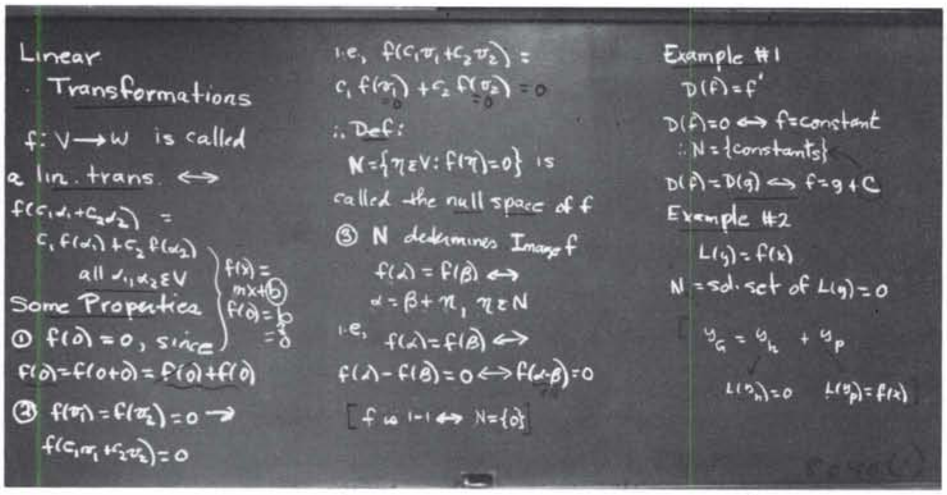
## Unit 4: Linear Transformations

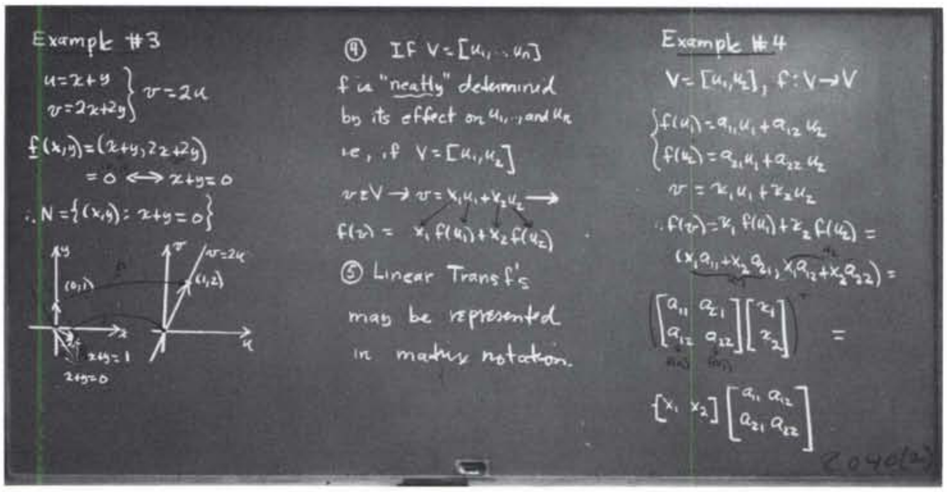
1. Overview

We have already seen many instances in our course where the concept of a linear function was most crucial. It turns out that the general concept of a linear transformation is best handled in terms of viewing them as special mappings of vector spaces into vector spaces. Thus, the aim of this unit is to show how this study is handled, and it is our hope that seeing the general structure will make it clear as to what common properties are shared by all linear transformations.

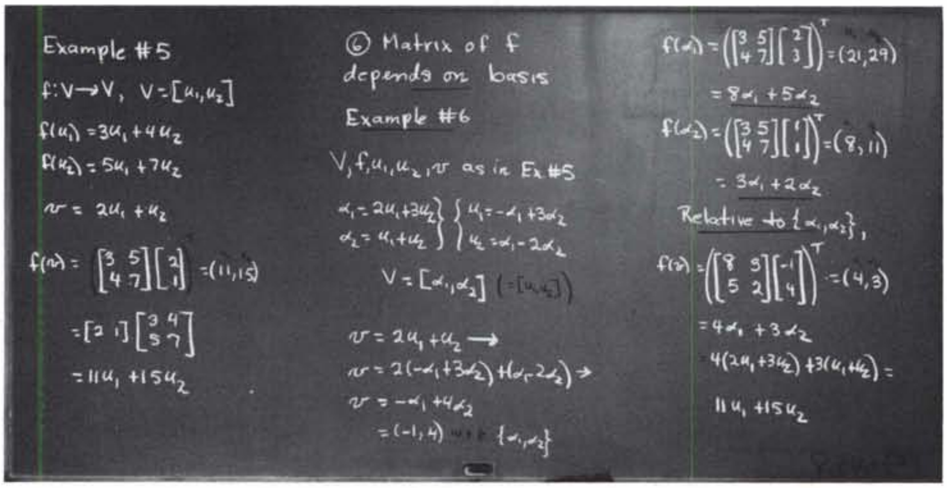
2. Lecture 3.040




a.




b.




c.

3. Exercises:

3.4.1(L)

Let $V = [u_1,u_2]$ and let $f:V \to V$ be the linear function defined by

$f(u_1) = -3u_1 + 2u_2$
$f(u_2) = 4u_1 - u_2$.

a. Letting $(x_1,x_2)$ denote $x_1u_1 + x_2u_2$, compute $f(x_1,x_2)$.

b. With f as above, let $v_1 = 7u_1 + 5u_2$ and $v_2 = 2u_1 + 3u_2$. Compute $f(v_1), f(v_2)$, and $f(v_1 + v_2)$; and show that $f(v_1 + v_2) = f(v_1) + f(v_2)$.

c. Identifying $u_1$ with $\vec{i}$ and $u_2$ with $\vec{j}$, describe f in terms of how it maps the xy-plane onto the uv-plane.

3.4.2

Let $V = [u_1,u_2,u_3]$; and let $\alpha_1 = (1,2,3)$, $\alpha_2 = (4,5,6)$, $\alpha_3 = (7,8,9) \varepsilon V$. Suppose $T:V \to w$ is linear where $W = [w_1,w_2,w_3,w_4]$.

a. Is it possible that $T(\alpha_1) = (3,1,2,4)$, $T(\alpha_2) = (4,2,1,5)$ and $T(\alpha_3) = (2,3,4,1)$? Explain.

b. Let $\gamma_1 = (1,1,1)$, $\gamma_2 = (1,2,3)$, $\gamma_3 = (2,3,5)$, and $\gamma_4 = (3,7,6)$ Express $T(\gamma_4)$ as a linear combination of $T(\gamma_1)$, $T(\gamma_2)$, and $T(\gamma_3)$.

3.4.3(L)

Define the linear transformation $f:V \to V$, where $V = [u_1,u_2]$, by $f(u_1) = -3u_1 + 2u_2$ and $f(u_2) = 4u_1 - u_2$.

a. Letting

$$\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

use the method described in the lecture to express $f(v) = f(x_1u_1 + x_2u_2)$ as a product of matrices.

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3.4.3(L) continued

b. Do the same as in (a) but now use the matrix

$$B = \begin{bmatrix} -3 & 2 \\ 4 & -1 \end{bmatrix} .$$

3.4.4

Let $V = [u_1, u_2, u_3]$ and let the linear transformation $f:V \to V$ be defined by

$$\begin{aligned} f(u_1) &= u_1 + u_2 + u_3 \\ f(u_2) &= 2u_1 + 3u_2 + 3u_3 \\ f(u_3) &= 3u_1 + 4u_2 + 6u_3 \end{aligned} .$$

Now, let $v = x_1u_1 + x_2u_2 + x_3u_3$.

a. Compute $f(v)$ without the use of matrices.

b. Compute $f(v)$ using the matrices

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 4 & 6 \end{bmatrix}$$

and

$$\vec{X} = [x_1 x_2 x_3].$$

c. Use $B^T$ and $\vec{X}^T$ to compute $f(v)$ in terms of

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 2 & 4 & 6 \end{bmatrix} \text{ and } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} .$$

3.4.5(L)

Let $V = [u_1, u_2]$ and let f be the linear transformation $f:V \to V$ defined by $f(u_1) = u_1 + 2u_2$, $f(u_2) = 3u_1 + 5u_2$. Let $v_1 = u_1 + u_2$ and $v_2 = 2u_1 + u_2$.

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3.4.5(L) continued

a. Show that $V = [v_1, v_2]$ and express $u_1$ and $u_2$ in terms of $v_1$ and $v_2$. Use this result to express $f(v_1)$ and $f(v_2)$ as linear combinations of $v_1$ and $v_2$. What is the matrix of coefficients of f relative to the basis $\{v_1, v_2\}$ ?

b. Let $v = 4u_1 + 7u_2$. Express f(v) as a linear combination of $u_1$ and $u_2$ and also as a linear combination of $v_1$ and $v_2$.

c. Suppose $V = [\alpha_1, \alpha_2]$ and that also $V = [\beta_1, \beta_2]$. Say

$$\beta_1 = b_{11}\alpha_1 + b_{12}\alpha_2$$
$$\beta_2 = b_{21}\alpha_1 + b_{22}\alpha_2 \; .$$

Suppose also that $T: V \to V$ is the linear transformation defined by

$$T(\alpha_1) = a_{11}\alpha_1 + a_{12}\alpha_2$$
$$T(\alpha_2) = a_{21}\alpha_1 + a_{22}\alpha_2 \; .$$

If

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} ,$$

show that the matrix $BAB^{-1}$ represents T relative to the basis $[\beta_1, \beta_2]$ .

3.4.6 (optional)

a. Show that if $X^{-1}AX = I$, then $A = I$.

b. Show that if $X^{-1}AX = 0$, then $A = 0$.

3.4.7

Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 5 & 4 \end{bmatrix}$$

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3.4.7 continued

be the matrix of coefficients of the linear transformation $f:V \to V$ relative to the basis $\{u_1,u_2,u_3\}$. Now, let $v_1 = u_1 + 2u_2 + 3u_3$, $v_2 = 2u_1 + 5u_2 + 6u_3$, and $v_3 = 3u_1 + 6u_2 + 10u_3$. Show that $V = [v_1,v_2,v_3]$, and use the method described in Exercise 8.4.5 to express the matrix of coefficients of f relative to the basis $\{v_1,v_2,v_3\}$.

3.4.8(L)

Let $V = [u_1,u_2,u_3]$ and let $f:V \to V$ be the linear transformation defined by

$$f(u_1) = u_1 + 2u_2 + 3u_3$$
$$f(u_2) = 2u_1 + 5u_2 + 8u_3$$
$$f(u_3) = u_1 + 4u_2 + 7u_3.$$

Describe the space f(V) and show that its dimension is 2. Also, describe the null space of V with respect to f.

3.4.9 (optional)

[This is a generalization of the previous exercise.]

Let $V = [v_1,v_2,v_3,v_4]$ and $W = [w_1,w_2]$. Suppose $f:V \to W$ is the linear transformation defined by

$$f(v_1) = w_1 + w_2$$
$$f(v_2) = 2w_1 + 3w_2$$
$$f(v_3) = 3w_1 + 5w_2$$
$$f(v_4) = 4w_1 + w_2.$$

a. Show that f(v) = W. In particular, find $\alpha_1$ and $\alpha_2 \varepsilon V$ such that $f(\alpha_1) = w_1$ and $f(\alpha_2) = w_2$. Also, find a basis for $N_f$.

b. Find a row-reduced basis for $N_f$ and show how $x_3$ and $x_4$ must be related to $x_1$ and $x_2$ if $(x_1,x_2,x_3,x_4) \varepsilon N_f$.

c. Find all $v \varepsilon V$ such that $f(v) = 5w_1 + 6w_2$.

3.4.10 (optional)

[This exercise is not crucial here but it is very important in Unit 6.]

Let V be a vector space and let c be a fixed real number. Suppose $f:V \rightarrow V$ is linear. Define $w = \{v \varepsilon V: f(v) = cv\}$. Prove that w is a subspace of V and, moreover, that $f(w) \subseteq W$.