

Unit 3: Additional Comments on Dimension

3.3.1(L)

Actually in our attempt to introduce ideas so that you would work with them prior to your seeing them discussed in the lecture, it turns out that the present exercise could have been assigned in the previous unit. Since it often takes time for the basic concepts involved in the study of a vector space to be absorbed, our hope is that if you had trouble with the exercises in the previous unit, you may now do similar problems without too much trouble.

We have made three parts to this exercise to help you review the various levels of sophistication that one can invoke in studying subspaces. In part (a), we use a rather simple version of row-reduced matrices in order to find the dimension of our subspace. In part (b) we go one step further and inquire as to how  $x_4$  must be related to  $x_1, x_2,$  and  $x_3$  if  $(x_1, x_2, x_3, x_4)$  is to be an element of  $W$ . Finally, in part (c) we use the augmented matrix technique to show how one set of basis vectors of  $W$  is related to another set of basis vectors.

$$\begin{array}{l}
 \text{a.} \\
 \left[ \begin{array}{cccc} 1 & 1 & 3 & 4 \\ 2 & 3 & 7 & 9 \\ 3 & -2 & 4 & 7 \\ 4 & -5 & 3 & 7 \\ 4 & 5 & 14 & 9 \end{array} \right] \sim \left[ \begin{array}{cccc} 1 & 1 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & -5 & -5 & -5 \\ 0 & -9 & -9 & -9 \\ 0 & 1 & 2 & -7 \end{array} \right] \\
 \sim \left[ \begin{array}{cccc} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -8 \end{array} \right] \\
 \sim \left[ \begin{array}{cccc} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -8 \end{array} \right] \\
 \sim \left[ \begin{array}{cccc} 1 & 0 & 0 & 19 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & -8 \end{array} \right]
 \end{array}$$

(1)

## Solutions

## Block 3: Selected Topics in Linear Algebra

## Unit 3: Additional Comments on Dimension

## 3.3.1(L) continued

From (1) we see that  $W = S(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = S(\beta_1, \beta_2, \beta_3)$

where

$$\left. \begin{aligned} \beta_1 &= (1, 0, 0, 19) \\ \beta_2 &= (0, 1, 0, 9) \\ \beta_3 &= (0, 0, 1, -8) \end{aligned} \right\} . \quad (2)$$

We also know from the form of  $\beta_1, \beta_2,$  and  $\beta_3$  that  $\{\beta_1, \beta_2, \beta_3\}$  is linearly independent. Namely,

$$\begin{aligned} x_1\beta_1 + x_2\beta_2 + x_3\beta_3 &= \begin{pmatrix} x_1, 0, 0, 19x_1 \\ 0, x_2, 0, 9x_2 \\ 0, 0, x_3, -8x_3 \end{pmatrix} \\ &= \underline{\begin{pmatrix} x_1, x_2, x_3, 19x_1 + 9x_2 - 8x_3 \end{pmatrix}}. \end{aligned} \quad (3)$$

Hence, from (3),

$$\begin{aligned} x_1\beta_1 + x_2\beta_2 + x_3\beta_3 = 0 &\leftrightarrow (x_1, x_2, x_3, 19x_1 + 9x_2 - 8x_3) = (0, 0, 0, 0) \\ &\leftrightarrow x_1 = x_2 = x_3 = 0. \end{aligned}$$

Since  $\{\beta_1, \beta_2, \beta_3\}$  is a linearly independent set which spans  $W$ , we conclude that  $\dim W = 3$ . [Notice that even though  $5 > \dim V = 4$ , the five vectors  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  and  $\alpha_5$  do not span all of  $V$ .]

b. Using (2) we have that

$$\begin{aligned} (x_1, x_2, x_3, x_4) \in W &\leftrightarrow \\ (x_1, x_2, x_3, x_4) &= x_1\beta_1 + x_2\beta_2 + x_3\beta_3. \end{aligned}$$

Thus, by (3)

$$(x_1, x_2, x_3, x_4) \in W \leftrightarrow x_4 = 19x_1 + 9x_2 - 8x_3.$$

3.3.1(L) continued

Check:

$$\begin{array}{l}
 \alpha_1 = \\
 \alpha_2 = \\
 \alpha_3 = \\
 \alpha_4 = \\
 \alpha_5 =
 \end{array}
 \begin{array}{c}
 x_1 \quad x_2 \quad x_3 \quad x_4 \quad 19x_1 + 9x_2 - 8x_3 \\
 \hline
 \left[ \begin{array}{ccccc}
 1 & 1 & 3 & 4 & 19 + 9 - 24 = 4 \\
 2 & 3 & 7 & 9 & 38 + 27 - 56 = 9 \\
 3 & -2 & 4 & 7 & 57 - 18 - 32 = 7 \\
 4 & -5 & 3 & 7 & 76 - 45 - 24 = 7 \\
 4 & 5 & 14 & 9 & 76 + 45 - 112 = 9 .
 \end{array} \right.
 \end{array}$$

a.

$$\begin{array}{c}
 u_1 \quad u_2 \quad u_3 \quad u_4 \quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \\
 \hline
 \left[ \begin{array}{ccccccccc}
 1 & 1 & 3 & 4 & 1 & 0 & 0 & 0 & 0 \\
 2 & 3 & 7 & 9 & 0 & 1 & 0 & 0 & 0 \\
 3 & -2 & 4 & 7 & 0 & 0 & 1 & 0 & 0 \\
 4 & -5 & 3 & 7 & 0 & 0 & 0 & 1 & 0 \\
 4 & 5 & 14 & 9 & 0 & 0 & 0 & 0 & 1
 \end{array} \right]
 \end{array}$$

$$\left[ \begin{array}{ccccccccc}
 1 & 1 & 3 & 4 & 1 & 0 & 0 & 0 & 0 \\
 0 & 1 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & -13 & 5 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & -22 & 9 & 0 & 1 & 0 \\
 0 & 0 & 1 & -8 & -2 & -1 & 0 & 0 & 1
 \end{array} \right] \tag{4}$$

The third and fourth rows of (4) tell us that  $0 = -13\alpha_1 + 5\alpha_1 + \alpha_3$  and  $0 = -22\alpha_1 + 9\alpha_2 + \alpha_4$ . That is,  $\alpha_3$  and  $\alpha_4$  are redundant since

$$\alpha_3 = 13\alpha_1 - 5\alpha_2 \tag{5}$$

and

$$\alpha_4 = 22\alpha_1 - 9\alpha_2. \tag{6}$$

[Notice how elegantly our matrix technique reveals the way  $\alpha_3$  and  $\alpha_4$  are expressed as linear combinations of  $\alpha_1$  and  $\alpha_2$ .]

At any rate, then (4) is equivalent to

3.3.1(L) continued

$$\begin{bmatrix} 1 & 0 & 2 & 3 & 3 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -8 & -2 & -1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 19 & 7 & 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 9 & 0 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & -8 & -2 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{matrix} \quad (7)$$

"Decoding" (7) yields

$$\begin{aligned} \beta_1 &= (1, 0, 0, 19) = 7\alpha_1 + \alpha_2 - 2\alpha_5 \\ \beta_2 &= (0, 1, 0, 9) = 2\alpha_2 - \alpha_5 \\ \beta_3 &= (0, 0, 1, -8) = -2\alpha_1 - \alpha_2 + \alpha_5 \end{aligned} \quad (8)$$

If we invert (8) to find  $\alpha_1, \alpha_2,$  and  $\alpha_5$  as linear combinations of  $\beta_1, \beta_2,$  and  $\beta_3,$  we should obtain

$$\begin{aligned} \alpha_1 &= \beta_1 + \beta_2 + 3\beta_3 \\ \alpha_2 &= 2\beta_1 + 3\beta_2 + 7\beta_3 \\ \alpha_5 &= 4\beta_1 + 5\beta_2 + 14\beta_3 \end{aligned} \quad (9)$$

Equation (9) may be written by inspection. Namely, if  $(x_1, x_2, x_3, x_4) \in W,$  then  $(x_1, x_2, x_3, x_4) = x_1\beta_1 + x_2\beta_2 + x_3\beta_3.$  In particular,

$$\begin{aligned} (1, 1, 3, 4) &= \beta_1 + \beta_2 + 3\beta_3 \\ (2, 3, 7, 9) &= 2\beta_1 + 3\beta_2 + 7\beta_3 \\ (4, 5, 14, 9) &= 4\beta_1 + 5\beta_2 + 14\beta_3 \end{aligned}$$

Note:

Had we wished to imitate the construction given in the proof that every finite set of vectors contains a linearly independent subset which spans the same space; we could have worked with the vectors  $\alpha_1, \alpha_2, \alpha_3, \alpha_4,$  and  $\alpha_5$  one at a time in the given order. More specifically we know that  $\alpha_1$  is independent since  $\alpha_1 \neq 0.$  We then look at  $S(\alpha_1, \alpha_2)$  and we see that

$$\begin{bmatrix} 1 & 1 & 3 & 4 \\ 2 & 3 & 7 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 & 4 \\ 0 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 1 \end{bmatrix} .$$

3.3.1(L) continued

$$\begin{aligned}
 \text{Hence } \chi &= (x_1, x_2, x_3, x_4) \in S(\alpha_1, \alpha_2) \\
 &= (x_1, 0, 2x_1, 3x_1) + (0, x_2, x_2, x_2) \\
 &= (x_1, x_2, 2x_1 + x_2, 3x_1 + x_2) \quad (10)
 \end{aligned}$$

Since  $\alpha_3$  and  $\alpha_4$  obey (10) they belong to  $S(\alpha_1, \alpha_2)$ ; but since  $\alpha_5$  doesn't obey (10)  $\alpha_5 \notin S(\alpha_1, \alpha_2)$ . Hence,

$$S(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = S(\alpha_1, \alpha_2, \alpha_5).$$

There are, of course, other paths of investigation open to us but our hope is that by now you are beginning to feel at home with the basic structure of a vector space.

3.3.2

$$\begin{aligned}
 \text{a. } \begin{bmatrix} 5 & 2 & 7 \\ -3 & 4 & 1 \\ -1 & -2 & -3 \end{bmatrix} &\sim \begin{bmatrix} -1 & -2 & -3 \\ -3 & 4 & 1 \\ 5 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ -3 & 4 & 1 \\ 5 & 2 & 7 \end{bmatrix} \sim \\
 &\begin{bmatrix} 1 & 2 & 3 \\ 0 & 10 & 10 \\ 0 & -8 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \sim \\
 &\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}
 \end{aligned}$$

Hence,  $w = S(\alpha_1, \alpha_2, \alpha_3) = S(\beta_1, \beta_2)$ ; where  $\beta_1 = (1, 0, 1)$  and  $\beta_2 = (0, 1, 1)$ . Therefore,  $\dim W = 2$ .

$$\begin{aligned}
 \text{b. } \begin{array}{cccccc} \underline{u_1} & \underline{u_2} & \underline{u_3} & \underline{\alpha_1} & \underline{\alpha_2} & \underline{\alpha_3} \\ \begin{bmatrix} 5 & 2 & 7 \\ -3 & 4 & 1 \\ -1 & -2 & -3 \end{bmatrix} & & & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \sim & \begin{bmatrix} -1 & -2 & -3 & 0 & 0 & 1 \\ -3 & 4 & 1 & 0 & 1 & 0 \\ 5 & 2 & 7 & 1 & 0 & 0 \end{bmatrix} \\ & & & & \sim & \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & -1 \\ -3 & 4 & 1 & 0 & 1 & 0 \\ 5 & 2 & 7 & 1 & 0 & 0 \end{bmatrix}
 \end{array}
 \end{aligned}$$

3.3.2 continued

$$\sim \begin{bmatrix} \bar{1} & 2 & 3 & 0 & 0 & -\bar{1} \\ 0 & 10 & 10 & 0 & 1 & -3 \\ 0 & -8 & -8 & 1 & 0 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 20 & 40 & 60 & 0 & 0 & -20 \\ 0 & 40 & 40 & 0 & 4 & -12 \\ 0 & -40 & -40 & 5 & 0 & 25 \end{bmatrix}$$

$$\begin{array}{ccc|ccc} \underline{u_1} & \underline{u_2} & \underline{u_3} & \underline{\alpha_1} & \underline{\alpha_2} & \underline{\alpha_3} \\ \hline 20 & 0 & 20 & | & 0 & -4 & -8 \\ 0 & 40 & 40 & | & 0 & 4 & -12 \\ 0 & 0 & 0 & | & 5 & 4 & 13 \end{array} \quad (1)$$

The last row of (1) tells us that

$$5\alpha_1 + 4\alpha_2 + 13\alpha_3 = 0. \quad (2)$$

c. By (2),  $5c\alpha_1 + 4c\alpha_2 + 13c\alpha_3 = 0$ . Hence,

$$\begin{aligned} \alpha_1 &= \alpha_1 + 0 \\ &= \alpha_1 + 5c\alpha_1 + 4c\alpha_2 + 13c\alpha_3 \\ &= (1 + 5c)\alpha_1 + 4c\alpha_2 + 13c\alpha_3, \text{ for each choice of } c. \end{aligned}$$

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3.3.3(L)

Here we indicate explicitly the role of vector spaces in the study of linear homogeneous differential equations. Namely, we let

$$W = \{f: f''(x) - 4f(x) = 0\} \quad (1)$$

Since  $f''$  exists,  $f$  is at least continuous, so  $W$  is a subset of the space of continuous functions. But  $W$  is more than a subset. It is a subspace. For, if  $f \in W$  and  $g \in W$ , we have

$$f''(x) - 4f(x) = 0$$

and

$$g''(x) - 4g(x) = 0.$$

3.3.3(L) continued

Hence,  $f''(x) + g''(x) - 4[f(x) + g(x)] = 0$  or, letting  $h = f + g$ ;  
 $h''(x) - 4h(x) = 0$ ; so by (1)  $h = f + g \in W$

A similar treatment shows that  $f \in W \rightarrow cf \in W$ , so that  $W$  is itself a vector space.

More specifically, notice that in the previous Block we showed that  $W = \{c_1 e^{2x} + c_2 e^{-2x}; c_1, c_2 \in \mathbb{R}\}$ . Since  $\{e^{2x}, e^{-2x}\}$  is linearly independent, we have that  $W = S[e^{2x}, e^{-2x}]$  so that  $W$  is a 2-dimensional vector space.

Quite in general, the set of all solution of

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y' + a_0y = 0 \quad (2)$$

is an  $n$ -dimensional vector space.

It is crucial that the right side of (2) be 0, otherwise the sum of two solutions of (2) would not be a solution of (2), etc.

3.3.4(L)

$$a. \begin{bmatrix} \bar{1} & 1 & 2 & \bar{3} \\ 2 & 3 & 5 & 7 \\ 2 & 1 & 3 & 5 \end{bmatrix} \sim \begin{bmatrix} \bar{1} & 1 & 2 & \bar{3} \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} \bar{1} & 0 & 1 & \bar{2} \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

Hence,

$$S = S(\alpha_1, \alpha_2, \alpha_3) = S(\beta_1, \beta_2)$$

where

$$\begin{cases} \beta_1 = (1, 0, 1, 2) \\ \beta_2 = (0, 1, 1, 1) . \end{cases}$$

In particular, then

$$\begin{aligned} (x_1, x_2, x_3, x_4) \in S &\leftrightarrow \\ (x_1, x_2, x_3, x_4) &= x_1\beta_1 + x_2\beta_2 \\ &= (x_1, 0, x_1, 2x_1) + (0, x_2, x_2, x_2) \\ &= (x_1, x_2, x_1 + x_2, 2x_1 + x_2) . \end{aligned} \quad (1)$$

## Solutions

## Block 3: Selected Topics in Linear Algebra

## Unit 3: Additional Comments on Dimension

## 3.3.4(L) continued

In other words,

$$(x_1, x_2, x_3, x_4) \in S \leftrightarrow \begin{cases} x_3 = x_1 + x_2 \\ x_4 = 2x_1 + x_2 \end{cases} \quad (2)$$

$$\begin{aligned} \text{b. } \begin{bmatrix} \underline{1} & 2 & 2 & \underline{3} \\ 2 & 5 & 4 & 7 \\ 3 & 7 & 7 & \underline{8} \end{bmatrix} &\sim \begin{bmatrix} \underline{1} & 2 & 2 & \underline{3} \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \\ &\sim \begin{bmatrix} \underline{1} & 0 & 2 & \underline{1} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \\ &\sim \begin{bmatrix} \underline{1} & 0 & 0 & \underline{5} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}. \end{aligned}$$

Hence,

$$T = T(\alpha_4, \alpha_5, \alpha_6) = T(\beta_3, \beta_4, \beta_5) \text{ where } \begin{cases} \beta_3 = (1, 0, 0, 5) \\ \beta_4 = (0, 1, 0, 1) \\ \beta_5 = (0, 0, 1, -2) \end{cases}$$

Therefore,

$$\begin{aligned} (x_1, x_2, x_3, x_4) \in T &\leftrightarrow \\ (x_1, x_2, x_3, x_4) &= x_1\beta_3 + x_2\beta_4 + x_3\beta_5 \\ &= (x_1, 0, 0, 5x_1) + (0, x_2, 0, x_2) + (0, 0, x_3, -2x_3) \\ &= (x_1, x_2, x_3, 5x_1 + x_2 - 2x_3). \end{aligned} \quad (3)$$

Thus,

$$(x_1, x_2, x_3, x_4) \in S \leftrightarrow x_4 = 5x_1 + x_2 - 2x_3. \quad (4)$$

c. For  $(x_1, x_2, x_3, x_4)$  to belong to  $S$ , we must have that

$$\left. \begin{aligned} x_3 &= x_1 + x_2 \\ x_4 &= 2x_1 + x_2 \end{aligned} \right\}; \quad (2')$$

while for  $(x_1, x_2, x_3, x_4)$  to belong to  $T$  we must have that



3.3.4(L) continued

$$x_4 = 5x_1 + x_2 - 2x_3. \quad (4)$$

Hence for  $(x_1, x_2, x_3, x_4)$  to belong to both  $S$  and  $T$ , conditions (2') and (4) must hold simultaneously.

Replacing  $x_3$  and  $x_4$  in (4) by their values in (2), we obtain

$$2x_1 + x_2 = 5x_1 + x_2 - 2(x_1 + x_2)$$

or

$$2x_1 + x_2 = 5x_1 + x_2 - 2x_1 - 2x_2$$

or

$$2x_2 = x_1. \quad (5)$$

Using (5) in (2) we conclude that

$$x_3 = x_1 + x_2 = 2x_2 + x_2 = 3x_2 \quad (6)$$

and

$$x_4 = 2(2x_2) + x_2 = 5x_2. \quad (7)$$

From (5), (6), and (7) we see that

$$\begin{aligned} (x_1, x_2, x_3, x_4) \in S \cap T &\leftrightarrow \\ (x_1, x_2, x_3, x_4) &= (2x_2, x_2, 3x_2, 5x_2) \\ &= x_2(2, 1, 3, 5). \end{aligned}$$

Hence,  $\dim S \cap T = 1$  and  $S \cap T$  is spanned by  $(2, 1, 3, 5)$ .

d.  $S + T = \{s + t : s \in S, t \in T\}$   
 $= \{(x_1\beta_1 + x_2\beta_2) + (x_3\beta_3 + x_4\beta_4 + x_5\beta_5) : x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}\}$   
 $= V(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)$ .

Using row-reduction we obtain

## Solutions

## Block 3: Selected Topics in Linear Algebra

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## 3.3.4(L) continued

$$\begin{bmatrix} \underline{1} & 0 & 1 & \underline{2} \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \underline{-2} \end{bmatrix} \sim \begin{bmatrix} \underline{1} & 0 & 1 & \underline{2} \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \underline{-2} \end{bmatrix} \sim \begin{bmatrix} \underline{1} & 0 & 1 & \underline{2} \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & \underline{-2} \end{bmatrix} \sim$$

$$\begin{bmatrix} \underline{1} & 0 & 0 & \underline{5} \\ 0 & 1 & 0 & 4 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & \underline{-2} \end{bmatrix} \sim \begin{bmatrix} \underline{1} & 0 & 0 & \underline{5} \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \underline{1} \end{bmatrix} \sim \begin{bmatrix} \underline{1} & 0 & 0 & \underline{0} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \underline{0} \end{bmatrix} .$$

Hence,

$$V(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5) = V[u_1, u_2, u_3, u_4].$$

In particular, then,

$$\dim(S + T) = 4.$$

$$\begin{aligned} \text{e. } \dim S &= 2 \\ \dim T &= 3 \\ \dim S \cap T &= 1 \\ \dim(S + T) &= 4. \end{aligned}$$

Hence, at least from this example, it appears that

$$\dim(S + T) = \dim S + \dim T - \dim S \cap T. \quad (8)$$

In this sense, we may be reminded of a similar rule for finite sets; namely

$$N(A \cup B) = N(A) + N(B) - N(A \cap B).$$

Equation (8) can be shown to hold in general (see Exercise 3.3.6). In particular, if  $\dim S \cap T = 0$  (i.e.,  $S \cap T = \{0\}$ ), then  $\dim(S + T) = \dim S + \dim T$ .

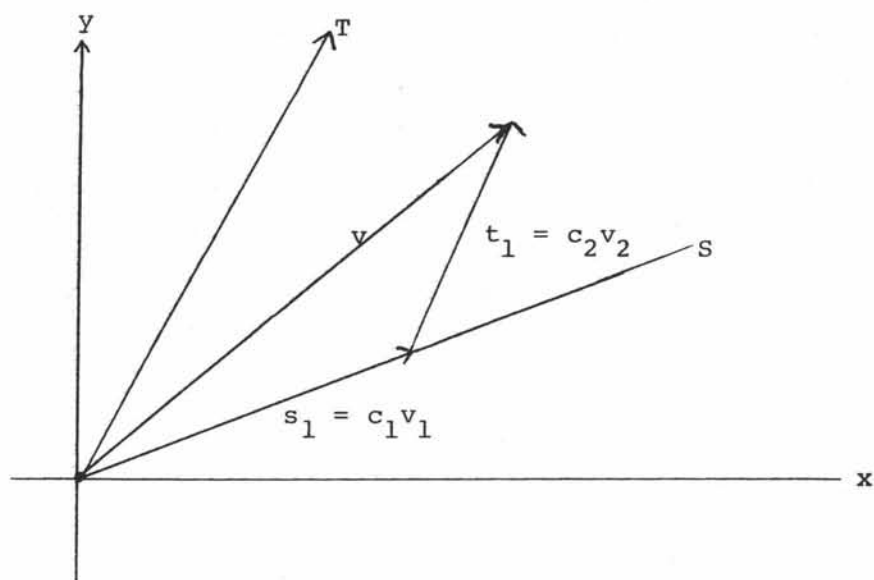
In the event that  $S \cap T = \{0\}$ ,  $S + T$  is written  $S \oplus T$  and is called the direct sum of  $S$  and  $T$ .

3.3.4(L) continued

A Geometric Interpretation of Linear Sums

Consider the case of two distinct lines in the plane, each of which passes through the origin. If we let  $v_1$  denote a non-zero vector on the first line and  $v_2$  a non-zero vector on the second line, then the first line may be viewed as the space spanned by  $v_1$  while the second line may be viewed as the space spanned by  $v_2$ . Let us use  $S = S[v_1]$  to denote the first line and  $T = T[v_2]$  to denote the second line. Then the plane itself is the linear sum of  $S$  and  $T$ ,  $S + T$ . That is, every vector in the plane may be written as a linear combination of  $v_1$  and  $v_2$ . Moreover, in this particular example, we may say that the plane is the direct sum of  $S$  and  $T$ , that is,  $S \oplus T$  since  $S \cap T = \{0\}$ . In other words, not only is each vector in the plane a sum of a vector in  $S$  and a vector in  $T$  but this sum can be represented in one and only one way. That is,  $s_1 + t_1 = s_2 + t_2$  if and only if  $s_1 = s_2$  and  $t_1 = t_2$ . [To prove this last assertion, notice that  $s_1 + t_1 = s_2 + t_2$  implies that  $s_1 - s_2$  is equal to  $t_2 - t_1$ . Since  $t_2 - t_1$  belongs to  $T$  so does  $s_1 - s_2$ ; but  $s_1 - s_2$  also belongs to  $S$ . Hence,  $s_1 - s_2 \in S \cap T$ , and since  $S \cap T = \{0\}$ , it follows that  $s_1 - s_2 = 0$ , or  $s_1 = s_2$ . A similar argument in which we reverse the roles of  $S$  and  $T$  establishes that  $t_1 = t_2$ .]

In terms of a picture:

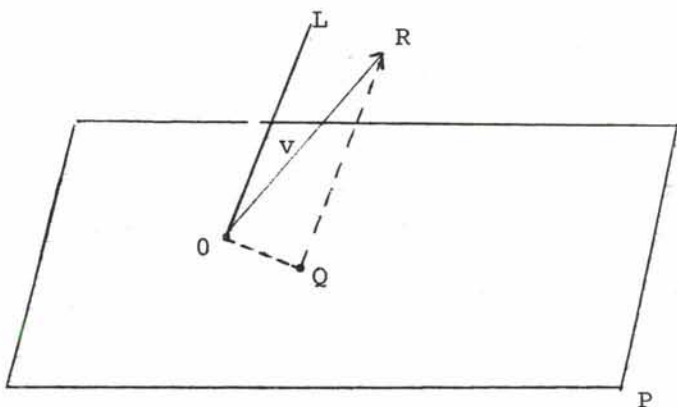


3.3.4(L) continued

In 3-dimensional space, we may view as an example, the case of two non-parallel planes  $P_1$  and  $P_2$ . Both  $P_1$  and  $P_2$  are 2-dimensional and their intersection,  $P_1 \cap P_2$  being a line is 1-dimensional. Moreover, every vector in 3-space may be written as the sum of a vector in  $P_1$  and a vector in  $P_2$ . Now, however, the same vector has infinitely many representations in the form  $p_1 + p_2$  where  $p_1 \in P_1$  and  $p_2 \in P_2$ . Thus, in this example  $E^3 = P_1 + P_2$ , but since  $P_1 \cap P_2 \neq \{0\}$ , we do not write  $E^3 = P_1 \oplus P_2$ .

Notice that in this example,  $\dim(P_1 + P_2) = 3$  and  $\dim P_1 + \dim P_2 - \dim P_1 \cap P_2 = 2 + 2 - 1 = 3$ .

If we now consider the plane  $P$  and a line  $L$  not parallel to  $P$ , the  $P \cap L = \{0\}$  \*. Now, notice that each vector in 3-space can be written in one and only one way in the form  $p + l$  where  $p \in P$  and  $l \in L$ . Pictorially:



1.  $v = \vec{OR}$
2. At  $R$  we draw a line parallel to  $L$  meeting the plane  $P$  at the point  $Q$ .
3.  $v = \vec{OQ} + \vec{QR}$ , but  $\vec{OQ} \in P$  and  $\vec{QR} \in L$ . Recall that as a space,  $L$  remains the same if it is displaced parallel to itself.

Thus, in this case

$$E^3 = P \oplus L.$$

\*As usual in linear algebra we assume that all lines pass through the origin. Thus, the intersection of  $P$  and  $L$ , which without this restriction could be any point, must be the origin.

3.3.5

---

$$\begin{aligned}
 \text{a. } & \begin{bmatrix} 1 & 1 & 2 & 3 & 3 \\ 2 & 3 & 4 & 5 & 7 \\ 3 & 4 & 7 & 8 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 3 & 3 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 & -1 \end{bmatrix} \\
 & \sim \begin{bmatrix} 1 & 0 & 2 & 4 & 2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -2 \end{bmatrix} \\
 & \sim \begin{bmatrix} 1 & 0 & 0 & 4 & 6 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -2 \end{bmatrix} .
 \end{aligned}$$

Hence,

$$\begin{aligned}
 S &= \{x_1(1,0,0,4,6) + x_2(0,1,0,-1,1) + x_3(0,0,1,0,-2)\} \\
 &= \{(x_1, x_2, x_3, 4x_1 - x_2, 6x_1 + x_2 - 2x_3)\} .
 \end{aligned}$$

That is

$$(x_1, x_2, x_3, x_4, x_5) \in S \leftrightarrow \begin{cases} x_4 = 4x_1 - x_2 \\ x_5 = 6x_1 + x_2 - 2x_3 \end{cases} \quad (1)$$

$\dim S = 3$ .

$$\begin{aligned}
 \text{b. } & \begin{bmatrix} 1 & 1 & 1 & 3 & 5 \\ 1 & 2 & 3 & 2 & 2 \\ 2 & 3 & 3 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 3 & 5 \\ 0 & 1 & 2 & -1 & -3 \\ 0 & 1 & 1 & 1 & -2 \end{bmatrix} \\
 & \sim \begin{bmatrix} 1 & 0 & -1 & 4 & 8 \\ 0 & 1 & 2 & -1 & -3 \\ 0 & 0 & -1 & 2 & 1 \end{bmatrix} \\
 & \sim \begin{bmatrix} 1 & 0 & 0 & 2 & 7 \\ 0 & 1 & 0 & 3 & -1 \\ 0 & 0 & -1 & 2 & 1 \end{bmatrix} \\
 & \sim \begin{bmatrix} 1 & 0 & 0 & 2 & 7 \\ 0 & 1 & 0 & 3 & -1 \\ 0 & 0 & 1 & -2 & -1 \end{bmatrix} .
 \end{aligned}$$

3.3.5 continued

Hence,

$$T = \{x_1(1,0,0,2,7) + x_2(0,1,0,3,-1) + x_3(0,0,1,-2,-1)\}$$

$$= \{(x_1, x_2, x_3, 2x_1 + 3x_2 - 2x_3, 7x_1 - x_2 - x_3)\}$$

That is,

$$(x_1, x_2, x_3, x_4, x_5) \in T \leftrightarrow \begin{cases} x_4 = 2x_1 + 3x_2 - 2x_3 \\ x_5 = 7x_1 - x_2 - x_3 \end{cases} \quad (2)$$

$\dim T = 3$ .

- c. For  $(x_1, x_2, x_3, x_4, x_5)$  to belong to  $S \cap T$ , equation (1) and equation (2) must both be satisfied.

In particular, equating the two values for  $x_4$  and the two values for  $x_5$ , we obtain

$$\left. \begin{aligned} 4x_1 - x_2 &= 2x_1 + 3x_2 - 2x_3 \\ \text{and} \\ 6x_1 + x_2 - 2x_3 &= 7x_1 - x_2 - x_3 \end{aligned} \right\} \quad (3)$$

Collecting like terms and simplifying, we see that equation (3) may be rewritten as

$$\left. \begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_1 - 2x_2 + x_3 &= 0 \end{aligned} \right\} \quad (3')$$

Thus, the equations in (3) are dependent and are equivalent to the single restriction that  $x_1 - 2x_2 + x_3 = 0$ . If we wish to think of  $x_1$  and  $x_2$  as being arbitrary, then (3') tells us that

$$x_3 = 2x_2 - x_1. \quad (4)$$

We may now use (4) with either (1) or (2) to determine how  $x_4$  and  $x_5$  are expressed in terms of  $x_1$  and  $x_2$ . Choosing (1) because it is simpler, we see that

$$x_4 = 4x_1 - x_2 \quad (5)$$

3.3.5 continued

and

$$x_5 = 6x_1 + x_2 - 2(2x_2 - x_1)$$

or

$$x_5 = 8x_1 - 3x_2. \quad (6)$$

From (4), (5), and (6) we see that  $(x_1, x_2, x_3, x_4, x_5) \in S \cap T$  if and only if

$$(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, 2x_2 - x_1, 4x_1 - x_2, 8x_1 - 3x_2) \quad (7)$$

Thus,  $S \cap T$  has "2 degrees of freedom" so that  $\dim S \cap T = 2$ . More formally, we may find a row-reduced basis for  $S \cap T$  by computing the right side of (7), once with  $x_1 = 1, x_2 = 0$  and once with  $x_1 = 0, x_2 = 1$ . We then obtain that  $\beta_1 = (1, 0, -1, 4, 8)$  and  $\beta_2 = (0, 1, 2, -1, -3)$  form a basis for  $S \cap T$ .

d.  $S + T = \{s + t : s \in S, t \in T\}$ .

But  $S$  is spanned by  $\alpha_1, \alpha_2$  and  $\alpha_3$  while  $T$  is spanned by  $\gamma_1, \gamma_2$ , and  $\gamma_3$ . Hence, since each  $s \in S$  is a linear combination of  $\alpha_1, \alpha_2, \alpha_3$  and each  $t \in T$  is a linear combination of  $\gamma_1, \gamma_2$ , and  $\gamma_3$ , it follows that each  $s + t \in S + T$  is a linear combination of  $\alpha_1, \alpha_2, \alpha_3, \gamma_1, \gamma_2$  and  $\gamma_3$ . To find the space spanned by these six vectors, we need only row-reduce the matrix

$$\begin{bmatrix} \bar{1} & 1 & 2 & 3 & \bar{3} \\ 2 & 3 & 4 & 5 & 7 \\ 3 & 4 & 7 & 8 & 8 \\ 1 & 1 & 1 & 3 & 5 \\ 1 & 2 & 3 & 2 & 2 \\ 2 & 3 & 3 & 7 & 8 \end{bmatrix}. \quad (8)$$

We could row-reduce (8) "from scratch" but if we want to take advantage of our work done in parts (a) and (b), we already know that

$$\begin{bmatrix} \bar{1} & 1 & 2 & 3 & \bar{3} \\ 2 & 3 & 4 & 5 & 7 \\ 3 & 4 & 7 & 8 & 8 \end{bmatrix} \sim \begin{bmatrix} \bar{1} & 0 & 0 & 4 & \bar{6} \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -2 \end{bmatrix}$$

3.3.5 continued

and

$$\begin{bmatrix} 1 & 1 & 1 & 3 & 5 \\ 1 & 2 & 3 & 2 & 2 \\ 2 & 3 & 3 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 & 7 \\ 0 & 1 & 0 & 3 & -1 \\ 0 & 0 & 1 & -2 & -1 \end{bmatrix}.$$

Hence,

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 3 \\ 2 & 3 & 4 & 5 & 7 \\ 3 & 4 & 7 & 8 & 8 \\ 1 & 1 & 1 & 3 & 5 \\ 1 & 2 & 3 & 2 & 2 \\ 2 & 3 & 3 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 4 & 6 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & 2 & 7 \\ 0 & 1 & 0 & 3 & -1 \\ 0 & 0 & 1 & -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 4 & 6 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & 3 & -1 \\ 0 & 0 & 1 & -2 & -1 \end{bmatrix} \\ \sim \begin{bmatrix} 1 & 0 & 0 & 4 & 6 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 4 & -2 \\ 0 & 0 & 1 & -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 4 & 6 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & -2 & 1 \end{bmatrix} \quad (9)$$

The last two rows of (9) may be discarded since they all name the same space that spanned by  $(0,0,0,-2,1)$  but even if we don't notice this in the next steps in the row-reduction, take care of it.

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 3 \\ 2 & 3 & 4 & 5 & 7 \\ 3 & 4 & 7 & 8 & 8 \\ 1 & 1 & 1 & 3 & 5 \\ 1 & 2 & 3 & 2 & 2 \\ 2 & 3 & 3 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 4 & 6 \\ 0 & 4 & 0 & -4 & 4 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & -4 & 2 \\ 0 & 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & -4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 8 \\ 0 & 4 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & -4 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 8 \\ 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix}. \quad (10)$$



3.3.5 continued

Since (10) is row-reduced, we see that  $\dim(S + T) = 4$  and in fact  $S + T = \text{span}\{\delta_1, \delta_2, \delta_3, \delta_4\}$ , where  $\delta_1 = (1, 0, 0, 0, 8)$ ,  $\delta_2 = (0, 1, 0, 0, \frac{1}{2})$ ,  $\delta_3 = (0, 0, 1, 0, -2)$  and  $\delta_4 = (0, 0, 0, 1, -\frac{1}{2})$ .

Hence,

$$\begin{aligned}(x_1, x_2, x_3, x_4, x_5) \in S + T &\leftrightarrow \\(x_1, x_2, x_3, x_4, x_5) &= x_1\delta_1 + x_2\delta_2 + x_3\delta_3 + x_4\delta_4 + x_5\delta_5 \\ &= (x_1, x_2, x_3, x_4, 8x_1 + \frac{1}{2}x_2 - 2x_3 - \frac{1}{2}x_4)\end{aligned}$$

that is,

$$\begin{aligned}(x_1, x_2, x_3, x_4, x_5) \in S + T &\leftrightarrow x_5 = 8x_1 + \frac{1}{2}x_2 - 2x_3 - \frac{1}{2}x_4 \\ &\leftrightarrow 16x_1 + x_2 - 4x_3 - x_4 - 2x_5 = 0.\end{aligned}$$

e. From parts (a), (b), (c), and (d) we have

$$\begin{aligned}\dim S &= 3 \\ \dim T &= 3 \\ \dim S \cap T &= 2 \\ \dim(S + T) &= 4\end{aligned}\tag{11}$$

Since  $4 = 3 + 3 - 2$ , it follows from (11) that  $\dim(S + T) = \dim S + \dim T - \dim S \cap T$ .

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3.3.6 (optional)

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Our main aim here is to prove that for all finite dimensional subspaces,  $S$  and  $T$ , of  $V$  that  $\dim(S + T) = \dim S + \dim T - \dim S \cap T$ .

Since the formal proof is rather abstract, we shall carry out the actual steps in terms of the concrete example worked out in the previous exercise. The gist is as follows:

Step 1:

In part (c) of the previous exercise, we obtained that  $\beta_1 = (1, 0, -1, 4, 8)$  and  $\beta_2 = (0, 1, 2, -1, -3)$  was a basis for  $S \cap T$ . Schematically,

3.3.6 continued

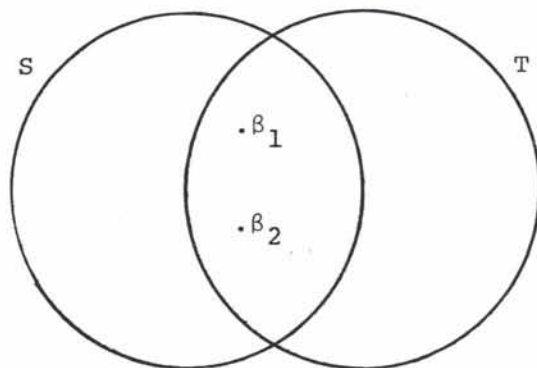


Figure 1

Step 2:

Since  $\{\alpha_1, \alpha_2, \alpha_3\}$  are a basis for S at least one of the  $\alpha$ 's cannot lie in  $S \cap T$ . Namely, since  $\{\alpha_1, \alpha_2, \alpha_3\}$  is linearly independent,  $\{\alpha_1, \alpha_2, \alpha_3\} \subset S \cap T$  implies that  $\dim S \cap T \geq 3$ , contrary to the fact that  $\dim S \cap T = 2$ .

In our particular case, we may check  $\alpha_1, \alpha_2$ , and  $\alpha_3$  explicitly for membership in  $S \cap T$ . Specifically,

$$\alpha_1 = (1, 1, 2, 3, 3) \in S \cap T \leftrightarrow \alpha_1 = \beta_1 + \beta_2$$

but  $\beta_1 + \beta_2 = (1, 1, 1, 3, 5)$ , hence  $\alpha_1 \notin S \cap T$  [but since  $\gamma_1 = (1, 1, 1, 3, 5)$ ,  $\gamma_1 \in S \cap T$ ; a fact we shall use later].  $\alpha_2 = (2, 3, 4, 5, 7) \leftrightarrow \alpha_2 = 2\beta_1 + 3\beta_2$ , but  $2\beta_1 + 3\beta_2 = (2, 0, -2, 8, 16) + (0, 3, 6, -3, -9) = (2, 3, 4, 5, 7)$ ; hence  $\alpha_2 \in S \cap T$ .

Finally,  $\alpha_3 = (3, 4, 7, 8, 8) \in S \cap T \leftrightarrow \alpha_3 = 3\beta_1 + 4\beta_2$  but  $3\beta_1 + 4\beta_2 = (3, 0, -3, 12, 24) + (0, 4, 8, -4, -12) = (3, 4, 5, 8, 12)$ ; hence  $\alpha_3 \notin S \cap T$ .

Thus, our picture now becomes

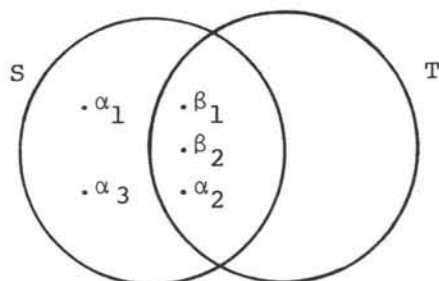


Figure 2

3.3.6 continued

Step 3:

Since  $\{\beta_1, \beta_2\}$  is already a basis for  $S \cap T$ , we may delete  $\alpha_2$  from the spanning set  $\{\alpha_2, \beta_1, \beta_2\}$  for  $S \cap T$ . Moreover, since  $\alpha_1 \notin S \cap T$  and  $S \cap T = [\beta_1, \beta_2]$ , then  $\{\beta_1, \beta_2, \alpha_1\}$  is linearly independent. [Namely, if not,  $\alpha_1$  would have to be a linear combination of  $\beta_1$  and  $\beta_2$  and this would mean  $\alpha_1 \in S \cap T$ , contrary to fact.] Thus, since  $\dim S = 3$  and  $\{\alpha_1, \beta_1, \beta_2\}$  is a linearly independent subset of  $S$ , we may conclude that  $\{\alpha_1, \beta_1, \beta_2\}$  is a basis for  $S$ . Again pictorially,

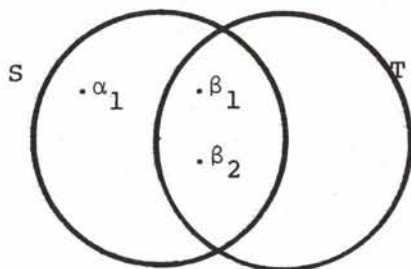


Figure 3

Note:

Although  $\alpha_3$  is now omitted from our diagram, all we mean is that the vectors in Figure 3 are a basis for  $S$ . We could just as easily conclude that since  $\alpha_3 \notin S \cap T$ ,  $\{\alpha_3, \beta_1, \beta_2\}$  is a basis for  $S$  in which case our picture would have been

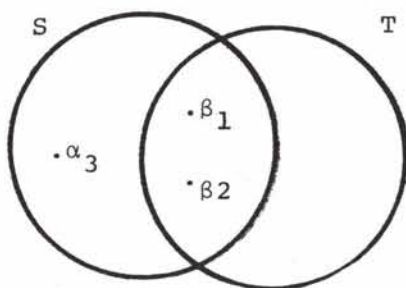


Figure 4

where now  $\alpha_1$  is redundant since  $\{\alpha_3, \beta_1, \beta_2\}$  is a basis for  $S$ .

The key point is that whether we refer to Figure 3 or Figure 4, we have produced a basis for  $S$  by augmenting our basis for  $S \cap T$ . In the more general (abstract) case, this is the key step.

3.3.6 continued

Namely, we form a basis for  $S$  by starting with a basis for  $S \cap T$  and extending it (for example, by the row-reduced matrix technique).

Step 4:

We now repeat steps 2 and 3, only now in reference to  $T$  rather than  $S$ . We know that since  $\{\gamma_1, \gamma_2, \gamma_3\}$  is linearly independent and  $\dim S \cap T = 2$ , then  $\{\gamma_1, \gamma_2, \gamma_3\} \not\subset S \cap T$ . We saw in step 2 that  $\gamma_1 \in S \cap T$ , so now we need only check  $\gamma_2$  and  $\gamma_3$ . Well,

$$\gamma_2 = (1, 2, 3, 2, 2) \in S \cap T \leftrightarrow \gamma_2 = \beta_1 + 2\beta_2$$

but  $\beta_1 + 2\beta_2 = (1, 0, -1, 4, 8) + (0, 2, 4, -2, -6) = (1, 2, 3, 2, 2)$ ;  
hence  $\gamma_2 \in S \cap T$  (so by default, it must be that  $\gamma_3 \notin S \cap T$ , but let's check just to be sure). Finally,

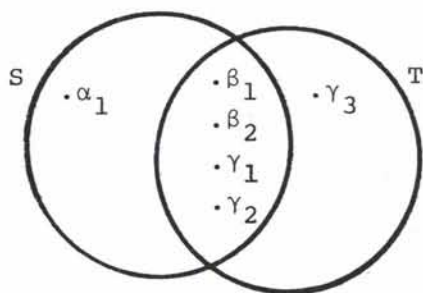
$$\gamma_2 = (2, 3, 3, 7, 8) \in S \cap T \leftrightarrow \gamma_3 = 2\beta_1 + 3\beta_2;$$

but

$$2\beta_1 + 3\beta_2 = (2, 3, 4, 5, 7)$$

[from step 2]; hence,  $\gamma_3 \notin S \cap T$ .

Pictorially,



or since  $\gamma_1$  and  $\gamma_2$  are redundant for spanning  $S \cap T$ , we have

3.3.6 continued

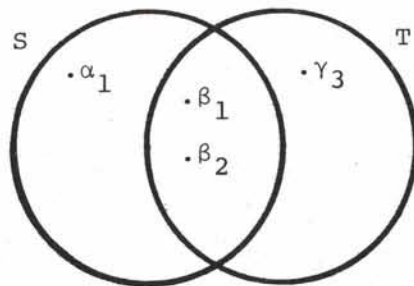


Figure 5

where Figure 5 shows how  $\{\beta_1, \beta_2\}$  is augmented into a basis for T.

The key point is that none of the  $\alpha$ 's ( $\gamma$ 's) can be expressed in terms of the  $\gamma$ 's ( $\alpha$ 's) since the  $\alpha$ 's belong to  $S \cap T$  while the  $\gamma$ 's belong to the  $S \cap T$ . (I.e., if  $\gamma_3 = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3$ , then  $\gamma_3 \in S$ , contrary to the fact that  $\gamma_3 \in T$  but  $\gamma_3 \notin S \cap T$ ).

From Figure 5 we begin to see where the formula

$$\dim(S + T) = \dim S + \dim T - \dim S \cap T$$

comes from.

Clearly,  $S + T = [\alpha_1, \beta_1, \beta_2, \gamma_3]$ , so  $\dim(S + T) = 4$  while

$$S \cap T = [\beta_1, \beta_2]$$

$$S = [\beta_1, \beta_2, \alpha_1]$$

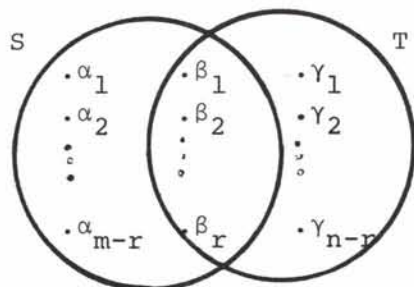
$$T = [\beta_1, \beta_2, \gamma_3]$$

$\beta_1, \beta_2$  and  $\beta_1, \beta_2$  are counted twice, once as being part of the basis for S and once as being part of the basis for T.

Note:

Figure 5 generalizes as follows. Suppose  $\dim S \cap T = r$ ,  $\dim S = m$  and  $\dim T = n$ . Let  $S \cap T = [\beta_1, \dots, \beta_r]$ . Then augment  $\{\beta_1, \dots, \beta_r\}$  by  $\alpha_1, \dots, \alpha_{m-r}$  to form a basis for S; and by  $\gamma_1, \dots, \gamma_{n-r}$  to form a basis for T. Thus,

3.3.6 continued



$$S + T = [\alpha_1, \dots, \alpha_{m-r}, \beta_1, \dots, \beta_r, \gamma_1, \dots, \gamma_{n-r}]$$

Hence

$$\begin{aligned} \dim(S + T) &= (m - r) + r + (n - r) \\ &= m + n - r \\ &= \dim S + \dim T - \dim S \cap T. \end{aligned}$$

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