, Chapter 2, Section A.

4. Do Exercise 1.2.1(L) [This is the quantitative counterpart of the reading in the supplementary notes, and thus except for the fact that it makes you get more involved by being written in the form of an exercise, this may be viewed as being the next section of supplementary notes.]

5. Read Thomas, Section 19.1 [The lecture, supplementary notes, and Exercise 1.2.1(L) should help make this section very easy to understand.]

6. Do the rest of the Exercises.

1.2.1(L)

a. Write i^{235} in the form a + bi, where a and b are real.

b. Express (3 + 5i) + (7 + 9i) in the form a + bi, where a and b are real.

c. Express (3 + 5i)(7 + 9i) in the form a + bi, where a and b are real.

d. Show that the real numbers are a subset of the numbers of the form a + bi where a and b are real.

e. Compute (a + bi)(a - bi).

f. Express $\frac{3 + 5i}{7 + 9i}$ in the form a + bi, where a and b are real.

1.2.2

a. Express $\frac{5 - 3i}{2 - 7i}$ in the form a + bi, where a and b are real.

b. Use the fact that division is the inverse of multiplication to check the result of part (a).

1.2.3(L)

Express $(1 + \sqrt{3}\, i)^3$ in the form a + bi, where a and b are real. [Now that we feel the point has been made, we shall henceforth refer to the "complex numbers" when we mean the set of numbers of the form a + bi, where a and b are real.]

1.2.4

Find all six roots of the polynomial equation $x^6 - 9x^3 + 8 = 0$.

1.2.5(L)

Suppose $r_1, \ldots,$ and r_n are the roots of the polynomial equation; $x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = 0$. Show how a_{n-1} is related to $r_1, \ldots, r_n$. In particular, explain why the sum of the roots of, for example, $x^4 - 1 = 0$ and $x^3 + 8 = 0$ is zero.

1.2.6(L)

If Z is the complex number a + bi, we define $\overline{Z}$ (called the complex conjugate of Z) to be the complex number a - bi.

a. Show that $Z + \overline{Z} = 2$ [real part of Z].

b. Suppose Z = a + bi; where a = 3 + 2i and b = 4 + 5i. Determine Z.

c. Prove that $Z = \overline{Z}$ if and only if Z is real.

d. Prove that $\overline{Z_1 + Z_2} = \overline{Z}_1 + \overline{Z}_2$.

e. Prove that $\overline{Z_1 Z_2} = \overline{Z}_1\, \overline{Z}_2$.

1.2.7

a. Prove that $\overline{(\overline{Z})} = Z$.

b. Prove that if $\overline{Z}_1 = \overline{Z}_2$ then $Z_1 = Z_2$.

1.2.8 (optional)

Use the results of the previous two exercises to prove that if Z is a root of the polynomial equation

$$a_n x^n + a_n x^{n-1} + \ldots + a_1 x + a_o = 0$$

where $a_o, a_1, \ldots,$ and a_n are all real numbers; then $\overline{Z}$ is also a root of the same equation. (This generalizes the result which is already well-known for quadratic equations.)

1.2.9 (optional)

a. If Z_1, Z_2 and Z_3 are any complex numbers prove that $Z_1(Z_2 + Z_3) = Z_1Z_2 + Z_1Z_3$.

b. If $Z \neq 0$, prove that $Z\left(\frac{1}{Z}\right) = 1$.

c. If $Z_1Z_2 = 0$ and $Z_1 \neq 0$ prove that $Z_2 = 0$.

d. Prove that for every complex number Z, $0Z = 0$. (This result is already known if Z is real.)

Unit 3: Complex Numbers From a Geometric Point of View

1. (Optional) Review Lecture 1.010.
2. Read Supplementary Notes, Chapter 2, Section B.
3. Read Thomas, Section 19.2.
4. Exercises:

1.3.1(L)

Use the Argand Diagram interpretation of the xy-plane to show that multiplying by i is equivalent to a 90° rotation of the plane in a counter-clockwise direction.

1.3.2

a. Describe the mapping of the xy-plane defined in terms of the Argand Diagram by saying that $a + bi$ is mapped into $(a + bi)(1 + i)$.

b. In particular, with respect to the mapping defined in part (a), what is the image of (a,b)?

c. Check your result by investigating the value of $(1 + i)^2$ and seeing what this means.

1.3.3

Use polar coordinates to determine $z_1z_2z_3$ where $z_1 = 1 + i$, $z_2 = 1 + i$, $z_3 = \frac{1}{2} + \frac{1}{2}\sqrt{3}i$.

1.3.4

Use DeMoivre's Theorem to find identities for $\sin 5\theta$ and $\cos 5\theta$ in terms of $\sin\theta$ and $\cos\theta$.

1.3.5(L)

a. Use the algebraic definition of absolute value to find the solution set of $|z - (1 + i)| = 2$.

b. Solve the same problem as in part (a) but now by using the geometric interpretation of absolute value.

c. Describe the locus of all points z in the Argand Diagram if $z \in \{z: |z - (1 + i)| < 2 \text{ and } |z - 2i| > \frac{3}{2}\}$.

1.3.6

a. Use algebraic methods to solve for the relationship between x and y if $z = x + iy$ and $|z - (1 + i)| = |z - (3 + 2i)|$.

b. Interpret the result of part (a) geometrically.

1.3.7(L)

Use polar coordinates to determine the <u>three</u> complex numbers z for which $z^3 = i$ (i.e., find $\sqrt[3]{i}$).

1.3.8

a. Find the eight eighth-roots of 1 (i.e., find all numbers z such that $z^8 = 1$).

b. Show, using polar coordinates, that if $z_2 = \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}i$, then all numbers z for which $z^8 = 1$ are integral powers of z_2.

1.3.9

Determine explicitly the members of the set

$\{z: z^3 = 1 + i\}$.

1.3.10 (Optional)

Use the results of Exercise 1.2.5(L) together with the Argand Diagram interpretation of the xy-plane to prove the following

1.3.10 continued

vector property:

Let P be a regular n-sided polygon inscribed in a circle of radius a. Then the sum of the vectors drawn from the center of the circle to the vertices of the polygon is zero.

1.3.11 (Optional)

So far we have not defined a product for vectors whereby the product of two planar vectors is a vector in the same plane. (The cross product was a vector perpendicular to the plane of the two vectors.) To remedy this situation, given $a\vec{i} + b\vec{j}$ and $c\vec{i} + d\vec{j}$, let us define $\otimes$ by

$$(a\vec{i} + b\vec{j}) \otimes (c\vec{i} + d\vec{j}) = (ac - bd)\vec{i} + (ad + bc)\vec{j}.$$

Use the Argand Diagram to prove that $\otimes$ is commutative, associative, and distributive; and also to describe $\otimes$ in geometrical terms.

Unit 4: Complex Functions of a Complex Variable

1. Overview

The main aim of this Block is to study the calculus of functions of a complex variable. Just as was the case when we studied real variables, our approach is first to discuss the number system, and then to apply the limit concept to these functions.

In our first three units we have developed the complex number system in some detail and you should now feel a bit more at home with the concept of this number system. In this unit, as our title implies, we shall discuss functions defined on the complex numbers; and the remainder of the Block will then be devoted to the topics usually identified with calculus.

You will notice that there is no lecture for this unit. The reason is that from a geometrical point of view, as we shall show in the exercises, the study of complex functions of a complex variable is equivalent to the real problem of describing mappings of the xy-plane into the uv-plane. The only difference, in terms of the language of the Argand diagram, is that the xy-plane becomes known as the z-plane while the uv-plane becomes known as the w-plane. Then in a way analogous to the notation of writing $y = f(x)$ in the study of real functions of a real variable, we write $w = f(z)$ when we are studying a complex functions of a complex variable.

2. Skim Thomas, Section 19.3. (After the exercises you may wish to re-read this section in greater detail, but our initial aim is for you to read just enough to get a quick overview of what this unit deals with. Then you should proceed directly to the exercises since our feeling is that the best way to learn this topic is in terms of working specific exercises. If there are still certain points bothering you after you have completed this unit, you may be helped by the lecture of the following unit which begins with a review of the concepts in this unit.)

3. Exercises

1.4.1(L)

a. In terms of the Argand diagram, discuss the set S if S is the subset of complex numbers defined by

$$S = \{z:z = \cos t + i \sin t,\ 0 \leq t \leq \pi\}.$$

b. Let C denote the complex numbers and suppose that $f:C \rightarrow C$ is defined by $f(z) = z^2$. With S as in part (a), describe the image of S with respect to f.

1.4.2

Let S be the region of the z-plane which consists of the unit circle centered at the origin between $\theta = 0°$ and $\theta = 90°$; let T be the line of slope 1 which passes through the origin; and lies in the first quadrant, and let $f:C \rightarrow C$ be defined by $f(z) = z^3$.

a. Use polar coordinates to find the image of S and T with respect to f.

b. Write f(z) in the form $u(x,y) + i\,v(x,y)$ where u and v are real-valued functions of the real variables x and y.

1.4.3

Find the real and imaginary parts of w as functions of x and y if $w = f(z)$ and

a. $f(z) = 2z$
b. $f(z) = \bar{z}$
c. $f(z) = |z|$
d. $f(z) = z^2 + 2z + i$
e. $f(z) = \frac{1}{z}$ $(z \neq 0)$

1.4.4(L)

Interpret each of the following geometrically.

a. $f(z) = \bar{z}$

b. $f(z) = z + C$ where C is a given complex number

c. $f(z) = Cz$ where C is a given complex number.

1.4.5

a. Describe, geometrically, the image of f if $f: C \to C$ is defined by $f(z) = \left(\frac{1 + i}{\sqrt{2}}\right) z + i$.

b. Describe f in the form

$$\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$$

1.4.6

S is the region of the z-plane consisting of the portion of the circle of radius 2 centered at the origin between $\theta = 0°$ and $\theta = 60°$. Describe the curve in the w-plane defined by $w = z^4 + 3 + 4i$ where $z \in S$.

1.4.7

Suppose $f(z) = z^3$, compute $\lim\limits_{z \to (1 + i)} f(z)$

1.4.8

Compute

$$\lim_{h \to 0} \left[\frac{(z + h)^2 - z^2}{h}\right]$$

where h is a complex variable.

The last two exercises are optional, but we hope that you will study them the same as if they had been required problems. The reason they are optional is that they have no direct bearing on the study of complex-valued functions of a complex variable. They are important, however, to explain why we do not spend too much time studying complex valued functions of a real variable or real valued functions of a complex variable. In other words, once the real numbers R and the complex numbers C each exist in their own right, there are four rather natural types of functions to study. Namely: (1) $f:R \to R$, (2) $f:R \to C$, (3) $f:C \to R$, and (4) $f:C \to C$. Case (1) was discussed in Part 1 of our course and Case (4) is the discussion of this Block. The optional exercises try to explain why we omit a special treatment of cases (2) and (3).

1.4.9

a. Let t denote a real variable. Compute $f'(t)$ if $f(t) = t + t^2 i$.

b. Find $f(t)$ if $f'(t) = t^2 + e^{3t} i$ and $f(0) = 1 + i$.

1.4.10

Suppose $y = f(z)$ [i.e., f is a real-valued function of a complex variable]. Show that if $f'(z)$ exists then $f'(z)$ must be identically equal to zero.