

Unit 4: Linear Transformations

1. Overview

We have already seen many instances in our course where the concept of a linear function was most crucial. It turns out that the general concept of a linear transformation is best handled in terms of viewing them as special mappings of vector spaces into vector spaces. Thus, the aim of this unit is to show how this study is handled, and it is our hope that seeing the general structure will make it clear as to what common properties are shared by all linear transformations.

Study Guide
 Block 3: Selected Topics in Linear Algebra
 Unit 4: Linear Transformations

2. Lecture 3.040

Linear Transformations

$f: V \rightarrow W$ is called a lin. trans. \leftrightarrow

$$f(c_1 v_1 + c_2 v_2) = c_1 f(v_1) + c_2 f(v_2)$$

all $v_1, v_2 \in V$

Some Properties

- $f(0) = 0$, since $f(0) = f(0+0) = f(0)+f(0) \Rightarrow f(0) = 0$
- $f(v_1) = f(v_2) = 0 \rightarrow f(c_1 v_1 + c_2 v_2) = 0$

i.e., $f(c_1 v_1 + c_2 v_2) = c_1 f(v_1) + c_2 f(v_2) = 0$

i. Def: $N = \{v \in V : f(v) = 0\}$ is called the null space of f

- N determines Image of f
 $f(\alpha) = f(\beta) \leftrightarrow \alpha = \beta + \eta, \eta \in N$
 i.e., $f(\alpha) = f(\beta) \leftrightarrow f(\alpha) - f(\beta) = 0 \leftrightarrow f(\alpha - \beta) = 0$
 $[f \text{ is } 1-1 \leftrightarrow N = \{0\}]$

Example #1
 $D(f) = f'$
 $D(f) = 0 \leftrightarrow f = \text{constant}$
 $\therefore N = \{\text{constants}\}$
 $D(f) = D(g) \leftrightarrow f = g + C$

Example #2
 $L(g) = f(x)$
 $N = \text{sol. set of } L(g) = 0$
 $y_g = y_h + y_p$
 $L(y_h) = 0 \quad L(y_p) = f(x)$

a.

Example #3

$$u = 2 + y \quad v = 2u$$

$$v = 2(2 + y) = 4 + 2y$$

$$f(x, y) = (2 + y, 2(2 + y)) = (2 + y, 4 + 2y)$$

$$= 0 \leftrightarrow 2 + y = 0 \text{ and } 4 + 2y = 0$$

$\therefore N = \{(x, y) : 2 + y = 0\}$

Example #4

 $V = [u_1, u_2], f: V \rightarrow V$

$$f(u_1) = a_{11}u_1 + a_{12}u_2$$

$$f(u_2) = a_{21}u_1 + a_{22}u_2$$

$$v = x_1 u_1 + x_2 u_2$$

$$f(v) = x_1 f(u_1) + x_2 f(u_2) = (x_1 a_{11} + x_2 a_{21}, x_1 a_{12} + x_2 a_{22})$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 a_{11} + x_2 a_{21} \\ x_1 a_{12} + x_2 a_{22} \end{bmatrix}$$

Example #5

 $f: V \rightarrow V, V = [u_1, u_2]$

$$f(u_1) = 3u_1 + 4u_2$$

$$f(u_2) = 5u_1 + 7u_2$$

$$v = 2u_1 + u_2$$

$$f(v) = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = (11, 15)$$

$$= [2 \ 1] \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix} = 11u_1 + 15u_2$$

Example #6

V, f, u_1, u_2, v as in Ex #5

$$u_1 = 2u_1 + 3u_2 \quad u_2 = -u_1 + 3u_2$$

$$u_2 = u_1 + u_2 \quad u_2 = u_1 - 2u_2$$

$$V = [u_1, u_2] (= [u_1, u_2])$$

$$v = 2u_1 + u_2 \rightarrow v = 2(-u_1 + 3u_2) + (u_1 - 2u_2) = -u_1 + 4u_2 = (-1, 4)$$

Example #6

$$f(v) = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = (8, 29)$$

$$= 8u_1 + 29u_2$$

Relative to $\{u_1, u_2\}$,

$$f(v) = \begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = (4, 3)$$

$$= 4u_1 + 3u_2 = 4(2u_1 + 3u_2) + 3(u_1 + u_2) = 11u_1 + 15u_2$$

b.

Example #5

 $f: V \rightarrow V, V = [u_1, u_2]$

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$$f(u_2) = 5u_1 + 7u_2$$

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Example #6

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Relative to $\{u_1, u_2\}$,

$$f(v) = \begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = (4, 3)$$

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c.

3. Exercises:

3.4.1(L)

Let $V = [u_1, u_2]$ and let $f: V \rightarrow V$ be the linear function defined by

$$f(u_1) = -3u_1 + 2u_2$$

$$f(u_2) = 4u_1 - u_2.$$

- Letting (x_1, x_2) denote $x_1u_1 + x_2u_2$, compute $f(x_1, x_2)$.
- With f as above, let $v_1 = 7u_1 + 5u_2$ and $v_2 = 2u_1 + 3u_2$. Compute $f(v_1)$, $f(v_2)$, and $f(v_1 + v_2)$; and show that $f(v_1 + v_2) = f(v_1) + f(v_2)$.
- Identifying u_1 with \vec{i} and u_2 with \vec{j} , describe f in terms of how it maps the xy -plane onto the uv -plane.

3.4.2

Let $V = [u_1, u_2, u_3]$; and let $\alpha_1 = (1, 2, 3)$, $\alpha_2 = (4, 5, 6)$,
 $\alpha_3 = (7, 8, 9) \in V$. Suppose $T: V \rightarrow W$ is linear where $W = [w_1, w_2, w_3, w_4]$.

- Is it possible that $T(\alpha_1) = (3, 1, 2, 4)$, $T(\alpha_2) = (4, 2, 1, 5)$ and $T(\alpha_3) = (2, 3, 4, 1)$? Explain.
- Let $\gamma_1 = (1, 1, 1)$, $\gamma_2 = (1, 2, 3)$, $\gamma_3 = (2, 3, 5)$, and $\gamma_4 = (3, 7, 6)$. Express $T(\gamma_4)$ as a linear combination of $T(\gamma_1)$, $T(\gamma_2)$, and $T(\gamma_3)$.

3.4.3(L)

Define the linear transformation $f: V \rightarrow V$, where $V = [u_1, u_2]$, by
 $f(u_1) = -3u_1 + 2u_2$ and $f(u_2) = 4u_1 - u_2$.

- Letting

$$\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

use the method described in the lecture to express $f(v) = f(x_1u_1 + x_2u_2)$ as a product of matrices.

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3.4.3(L) continued

- b. Do the same as in (a) but now use the matrix

$$B = \begin{bmatrix} -3 & 2 \\ 4 & -1 \end{bmatrix} .$$

3.4.4

Let $V = [u_1, u_2, u_3]$ and let the linear transformation $f: V \rightarrow V$ be defined by

$$\begin{aligned} f(u_1) &= u_1 + u_2 + u_3 \\ f(u_2) &= 2u_1 + 3u_2 + 3u_3 \\ f(u_3) &= 3u_1 + 4u_2 + 6u_3 . \end{aligned}$$

Now, let $v = x_1u_1 + x_2u_2 + x_3u_3$.

- a. Compute $f(v)$ without the use of matrices.
b. Compute $f(v)$ using the matrices

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 4 & 6 \end{bmatrix}$$

and

$$\vec{X} = [x_1 x_2 x_3] .$$

- c. Use B^T and \vec{X}^T to compute $f(v)$ in terms of

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 2 & 4 & 6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} .$$

3.4.5(L)

Let $V = [u_1, u_2]$ and let f be the linear transformation $f: V \rightarrow V$ defined by $f(u_1) = u_1 + 2u_2$, $f(u_2) = 3u_1 + 5u_2$. Let $v_1 = u_1 + u_2$ and $v_2 = 2u_1 + u_2$.

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3.4.5(L) continued

- a. Show that $V = [v_1, v_2]$ and express u_1 and u_2 in terms of v_1 and v_2 . Use this result to express $f(v_1)$ and $f(v_2)$ as linear combinations of v_1 and v_2 . What is the matrix of coefficients of f relative to the basis $\{v_1, v_2\}$?
- b. Let $v = 4u_1 + 7u_2$. Express $f(v)$ as a linear combination of u_1 and u_2 and also as a linear combination of v_1 and v_2 .
- c. Suppose $V = [\alpha_1, \alpha_2]$ and that also $V = [\beta_1, \beta_2]$. Say

$$\begin{aligned}\beta_1 &= b_{11}\alpha_1 + b_{12}\alpha_2 \\ \beta_2 &= b_{21}\alpha_1 + b_{22}\alpha_2 .\end{aligned}$$

Suppose also that $T:V \rightarrow V$ is the linear transformation defined by

$$\begin{aligned}T(\alpha_1) &= a_{11}\alpha_1 + a_{12}\alpha_2 \\ T(\alpha_2) &= a_{21}\alpha_1 + a_{22}\alpha_2 .\end{aligned}$$

If

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} ,$$

show that the matrix BAB^{-1} represents T relative to the basis $[\beta_1, \beta_2]$.

3.4.6 (optional)

- a. Show that if $X^{-1}AX = I$, then $A = I$.
- b. Show that if $X^{-1}AX = 0$, then $A = 0$.

3.4.7

Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 5 & 4 \end{bmatrix}$$

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3.4.7 continued

be the matrix of coefficients of the linear transformation $f:V \rightarrow V$ relative to the basis $\{u_1, u_2, u_3\}$. Now, let $v_1 = u_1 + 2u_2 + 3u_3$, $v_2 = 2u_1 + 5u_2 + 6u_3$, and $v_3 = 3u_1 + 6u_2 + 10u_3$. Show that $V = [v_1, v_2, v_3]$, and use the method described in Exercise 8.4.5 to express the matrix of coefficients of f relative to the basis $\{v_1, v_2, v_3\}$.

3.4.8(L)

Let $V = [u_1, u_2, u_3]$ and let $f:V \rightarrow V$ be the linear transformation defined by

$$\begin{aligned}f(u_1) &= u_1 + 2u_2 + 3u_3 \\f(u_2) &= 2u_1 + 5u_2 + 8u_3 \\f(u_3) &= u_1 + 4u_2 + 7u_3.\end{aligned}$$

Describe the space $f(V)$ and show that its dimension is 2. Also, describe the null space of V with respect to f .

3.4.9 (optional)

[This is a generalization of the previous exercise.]

Let $V = [v_1, v_2, v_3, v_4]$ and $W = [w_1, w_2]$. Suppose $f:V \rightarrow W$ is the linear transformation defined by

$$\begin{aligned}f(v_1) &= w_1 + w_2 \\f(v_2) &= 2w_1 + 3w_2 \\f(v_3) &= 3w_1 + 5w_2 \\f(v_4) &= 4w_1 + w_2.\end{aligned}$$

- Show that $f(V) = W$. In particular, find α_1 and $\alpha_2 \in V$ such that $f(\alpha_1) = w_1$ and $f(\alpha_2) = w_2$. Also, find a basis for N_f .
- Find a row-reduced basis for N_f and show how x_3 and x_4 must be related to x_1 and x_2 if $(x_1, x_2, x_3, x_4) \in N_f$.
- Find all $v \in V$ such that $f(v) = 5w_1 + 6w_2$.

3.4.10 (optional)

[This exercise is not crucial here but it is very important in Unit 6.]

Let V be a vector space and let c be a fixed real number. Suppose $f: V \rightarrow V$ is linear. Define $w = \{v \in V : f(v) = cv\}$. Prove that w is a subspace of V and, moreover, that $f(w) \subseteq w$.

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