Unit 4: (Optional) The Directional Derivative in n-Dimensional Vector Spaces

3.4.1

a. By definition, the set $\frac{15}{(4,3,2,1)}$ is

$$\{\underline{x}:(4,3,2,1)\cdot\underline{x} = 15\} \qquad (1)$$

If we now write $\underline{x}$ as (x_1,x_2,x_3,x_4), (1) becomes

$$\{(x_1,x_2,x_3,x_4):(4,3,2,1)\cdot(x_1,x_2,x_3,x_4) = 15\}$$

or

$$\{(x_1,x_2,x_3,x_4):4x_1+3x_2+2x_3+x_4 = 15\} \qquad (2)$$

But (2) is the solution set of the linear algebraic equation

$$4x_1+3x_2+2x_3+x_4 = 15 \qquad (3)$$

b. Any vector having the same direction as (4,3,2,1) must have the form, t(4,3,2,1), or, (4t,3t,2t,t). For such a vector to satisfy (3) [which is what it means for the vector to belong to $\frac{15}{(4,3,2,1)}$] we must have

$$4(4t)+3(3t)+2(2t)+(t) = 15$$

From this we see that $30t = 15$, or $t = \frac{1}{2}$. With $t = \frac{1}{2}$, t(4,3,2,1) becomes $\frac{1}{2}(4,3,2,1)$ or $(2,\frac{3}{2},1,\frac{1}{2})$. Thus by our definition of <u>the</u> vector $\frac{15}{(4,3,2,1)}$,

$$\frac{15}{(4,3,2,1)} = (2,\tfrac{3}{2},1,\tfrac{1}{2}) \qquad (4)$$

c. More generally, if c is any number and $\underline{a}=(a_1,a_2,a_3,a_4)$ any vector,

3.4.1 continued

$$\frac{c}{(a_1,a_2,a_3,a_4)} = \{(x_1,x_2,x_3,x_4):a_1x_1+a_2x_2+a_3x_3+a_4x_4 = c\}$$

If (ta_1,ta_2,ta_3,ta_4) belongs to this set we must have

$$a_1(ta_1)+a_2(ta_2)+a_3(ta_3)+a_4(ta_4) = c,$$

whence

$$t(a_1^2+a_2^2+a_3^2+a_4^2) = c \quad .$$

Since $a_1^2+a_2^2+a_3^2+a_4^2 = \|\underline{a}\|^2$ (This is why we use the Euclidean metric.), it follows that

$$t = \frac{c}{\|\underline{a}\|^2} \quad .$$

Hence the vector in the direction of $\underline{a}$ which belongs to $\frac{c}{\underline{a}}$ is

$$\frac{c}{\|\underline{a}\|^2}(a_1,a_2,a_3,a_4), \text{ or, } \frac{c}{\|\underline{a}\|^2}\underline{a} \ .$$

Finally, this vector may be rewritten as

$$\frac{c}{\|\underline{a}\|} \ \frac{\underline{a}}{\|\underline{a}\|}$$

and this, in turn is

$\frac{c}{\|\underline{a}\|}\underline{u}$, where $\underline{u}$ is the unit vector in the direction of $\underline{a}$.

This result checks with our general result in the notes.

In terms of (4), c=15, $\underline{a}=(4,3,2,1)$. Hence $\|\underline{a}\|^2=4^2+3^2+2^2+1^2=30$. Therefore,

$$\underline{u} = \frac{\underline{a}}{\underline{a}} = \frac{(4,3,2,1)}{\sqrt{30}} \quad ,$$

and

3.4.1 continued

$$\frac{c}{\|\underline{a}\|}\ \underline{u} = \frac{15}{\sqrt{30}}\left(\frac{(4,3,2,1)}{\sqrt{30}}\right) = \frac{1}{2}(4,3,2,1) = (2,\tfrac{3}{2},1,\tfrac{1}{2})$$

3.4.2

If $\underline{x} = (x_1,x_2,x_3,x_4)$ and $\underline{y} = (y_1,y_2,y_3,y_4)$ each belongs to $\frac{c}{(a_1,a_2,a_3,a_4)}$, then, by definition,

$$a_1x_1+a_2x_2+a_3x_3+a_4x_4 = c \qquad (1)$$

and

$$a_1y_1+a_2y_2+a_3y_3+a_4y_4 = c \qquad (2)$$

Adding equations (1) and (2), we obtain

$$a_1(x_1+y_1)+a_2(x_2+y_2)+a_3(x_3+y_3)+a_4(x_4+y_4) = 2c \qquad (3)$$

From (3) we see that $\underline{x}+\underline{y}$ belongs to $\frac{2c}{(a_1,a_2,a_3,a_4)}$

$$\begin{aligned}[\text{I.e., } \underline{a}\cdot(\underline{x}+\underline{y}) &= (a_1,a_2,a_3,a_4)\cdot(x_1+y_1,x_2+y_2,x_3+y_3,x_4+y_4)\\ &= a_1(x_1+y_1)+a_2(x_2+y_2)+a_3(x_3+y_3)+a_4(x_4+y_4)\\ &= 2c] .\end{aligned}$$

Thus, unless c=2c, it is impossible for the sum of two vectors in $\frac{c}{\underline{a}}$ to also be in $\frac{c}{\underline{a}}$.

Of course, if c=0 then c=2c. In other words, the sum of any two members of $\frac{0}{\underline{a}}$ is also a member of $\frac{0}{\underline{a}}$.

A Note on Exercise 3.4.2

Given the linear algebraic equation

$$a_1x_1+\ldots+a_nx_n = c \qquad (1)$$

we call the equation homogeneous if c=0.

Before we discuss the significance of such an equation, let us review what the solution set of (1) looks like in the language of vectors. Quite simply, if we let $\underline{x} = (x_1,\ldots,x_n)$ and $a = (a_1,\ldots a_n)$, then the solution set of (1) is the set $\frac{c}{\underline{a}}$. In Exercise 3.4.2 we showed that unless c=0, the sum of two solutions* of (1) was not a solution of (1). On the other hand, if c=0, the solution set of (1) is closed with respect to addition. In a similar way, it can be shown that if c=0, the solution set of (1) has the additional property that any scalar multiple of a member of the set is also a member of the set. That is, if $a_1x_1+\ldots+a_nx_n = 0$, then $t(a_1x_1+\ldots+a_nx_n) = t(0) = 0$, or $a_1(tx_1)+\ldots+a_n(tx_n) = 0$. In other words, if $\underline{x}$ belongs to $\frac{0}{\underline{a}}$, so also does $t\underline{x}$, for any number, t. Since $\frac{0}{\underline{a}}$ is closed with respect to the two important vector properties of addition and scalar multiplication, one often refers to the solution set of a homogeneous linear equation as a solution space.

3.4.3

a. (1) $\frac{c}{(a_1,a_2)} = \{(x,y):a_1x+a_2y = c\}$

Thus, while $\frac{c}{(a_1,a_2)}$ is the solution set of the linear algebraic

*To say that $\underline{s}$ and $\underline{t}$ are solutions of (1) means that $a_1s_1+\ldots+a_ns_n$ and $a_1t_1+\ldots+a_nt_n$ both equal c, and by the sum of the two solutions we mean the usual notion, as described in the exercise, of adding the two solution vectors component by component. What the exercise showed us was that if $c\neq 0$ then $\underline{x}+\underline{t}$ was not a member of the solution set of (1).

3.4.3 continued

equation $a_1x+a_2y = c$, it is also the Cartesian equation of a line in the xy-plane.

That is, in 2-dimensional space, $\frac{c}{\underline{a}}$ may be viewed as the set of points in the xy-plane which lie on the line $a_1x+a_2y = c$.

(2) Similarly,

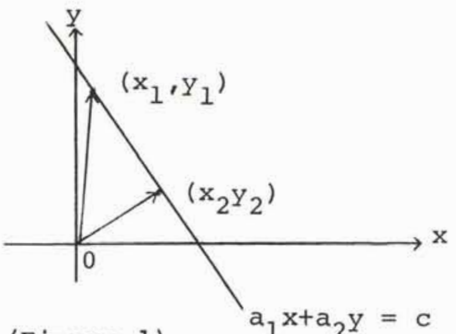
$$\frac{c}{(a_1,a_2,a_3)} = \{(x,y,z):a_1x+a_2y+a_3z = c\}$$

so that, geometrically, we may view this as a set of points which lie in the plane whose Cartesian equation is $a_1x+a_2y+a_3z = c$.

Notice that while for n=2 and n=3, $\frac{c}{\underline{a}}$ has a nice geometric interpretation (for n=1, the interpretation is even simpler since the $\frac{c}{\underline{a}} = \frac{c}{a}$ which is a single point on the x-axis) the meaning of $\frac{c}{\underline{a}}$ for n>3 is as real as in the cases n=2 or n=3, especially in terms of being the solution set of the linear algebraic equation

$$a_1x_1+\ldots+a_nx_n = c$$

b.



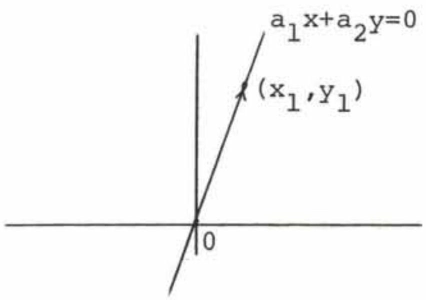
(Figure 1)

3.4.3 continued

The points (x_1,y_1) and (x_2,y_2) as drawn in Figure 1 belong to the given line; hence, they are members of $\frac{c}{(a_1,a_2)}$. The sum of these two points is the point (x_1+x_2,y_1+y_2), or, in terms of vectors, the sum is the vector sum $(x_1,y_1)+(x_2,y_2)$ where in this context (x_1,y_1) is the arrow from $(0,0)$ to the point (x_1,y_1).

It should be clear from (Figure 1) that the sum of the two vectors, if it begins at the origin, will not terminate on the given line. (In terms of Exercise 3.4.2, (x_1+x_2,y_1+y_2) satisfies $a_1x+a_2y=2c$ rather than $a_1x + a_2y = c$).

If $c = 0$, however, our line passes through the origin



(Figure 2)

In this case every vector from 0 to a point (x_1,y_1) on the line, lies on the line. Thus, the sum of two such vectors will also terminate on the line.

3.4.4

We already know that as a set, $\frac{c}{\underline{a}}$ is the line $a_1x+a_2y = c$.

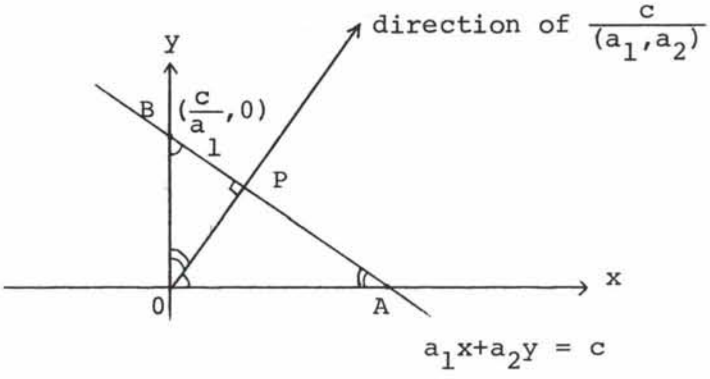
Now, by definition, <u>the vector</u> $\frac{c}{\underline{a}}$ is the vector in the direction of $\underline{a}$ with magnitude $\frac{c}{\|\underline{a}\|}$.

Notice that the direction of $\underline{a}$ is at right angles to the line

3.4.4 continued

$a_1x+a_2y = c$. Namely, rewriting the equation of the line in the form $y = -\frac{a_1}{a_2}x + \frac{c}{a_2}$ (if $a_2\neq 0^*$), we see that the line has slope $-\frac{a_1}{a_2}$, whereas the slope of $\underline{a} = (a_1,a_2) = a_1\vec{i}+a_2\vec{j}$, is $\frac{a_2}{a_1}$. Since the slopes are negative reciprocals of one another the lines are at right angles.

So, thus far, we have (and our diagram is for the case that a_1, a_2, and c are all positive, although similar results hold for all other cases)



(Figure 1)

Applying some elementary geometry to Figure 1, we see that triangles OPB and AOB are similar. Hence

$$\frac{\overline{OP}}{\overline{OB}} = \frac{\overline{OA}}{\overline{AB}} \quad (1)$$

*If $a_2=0$ then the line is given by $a_1x=c$ which is parallel to the y-axis, and the vector $\underline{a}=(a_1,a_2)$ is then $a_1\vec{i}$ which is parallel to the x-axis. Thus, in this case, $\underline{a}$ is still perpendicular to $a_1x+a_2y = c$.

3.4.4 continued

But, $\overline{OB} = \frac{c}{a_1}$; $\overline{OA} = \frac{c}{a_2}$, and

$$\overline{AB} = \sqrt{\frac{c^2}{a_1^{\,2}} + \frac{c^2}{a_2^{\,2}}} = \frac{c}{a_1\, a_2}\sqrt{a_1^{\,2}+a_2^{\,2}} .$$

Therefore, from (1) we have

$$\overline{OP} = \frac{\overline{OA}}{\overline{AB}}\,\overline{OB} = \frac{\left(\frac{c}{a_2}\right)\left(\frac{c}{a_1}\right)}{\frac{c}{a_1\, a_2}\sqrt{a_1^{\,2}+a_2^{\,2}}} = \frac{c}{\sqrt{a_1^{\,2}+a_2^{\,2}}} = \frac{c}{\|\underline{a}\|}$$

Hence $\overline{OP}$ equals the magnitude of the vector $\frac{c}{\underline{a}}$.
In other words, the vector $\frac{c}{\underline{a}}$ is the vector which is perpendicular to the line $\frac{c}{\underline{a}}$ and extends from the origin to the line. In other words the magnitude of the vector $\frac{c}{\underline{a}}$ represents the (perpendicular) distance from the origin to the line. In the event c=0, the line passes through the origin, in which case this distance is zero, and this checks with the fact that $\frac{0}{\underline{a}}$ is the zero-vector.
While we do not want to pursue the geometry of n-dimensions further here, in general, for any n-dimensional vector space the vector $\frac{c}{\underline{a}}$ represents the minimum distance between the origin, $\underline{0}=(0,\ldots,0)$, and any point (member) in the set $\frac{c}{\underline{a}}$.

3.4.5

a. We have $f(\underline{x}) = \|\underline{x}\|^2$.

Therefore, since, in general

$$f'_{\underline{u}}(\underline{a}) = \lim_{t\to 0}\left[\frac{f(\underline{a}+t\underline{u})-f(\underline{a})}{t}\right]\underline{u} ,$$

3.4.5 continued

in this case we have

$$f'_{\underline{u}}(\underline{a}) = \lim_{t\to 0}\left[\frac{\|\underline{a}+t\underline{u}\|^2-\|\underline{a}\|^2}{t}\right]\underline{u} \qquad (1)$$

We should point out that the bracketed expression in (1) makes perfectly good sense without our having to resort to n-tuple notation. However, if we feel more at home with the n-tuple notation, we may think of $\underline{x}$ as being the n-tuple $(x_1,\ldots,x_n)$, $\underline{a}$ as being $(a_1,\ldots,a_n)$, and $\underline{u}$ as being $(u_1,\ldots,u_n)$. If we do this, then we have:

$$\underline{a} + t\underline{u} = (a_1 + tu_1,\ldots,a_n + tu_n),$$

whereupon

$$\|\underline{a}+t\underline{u}\|^2 = (a_1+tu_1)^2+\ldots+(a_n+tu_n)^2$$

$$= (a_1^{\,2}+\ldots+a_n^{\,2})+2t(a_1u_1+\ldots+a_nu_n)+t^2(u_1^{\,2}+\ldots+u_n^{\,2})$$

Accordingly, since $\|\underline{a}\|^2 = (a_1^{\,2}+\ldots+a_n^{\,2})$,

$$\|\underline{a}+t\underline{u}\|-\|\underline{a}\|^2 = 2t(a_1u_1+\ldots a_nu_n)+t^2(u_1^{\,2}+\ldots u_n^{\,2}).$$

Therefore, if $t\neq 0$,

$$\frac{\|\underline{a}+t\underline{u}\|^2-\|\underline{a}\|^2}{t} = 2(a_1u_1+\ldots+a_nu_n)+t(u_1^{\,2}+\ldots+u_n^{\,2}) \qquad (2)$$

From (2), we obtain

3.4.5 continued

$$\lim_{t\to 0^+}\left[\frac{\|\underline{a}+t\underline{u}\|^2-\|\underline{a}\|^2}{t}\right] = 2(a_1u_1+\ldots+a_nu_n) = 2\underline{a}\cdot\underline{u} \qquad (3)$$

[As an aside notice that (3) could have been derived in n-space without recourse to n-tuple notation. Namely,

$\|\underline{a}+t\underline{u}\|^2 = (\underline{a}+t\underline{u})\cdot(\underline{a}+t\underline{u}) = \underline{a}\cdot\underline{a}+2t\underline{a}\cdot\underline{u}+t^2\underline{u}\cdot\underline{u},$

and since $\underline{a}\cdot\underline{a} = \|\underline{a}\|^2$ and $\underline{u}\cdot\underline{u} = \|\underline{u}\|^2$, this means that

$$\|\underline{a}+t\underline{u}\|^2-\|\underline{a}\|^2 = \|\underline{a}\|^2+2t\underline{a}\cdot\underline{u}+t^2\|\underline{u}\|^2-\|\underline{a}\|^2 = 2t\underline{a}\cdot\underline{u}+t^2\|\underline{u}\|^2 .$$

Hence,

$$\frac{\|\underline{a}+t\underline{u}\|^2-\|\underline{a}\|^2}{t} = 2\underline{a}\cdot\underline{u} + t\|\underline{u}\|^2$$

and letting $t\to 0^+$ we obtain the same result as in (3)].

In any event if the result of (3) is substituted into (1), we obtain

$$f'_{\underline{u}}(\underline{a}) = (2\underline{a}\cdot\underline{u})\underline{u} \qquad (4)$$

Hopefully, (4) clearly illustrates how, once $\underline{a}$ is chosen, the direction <u>and the magnitude</u> of the directional derivative, $f'_{\underline{u}}(\underline{a})$, depends on $\underline{u}$.

In particular, for both parts (b) and (c) of this exercise, $\underline{a}$ = (1,2,5,3,1). In part (b) our direction is that of (1,3,4,-1,2), so that

$$\underline{u} = \frac{(1,3,4,-1,2)}{\|(1,3,4,-1,2)\|} = \frac{(1,3,4,-1,2)}{\sqrt{31}} \quad ;$$

3.4.5 continued

while in (c) our direction is that of (2,1,2,1,2), so that

$$\underline{u} = \frac{(2,1,2,1,2)}{\|(2,1,2,1,2)\|} = \frac{(2,1,2,1,2)}{\sqrt{14}} .$$

Putting these results into (4) we have

b.

$$f'_{\underline{u}}(\underline{a}) = \left[2(1,2,5,3,1)\cdot\frac{(1,3,4,-1,2)}{\sqrt{31}} \right] \frac{(1,3,4,-1,2)}{\sqrt{31}} \tag{5}$$

$$= \frac{2}{31} [1+6+20-3+2](1,3,4,-1,2)$$

$$= \frac{2}{31} [26](1,3,4,-1,2)$$

$$= \frac{52}{31} (1,3,4,-1,2)$$

$$= (\tfrac{52}{31}, \tfrac{156}{31}, \tfrac{208}{31}, -\tfrac{52}{31}, \tfrac{104}{31}) \tag{6}$$

[Of course we could have computed (5) in the form

$$\left[\frac{2}{\sqrt{31}} (1+6+20-3+2) \right] \frac{(1,3,4,-1,2)}{\sqrt{31}} = \frac{52}{\sqrt{31}} \underline{u} \quad , \tag{7}$$

from which we may now read at once that the directional derivative is the vector in the direction of $\underline{u}$ with magnitude equal to $\frac{52}{\sqrt{31}}$. While (6) and (7) are equivalent, (6) stresses the answer in n-tuple notation.]

c.

$$f'_{\underline{u}}(\underline{a}) = \left[2(1,2,5,3,1)\cdot\frac{(2,1,2,1,2)}{\sqrt{14}} \right] \frac{(2,1,2,1,2)}{\sqrt{14}}$$

$$= \frac{2}{\sqrt{14}} (2+2+10+3+2) \underline{u}$$

$$= \frac{38}{\sqrt{14}} \underline{u} \tag{8}$$

3.4.5 continued

Notice that the $\underline{u}$'s in (7) and (8) refer to different directions, and that in (7) and (8) the magnitudes (as well as the directions) are unequal.

d. Using traditional notation, we would have

$$f(x_1,x_2,x_3,x_4,x_5) = {x_1}^2+{x_2}^2+{x_3}^2+{x_4}^2+{x_5}^2 \quad ,$$

whence

$$\vec{\nabla}f(\underline{a}) = (2x_1,2x_2,2x_3,2x_4,2x_5)_{\underline{x}=\underline{a}}$$

$$= (2a_1,2a_2,2a_3,2a_4,2a_5)$$

$$= 2(a_1,a_2,a_3,a_4,a_5)$$

Therefore

$$f'_{\underline{u}}(\underline{a}) = \vec{\nabla}f(\underline{a})\cdot\underline{u}$$

$$= 2(a_1,a_2,a_3,a_4,a_5)\cdot(u_1,u_2,u_3,u_4,u_5)$$

$$= 2\underline{a}\cdot\underline{u}$$

which checks with (3).

3.4.6

From Exercise 3.4.5, we know that

$$f'_{\underline{u}}(\underline{a}) = (2\underline{a}\cdot u)\underline{u} \quad .$$

Therefore,

3.4.6 continued

$$\|f'_{\underline{u}}(\underline{a})\|^{*} = \|(2\underline{a}\cdot\underline{u})\underline{u}\|$$
$$= |2\underline{a}\cdot\underline{u}|\ \|\underline{u}\|$$

or, since $\|\underline{u}\|=1$,

$$\|f'_{\underline{u}}(\underline{a})\| = |2\underline{a}\cdot\underline{u}|$$
$$= 2|\underline{a}\cdot\underline{u}| \qquad (1)$$

Recalling that for the Euclidean metric,

$$|\underline{a}\cdot\underline{u}| \leqslant \|\underline{a}\|\ \|\underline{u}\| = \|\underline{a}\|$$

we see from (1) that

$$\|f'_{\underline{u}}(\underline{a})\| \leqslant 2\|\underline{a}\| \qquad (2)$$

As a partial check of (2), we may refer to parts (b) and (c) of Exercise 3.4.5 in which we saw that $\|f'_{\underline{u}}(\underline{a})\| = \frac{52}{\sqrt{31}}$ and $\frac{38}{\sqrt{14}}$ respectively. In this case $\underline{a}$ was (1,2,5,3,1). Hence

$$\|\underline{a}\| = \sqrt{1+4+25+9+1} = \sqrt{40}\ ,$$

or

$$2\|\underline{a}\| = 2\sqrt{40} = 4\sqrt{10}$$

Equation (2) tells us that the magnitude of each directional

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*Remember, according to our definition, $f'_{\underline{u}}(\underline{a})$ is a vector, not a number. This differs from the usage in the text where what is there called the directional derivative in the direction $\underline{u}$ is what we would call $\|f'_{\underline{u}}(\underline{a})\|$. The difference is quite trivial, but worth noticing.

3.4.6 continued

derivative cannot exceed $4\sqrt{10}$. In particular, then, both $\frac{52}{\sqrt{31}}$ and $\frac{38}{\sqrt{14}}$ must be less than $4\sqrt{10}$, and a check shows that this is indeed the case. (By the way, when comparing the magnitudes of numbers involving square roots, it is often convenient to square each member, realizing that the larger square corresponds to the larger number.)

The next subtlety concerns whether there is a direction $\underline{u}$ for which $\|f'_{\underline{u}}(\underline{a})\|$ will take on the upper bound, $2\|\underline{a}\|$.

If we return to (1) and pick $\underline{u}$ to be in the direction of $\underline{a}$, i.e.,

$$\underline{u} = \frac{\underline{a}}{\|\underline{a}\|} \quad ,$$

we find that

$$\|f'_{\underline{u}}(\underline{a})\| = 2\left|\underline{a}\cdot\frac{\underline{a}}{\|\underline{a}\|}\right|$$

$$= \frac{2\|\underline{a}\|^2}{\|\underline{a}\|}$$

$$= 2\|\underline{a}\|$$

Hence, <u>in this example</u>, the maximum magnitude of any directional derivative occurs in the direction of $\underline{a}$ and the maximum magnitude is $2\|\underline{a}\|$.

Therefore, according to our definition of $f'(\underline{a})$ as given in the notes,

$$f'(\underline{a}) = 2\|\underline{a}\|\left(\frac{\underline{a}}{\|\underline{a}\|}\right)$$
$$= 2\underline{a}$$

(I.e., if $f(\underline{x}) = \|\underline{x}\|^2$ then $f'(\underline{a}) = 2\underline{a}$).

3.4.6 continued

In terms of the approach used in the text, our $f'(\underline{a})$ should correspond to the gradient of f at $\underline{x} = \underline{a}$, i.e., $\vec{\nabla}f(\underline{a})$. To check this, we have:

$f(\underline{x}) = \|\underline{x}\|^2$ implies $f(x_1,\ldots x_n) = x_1{}^2+\ldots+x_n{}^2$.

Hence,

$\vec{\nabla}f(x_1,\ldots,x_n) = (2x_1,\ldots,2x_n) = 2(x_1,\ldots,x_n)$

Therefore

$\vec{\nabla}f(a_1,\ldots,a_n) = 2(a_1,\ldots,a_n)$; and letting $(a_1,\ldots,a_n) = \underline{a}$,

$\vec{\nabla}f(\underline{a}) = 2\underline{a}$.

3.4.7

This exercise is perhaps the most significant in our entire discussion of what it means for a function of several real variables to be differentiable. The main point that arises is that, from a very important point of view, the directional derivatives do not tell the whole story, and, of even more significance, there is much strong feeling that the definition of a derivative should be independent of the concept of a directional derivative. That is, from a logical point of view, there are many reasons why one should first find a definition of a derivative, and <u>then</u>, as a special case, define the notion of a directional derivative.

The usual approach for doing this involves revisiting the concept of a differential in 1-dimensional space. Recall, that we showed that if f was differentiable at x=a, then there was a neighborhood of a, say N(a), such that for <u>every</u> number, a+h, in N(a),

$$f(a+h)-f(a) = f'(a)h+kh \quad \text{where } \lim_{h\to 0} k=0 \tag{1}$$

3.4.7 continued

(Perhaps (1) will look more familiar to you if you notice that h simply is what we called Δx previously.)

If we look at (1), we notice that once x=a is fixed, f'(a) is a fixed constant, whose value depends only on a, (and, of course, f) and not on h. The interesting logical point (even though we might not be able to give any reason why we might want to do so) is that we can begin the study of 1-dimensional calculus by actually using a slight refinement of (1) to define a derivative.

More specifically, suppose a given function f is defined in a neighborhood of x=a. We then define f to be differentiable at x=a, if there exists a neighborhood of a, N(a), such that for every number a+h in N(a), there exists a constant, C, such that

$$f(a+h)-f(a) = Ch+kh, \text{ where } \lim_{h\to 0} k = 0 \tag{2}$$

Notice that the only difference between (1) and (2) is that in (2) f'(a) is replaced by C. Certainly, we had better do something like this if we are assuming in equation (2) that the derivative has not yet been defined!

The main point is that our original definition of differentiable is equivalent to the definition given in (2). That is, either definition implies the other. For example, if we accept our original definition, we know from Part 1 of our course that (1) is true, and once (1) is true we need only choose C to equal f'(a) to establish the truth of (2).

Conversely, if we assume that (2) is true, we can show that our constant C must be what we would have previously called f'(a). To see this in more detail we need only divide both sides of (2) by h (which is permissible in the sense that since we shall let $h\to 0$, $h\neq 0$) to obtain

$$\frac{f(a+h)-f(a)}{h} = C+k, \quad \text{where } \lim_{h\to 0} k = 0 \tag{3}$$

3.4.7 continued

If we now let $h \to 0$ in (3), we obtain

$$\lim_{h\to 0} \left[\frac{f(a+h)-f(a)}{h} \right] = \lim_{h\to 0} C + \lim_{h\to 0} k \quad ,$$

and since C is a constant and $\lim_{h\to 0} k = 0$, it follows that

$$\lim_{h\to 0} \left[\frac{f(a+h)-f(h)}{h} \right] = C \qquad (4)$$

But, by our original definition of derivative,

$$f'(a) = \lim_{h\to 0} \left[\frac{f(a+h)-f(h)}{h} \right] ,$$

so that (4) implies

$$C = f'(a) \qquad (5)$$

A further implication of (5) is that if the constant C, as stipulated by (2), exists it is unique. That is, C must equal f'(a).

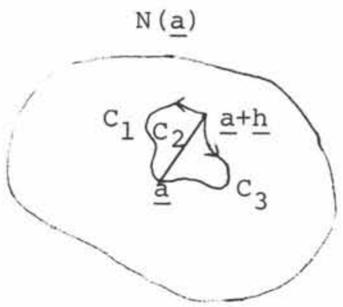
With these preliminaries out of the way, let us now ask what is so important about the definition of a derivative as implied by (2). While the main reason will not seem too important in the relatively simple 1-dimensional case (simple, because there is only one degree of freedom, and this means that, geometrically, there is only one direction), it will shed some light on the other dimensional vector spaces.

The key idea is that if we look at the left side of equation (1) or (2), all we see is the difference between the two numbers f(a+h) and f(a). No mention is made of any direction. Well, from this point of view, it seems that it would be nice if we could define

3.4.7 continued

a derivative in n-space so that an expression such as f(a+h)-f($\underline{a}$) makes sense without reference to direction, especially since even in n-space we want $\underline{h}\to\underline{0}$ to connote the idea that the answer does not depend on the path by which $\underline{h}\to\underline{0}$.

A second, although not necessarily an important, problem is that our definition of directional derivative does not cover all possible paths. That is, when we wrote a direction in the form t$\underline{u}$ and then let $t\to 0^+$, we were in effect limiting our approach to straight lines. For example in the case n=2, why couldn't $\underline{h}\to\underline{0}$ along some path other than a straight line? Figure 1 discusses this idea in a bit more detail.



(1) In the expression f($\underline{a}$+$\underline{h}$)-f($\underline{a}$), no direction is mentioned. That is, for the given f, $\underline{a}$, and $\underline{h}$, f($\underline{a}$+$\underline{h}$)-f($\underline{a}$) regardless of whether we view the path as C_1,C_2, or C_3.

(2) The directional derivative <u>requires</u> that the path be C_2.

(Figure 1)

In other words, in referring to Figure 1, how would we indicate that we wanted to take a limit of f($\underline{a}$+$\underline{h}$)-f($\underline{a}$) along the curve C_3? We could invent some nice excuses, such as saying that near $\underline{a}$, C_3 is approximated adequately by the tangent to C_3 at $\underline{a}$, but is this what we really want to say? Or, if it is what we want to say, does the concept carry over into all n-dimensional spaces?

In any event, without worrying further about the reasons for our investigation, it turns out that a very acceptable definition of a derivative in n-space can be obtained by mimicking the definition given in (2) for 1-dimensional space.

3.4.7 continued

Given that $\underline{a}\varepsilon E^n$, we define a function f to be differentiable at $\underline{x}=\underline{a}$ if there exists a neighborhood $N(\underline{a})$ of $\underline{a}$ such that for every point (n-tuple) $\underline{a}+\underline{h}$ in $N(\underline{a})$

$$f(\underline{a}+\underline{h})-f(\underline{a}) = \underline{C}\cdot\underline{h}+k\|\underline{h}\| \tag{6}$$

where $\lim_{\|\underline{h}\|\to 0} k = 0$ and $\underline{C}$ is a constant vector which depends on f and $\underline{a}$ <u>but</u> <u>not</u> on $\underline{h}$.

While (6) may be accepted as a definition without further question, the more serious-minded among us might wonder how we decided to vectorize (2) to obtain (6). Clearly, once $\underline{C}$ replaced C and $\underline{h}$ replaced h, in terms of existing operations, we have no choice but to interpret the product as a dot product. Namely, since the left side of (6) is a number, the right side must also be a number, and only the dot product combines two vectors to produce a number.

As for the second term on the right side of (6) we observe that a very easy way to make sure that $\underline{h}\to\underline{0}$ independently of any direction is to make sure that its magnitude approaches 0. That is, $\|\underline{h}\|\to 0$ is equivalent to saying that $\underline{h}\to\underline{0}$ in every direction. Once we agree to replace h by $\|h\|$, k must also be a number since the right side of (6), as we have previously mentioned, is a number.

[Although we will certainly admit that one might have elected to use a more complete vectorization of (2) to obtain

$$f(\underline{a}+\underline{h})-f(\underline{a}) = \underline{C}\cdot\underline{h}+\underline{k}\cdot\underline{h}, \quad \lim_{\underline{h}\to 0} \underline{k} = \underline{0} \tag{6'}$$

our claim is that (6) serves our needs as well as (6').]

At any rate, suppose we use either (6) or (6') to determine the directional derivative of f at $\underline{a}$ in the direction of $\underline{h}$. We would let

$$\underline{u} = \frac{\underline{h}}{\|\underline{h}\|}$$

or $\underline{h} = \|\underline{h}\|\underline{u}$ (Here, $\|\underline{h}\|$ plays the role of t in our notes), whereupon equation (6) yields

3.4.7 continued

$$\frac{f(\underline{a}+\|\underline{h}\|\underline{u})-f(\underline{a})}{\|\underline{h}\|} = \underline{C} \cdot \frac{\underline{h}}{\|\underline{h}\|} + k \qquad (7)$$

(Had we used (6'), we would have obtained

$$\frac{f(\underline{a}+\|\underline{h}\|\underline{u})-f(\underline{a})}{\|\underline{h}\|} = (\underline{C}+\underline{k}) \cdot \frac{\underline{h}}{\|\underline{h}\|} \qquad)$$

In any event, from (7), we may deduce that

$$\lim_{\|h\|\to 0^+} \left[\frac{f(\underline{a}+\|\underline{h}\|\underline{u})-f(\underline{a})}{\|\underline{h}\|} \right] = \underline{C}\cdot\underline{u} \quad \left(\text{since } \lim_{\|h\|\to 0^+} k = 0\right) \qquad (8)$$

Now, we already know that

$$f'_{\underline{u}}(\underline{a}) = \lim_{\|h\|\to 0^+} \left[\frac{f(\underline{a}+\|\underline{h}\|\underline{u})-f(\underline{a})}{\|\underline{h}\|} \right] \underline{u}$$

so that (8) yields

$$f'_{\underline{u}}(\underline{a}) = (\underline{C}\cdot\underline{u})\underline{u} \qquad (9)$$

From (9), it follows that

$$\|f'_{\underline{u}}(\underline{a})\| = |\underline{C}\cdot\underline{u}| \qquad (10)$$

From (10), we see that $\|f'_{u}(\underline{a})\|$ is maximum when $\underline{u}$ has the same direction as $\underline{C}$ and when this happens

3.4.7 continued

$$|\underline{C}\cdot\underline{u}| = |\ \|\underline{C}\|\underline{u}\cdot\underline{u}\ | = |\ \|\underline{C}\|\ \|\underline{u}\|^2| = \|\underline{C}\| \text{ (since } \|\underline{u}\|=1)$$

In other words, $\underline{C}$ is the vector in whose direction $f'_u(\underline{a})$ has the maximum magnitude, and this maximum magnitude is precisely $\|\underline{C}\|$.

This, in terms of (6), says that $\underline{C}$ is what we have called $f'(\underline{a})$. The true beauty of (6), however, lies in the fact that it never uses the concept of direction and that, from it, we can derive the directional derivative in every direction. It is for this reason that definition (6) is given in the more elegant textbooks. In still other words, one can compute with (6) without any restriction as to direction.

As a final note, it should be observed that if one were to write (6) in n-tuple notation, we could let $\underline{a} = (a_1,\ldots,a_n)$, $\underline{C} = (c_1,\ldots,c_n)$, and $\underline{h} = (h_1,\ldots,h_n)$. Then (6) would become:

$$f(a_1+h_1,\ldots,a_n+h_n) - f(a_1,\ldots,a_n)$$

$$= c_1h_1+\ldots+c_nh_n + k\sqrt{h_1^2+,,,+h_n^2}$$

If we now divide each side by say, h_1 and let $h_2=\ldots=h_n = 0$, we obtain

3.4.7 continued

$$\frac{f(a_1+h_1,a_2,\ldots,a_n)-f(a_1,\ldots,a_n)}{h_1} = c_1 \pm k^* \quad ,$$

and now letting $h_1 \to 0$, we see that

$$f_{x_1}(a_1,\ldots,a_n) = \lim_{h\to 0} \frac{f(a_1+h_1,a_2,\ldots,a_n)-f(a_1,\ldots,a_n)}{h_1}$$

$$= c_1 \quad (\text{since } \lim_{\|h\|\to 0} k = 0)$$

In a similar way we see that

$c_2 = f_{x_2}(a_1,\ldots,a_n),\ldots,$ and $c_n = f_{x_n}(a_1,\ldots,a_n)$,

so that

$\underline{C} = (c_1,\ldots,c_n) = [f_{x_1}(\underline{a}),\ldots,f_{x_n}(\underline{a})]$

which agrees with the text's definition of gradient.

This is another way of showing that

$f'(\underline{a})$ and $\vec{\nabla} f(\underline{a})$

are equivalent.

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*Notice that $\frac{\sqrt{h_1^{\,2}}}{h_1} = 1$ if h_1 is positive but -1 if h_1 is negative. That is, $\sqrt{h_1^{\,2}} = |h_1|$.