Unit 2: An Introduction to Partial Derivatives

3.2.1(L)

As is usually the case, any amount of theory is of minimum value to the student unless he feels at ease with the type of computations that are involved in the implementation of the theory. For this reason, we feel it mandatory that this unit begin with an exercise which involves the mechanics of taking partial derivatives. Among other things, it will insure that you understand that our rather formal definition of a partial derivative translates into a relatively easy computational recipe.

a. We wish to compute $f_x(1,2)$ where f is defined by

$$f(x,y) = x^2 + y^3. \tag{1}$$

Now the definition of $f_x(1,2)$ $\left[= \left(\frac{\partial f}{\partial x}\right)_{(1,2)} \right]$ is

$$\lim_{\Delta x \to 0} \left[\frac{f(1 + \Delta x, 2) - f(1,2)}{\Delta x} \right]. \tag{2}$$

Notice that (2) essentially says that f is a function of x alone since y is being held constant at 2. (This is why we must insist that x and y be independent, for if x and y were dependent upon one another, we could not vary x and still hold y constant. In other words, if x and y are independent, we can let $\Delta y = 0$ while Δx can be unequal to 0, or we can let $\Delta x = 0$ while Δy varies.)

In other words, as far as (2) is concerned, (1) may have been written as

$$f(x,2) = x^2 + 8 = g(x). \tag{3}$$

[We write f(x,2) = g(x) to indicate that f depends only on x. We do not write f(x,2) = f(x) since f would then denote two different functions. Among other things, the domain of f in the context of f(x,2) would be E^2 while the domain of f in the context of f(x) would be E^1.]

3.2.1(L) continued

The connection between (1) and (3) is

$$f_x(1,2) = g'(1)$$

since in terms of g, (2) may be rewritten as

$$\lim_{\Delta x \to 0} \left[\frac{g(1 + \Delta x) - g(1)}{\Delta x} \right] = g'(x).$$

Trivially, $g'(x) = 2x$; hence, $g'(1) = 2$. Therefore,

$$f_x(1,2) = 2.$$

The point is that y could have been held fixed at any value, not necessarily 2. As long as y is being held constant, we may think of $x^2 + y^3$ in the same way as we would think of $x^2 + c^3$. In fact, this is how we obtained (3), with c = 2.

With respect to x, the derivative of $x^2 + c^3 = 2x$. Hence, $f_x(x,c) = 2x$.

We usually write this as $f_x(x,y) = 2x$, where it is understood that y is being held constant. We call the result a <u>partial</u> derivative in the sense that since the constant value of y was arbitrarily chosen, the value of $f_x(x,y)$ will usually depend on y as well as on x (even though in this part we chose an example where f_x depended only on x, in order to keep the arithmetic simple).

The advantage of beginning with an f whose domain was E^2 is that we can capitalize on the geometric interpretation. Namely, suppose we view f in terms of the graph defined by the surface z = f(x,y) shown in Figure 1. If we now say that we want to hold y at the constant value y_o, we are really defining the <u>plane</u> $y = y_o$. This plane intersects our surface along some curve C. We now pick some point (x_o, y_o) in the xy-plane (where y_o is the same value as above) and we look at the point on the surface which corresponds to this point in the plane. That is, we look at $P[x_o, y_o, f(x_o, y_o)]$. Then $f_x(x_o, y_o)$ is simply the slope of the curve C at the point P, provided, of course, that C is smooth at P. (Notice the possibility that the surface can be cut by infinitely many planes which pass

3.2.1(L) continued

through (x_o, y_o) and are perpendicular to the xy-plane. Each such plane will give rise to another curve of intersection C, each of which must pass through P. Depending on the "pathology" of the surface, it is possible that some of the C's are smooth at P while others aren't. The conditions which guarantee that every C is smooth at P are discussed in the next unit.)
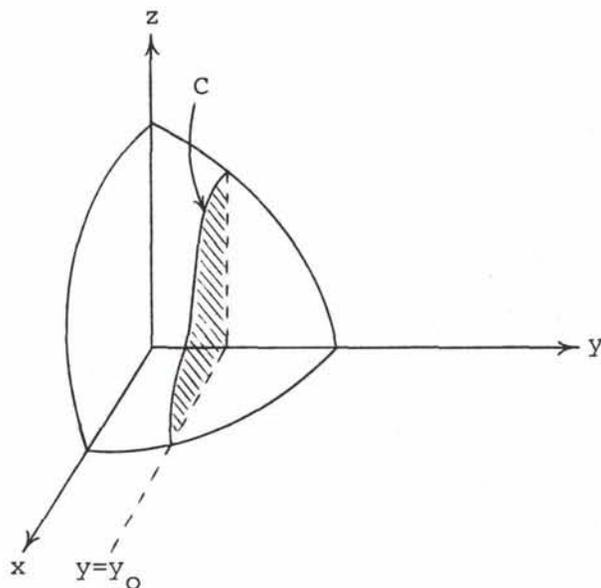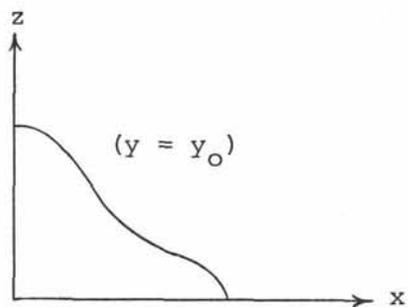

Figure 1

As viewed along the y-axis, C would look like

Figure 2

In Figure 2, C appears to be in the xz-plane and have an equation of the form $z = g(x)$. This is not quite true since C is in the

3.2.1(L) continued

plane $y = y_o$ which is parallel to the xz-plane, but displaced by y_o units from it. That is why Figure 2 must include the descriptive label $y = y_o$.

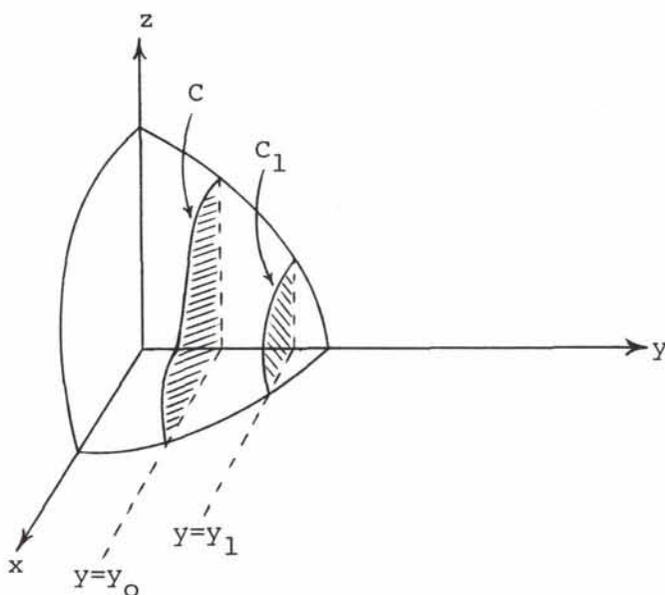Had we chosen a different value of y, say $y = y_1$, we would have obtained

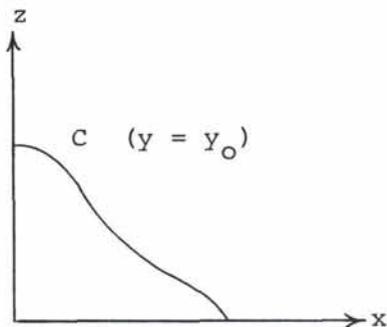Figure 3

so that from the y-axis, we would see

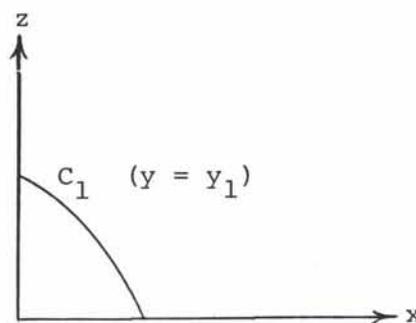Figure 4 - which is the same as Figure 2

Figure 5

3.2.1(L) continued

As we look at either Figure 4 or Figure 5, we see that our curves C and C_1 are of the form $z = g(x)$ (i.e. independent of y). Yet while this is true, we see that the shape of the curve does depend on the constant value we choose for y.

The next stage is that once $y = y_0$ is chosen, P is not specifically chosen on C until we fix a value of x, say $x = x_0$. Once this is done, we have

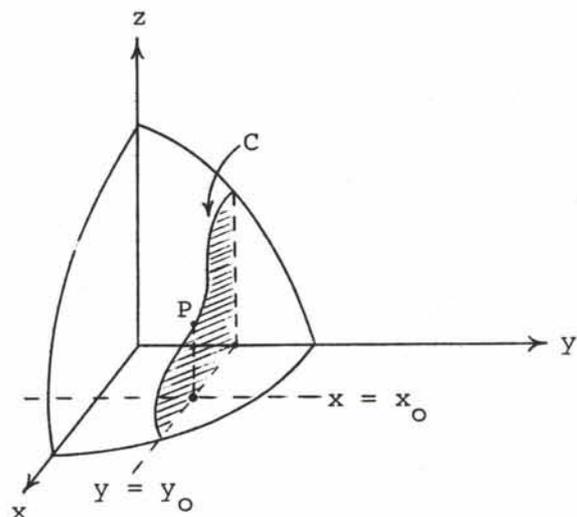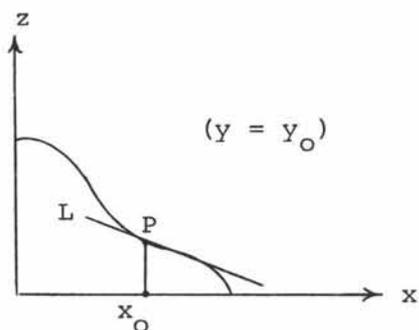

Figure 6

Now, $f_x(x_0, y_0)$, if it exists, is simply the slope of C at P. That is,

If L is tangent to C at P, its slope is

$$f_x(x_0, y_0) = \left(\frac{\partial z}{\partial x}\right)_{(x_0, y_0)}$$

Figure 7

3.2.1(L) continued

Notice from Figure 7 that P depends on the value of x_o. Hence, even when y is fixed, f_x still depends on x. Thus, $f_x(x_o, y_o)$ depends on both x_o and y_o. Thus, f_x is a function of both x and y in general.

b. $f_{xx}(1,2)$ means

$$\frac{\partial\left[\frac{\partial f}{\partial x}\right]}{\partial x}\Bigg]_{(1,2)}.$$

Now, since

$$f(x,y) = x^3y + x^4 + y^5 \tag{4}$$

$$f_x(x,y) = 3x^2y + 4x^3 \tag{5}$$

[where (5) is obtained from (4) by "pretending" y is some constant in (4)].

From (5), we see that f_x is also a function of x and y [say, $f_x(x,y) = h(x,y)$]. Hence, $f_{xx}(x,y) = h_x(x,y)$. Treating y as a constant in (5), we obtain

$$h_x(x,y) = f_{xx}(x,y) = 6xy + 12x^2$$

Therefore

$$f_{xx}(1,2) = 6(1)(2) + 12(1)^2 = 24.$$

c. Our main aim here is to make sure that you understand that the concept of a partial derivative holds for any number of <u>independent</u> variables.

We have

$$f(w,x,y,z) = w^2xy + z^3y^2 + x^3zw. \tag{7}$$

Again, $f_y(1,2,3,4)$ simply means to differentiate f as if y were the only variable (i.e. vary y while x, z, and w are held constant,

3.2.1(L) continued

and this is possible as long as our variables are independent).
This leads to

$$f_y(w,x,y,z) = w^2x + 2z^3y$$

Therefore,

$$f_y(1,2,3,4) = (1)^2(2) + 2(4)^3 3 = 386. \tag{8}$$

In terms of the basic definition, $f_y(1,2,3,4)$ could be computed
from

$$f_y(1,2,3,4) = \lim_{\Delta y \to 0} \left[\frac{f(1,2,3+\Delta y,4) - f(1,2,3,4)}{\Delta y} \right]. \tag{9}$$

As far as the bracketed expression in (9) is concerned, f is a
function of y alone since x, w, and z are "frozen" at w = 1, x = 2,
z = 4.

That is, from (7)

$$f(1,2,y,4) = (1^2)(2)(y) + (4)^3 y^2 + (2)^3(4)(1)$$

$$= 2y + 64y^2 + 32$$

$$= h(y).$$

Therefore,

$$f_y(1,2,y,4) = h'(y) = 2 + 128y.$$

Therefore,

$$f_y(1,2,3,4) = h'(3) = 2 + 384 = 386,$$

which agrees with (8).

3.2.1(L) continued

This is precisely what we would have obtained by evaluating the
limit in (9), just as we did to find derivatives when we first
studied them in Part 1 of our course.

In fact, to see this pictorially (and this will also show that
there are graphical ways of interpreting functions of n variables,
even though these are not what we usually mean by "graphical
ways"), notice that once w, x, and z are "frozen" at any values,
we have a function of y only. In our particular example, with
w = 1, x = 2, and z = 4, we saw that

$$h(y) = f(1,2,y,4) = 2y + 64y^2 + 32,$$

and if we were given no further details other than this equation,
we would not hesitate (we hope!) to graph it as
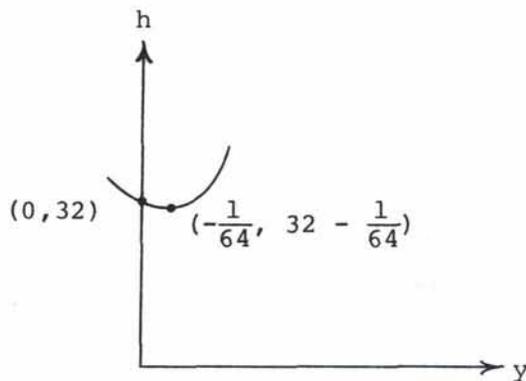
Then to indicate that h(y) was obtained only by specifying the
particular values w = 1, x = 2, and z = 4, we label the above
graph as:

Graph of f(w,x,y,z)
for w = 1, x = 2, z = 4.

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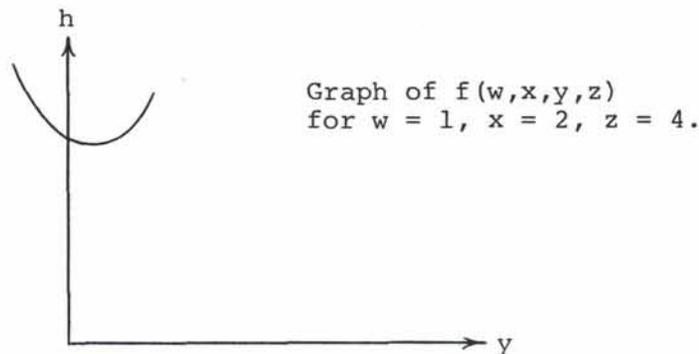
3.2.2

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a. $f(w,x,y,z) = w^3x^2y + x^3y^2z + wz^4$

   $f_w(w,x,y,z) = 3w^2x^2y + z^4$

   $f_{ww}(w,x,y,z) = 6wx^2y$

   $f_{ww}(1,2,3,4) = 6(1)(2)^2(3) = 72.$

b. We have

   $$z^3xy + z^5y + \cos z = 1. \tag{1}$$

   This defines z implicitly as a function of x and y. Thus, to find
   $\frac{\partial z}{\partial x}$ we treat y as a constant and differentiate (1) as we usually
   would by implicit differentiation with respect to x. We obtain

   $$3z^2 \frac{\partial z}{\partial x} xy + z^3y^* + 5z^4 \frac{\partial z}{\partial x} y - \sin z \frac{\partial z}{\partial x} = 0.$$

   Therefore,

   $$(3z^2xy + 5z^4y - \sin z) \frac{\partial z}{\partial x} = -z^3y.$$

   Therefore,

   $$\frac{\partial z}{\partial x} = \frac{-z^3y}{3z^2xy + 5z^4y - \sin z}.$$

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*Notice that since z depends on x, the partial derivative of z^3xy
involves a product of two functions of x (since y is being held
constant). Hence, to differentiate this term with respect to x,
we must use the product rule.

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3.2.3(L)

The main aim of this exercise is to emphasize a point that may not be as clear as it should be. Namely, it is very important when we take a partial derivative with respect to another variable to know exactly what the other independent variables are. Not understanding this causes a misinterpretation about the relationship between, say, $\frac{\partial u}{\partial x}$ and $\frac{\partial x}{\partial u}$.

Given that u = 2x - 3y, it is easy to see that if we assume that x and y are independent variables, $\frac{\partial u}{\partial x}$ = 2. Notice that when we say this we assume when we differentiate with respect to x that y is the variable that is being held constant. That is, it might have been wiser had we written, say, $(\frac{\partial u}{\partial x})_y$ = 2 to indicate that we are differentiating u with respect to x while y is being held constant. Admittedly, this might certainly seem clear from context without the more elaborate notation.

At any rate, we now let v = 3x - 4y. The preliminary point we want to make in part (a) is that u and v are also a pair of independent variables if x and y are.

a. We must first define what we mean by saying that u and v are independent, and, as we mentioned earlier, this means simply that we may pick a value for either u or v without being committed to a value for the other.

For example, suppose we let u equal a particular constant, say, c. Is v in any way restricted by this choice? (A similar discussion, of course, would hold if we picked a particular value for v and asked whether this restricted the choice of u.) From an algebraic point of view, the fact that u = 2x - 3y tells us that once we let u = c, $x = \frac{c + 3y}{2}$. Using this relationship and the fact that v = 3x - 4y, tells us that $v = 3(\frac{c + 3y}{2}) - 4y$, and this in turn tells us that no matter what constant value is assigned to c, we can make v equal to anything we want just by appropriately choosing a value for y. Thus, if we would like v to equal b when u = c, we have

$$v = \frac{3c + 9y - 8y}{2} = \frac{3c + y}{2}.$$

3.2.3(L) continued

Therefore,

$$b = \frac{3c + y}{2}$$

or

$$y = 2b - 3c.$$

While the algebraic approach works for any number of independent variables, we again have a nice pictorial representation in the case $n = 2$. $u = c$ means $2x - 3y = c$.

For arbitrary values of c, $2x - 3y = c$ is a family of parallel lines. (In particular, $2x - 3y = c$ implies $y = \frac{2}{3}x - \frac{c}{3}$, so that the slope of the line is $\frac{2}{3}$ and the y-intercept is $-\frac{c}{3}$.) $v = b$ means $3x - 4y = b$, and for arbitrary values of b, $3x - 4y = b$ is another family of parallel lines. (i.e., $3x - 4y = b \rightarrow y = \frac{3}{4}x - \frac{b}{4}$, so that the slope of each line is $\frac{3}{4}$ and the y-intercept is $-\frac{b}{4}$.)

Thus, without specifying either c or b, our lines look like
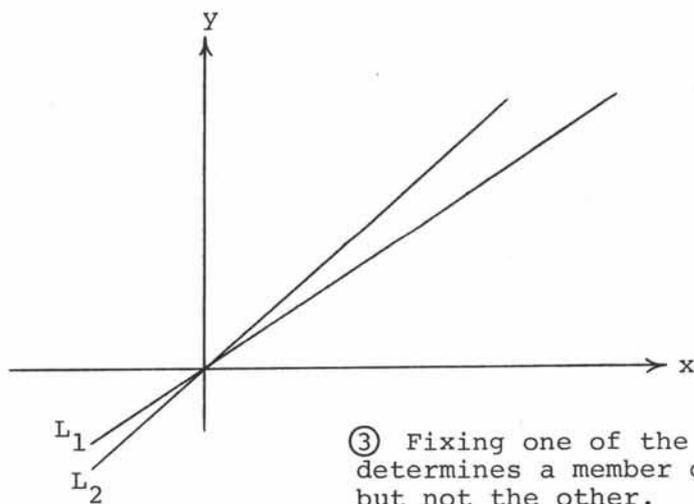
① Every line of the form $2x - 3y = c$ is parallel to L_1.

② Every line of the form $3x - 4y = b$ is parallel to L_2.

③ Fixing one of the two numbers b or c determines a member of one family of lines but not the other.

b. If we now "invert" the pair of equations

$$\left. \begin{array}{l} u = 2x - 3y \\ v = 3x - 4y \end{array} \right\} \tag{1}$$

3.2.3(L) continued

(i.e., we solve for x and y in terms of u and v) we see from (1)
that

$$4u = 8x - 12y$$
$$\underline{-3v = -9x + 12y}$$
$$4u - 3v = -x$$

Therefore,

$$x = -4u + 3v.$$

Similarly,

$$3u = 6x - 9y$$
$$\underline{-2v = -6x + 8y}$$
$$3u - 2v = -y$$

In other words,

$$\left. \begin{array}{l} x = -4u + 3v \\ y = -3u + 2v \end{array} \right\} . \tag{2}$$

Since we showed in (a) that u and v are independent variables, we
may compute $\frac{\partial x}{\partial u}$ in (2) to conclude that

$$\frac{\partial x}{\partial u} = -4.$$

Since $\frac{\partial u}{\partial x} = 2$ and $\frac{\partial x}{\partial u} = -4$, it appears that (as our text says)

$$\frac{\partial u}{\partial x} \neq \frac{1}{\frac{\partial x}{\partial u}}. \tag{3}$$

Our feeling is that (3) is a disturbing fact to most students.
After all, up to now, we have been showing how closely the calcu-
lus of a single variable parallels that of several variables, yet
(3) seems to contradict the corresponding result of 1-dimensional
calculus that $\frac{du}{dx} = (\frac{dx}{du})^{-1}$.

3.2.3(L) continued

This is the point of the elaborate notation when we said that $\frac{\partial u}{\partial x} = 2$ meant $\left(\frac{\partial u}{\partial x}\right)_y = 2$; and when we said that $\frac{\partial x}{\partial u} = -4$, we meant $\left(\frac{\partial x}{\partial u}\right)_v = -4$. Notice that if we change the pairing of the variables, algebraic changes take place. That is, $\left(\frac{\partial u}{\partial x}\right)_y$ indicates that u and v are expressed in terms of x and y, while $\left(\frac{\partial x}{\partial u}\right)_v$ indicates that x and y are expressed in terms of u and v.

As we shall show in parts (c) and (d), what <u>is</u> true is that $\left(\frac{\partial u}{\partial x}\right)_y = \frac{1}{\left(\frac{\partial x}{\partial u}\right)_y}$ and $\left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{\left(\frac{\partial u}{\partial x}\right)_v}$. That is, the result of 1-dimensional calculus is true <u>if we keep the variables properly aligned.</u>

c. To compute $\left(\frac{\partial x}{\partial u}\right)_y$, we are assuming that u and y are being considered as the independent variables and that v and x are being expressed in terms of u and y. (We leave it for you to check that y and 2x - 3y are indeed independent.)

To this end, $u = 2x - 3y \rightarrow$

$$x = \frac{u + 3y}{2} = \frac{1}{2}u + \frac{3}{2}y.$$

Therefore,

$$\left(\frac{\partial x}{\partial u}\right)_y = \frac{1}{2}.$$

This is in accord with the fact that $\left(\frac{\partial u}{\partial x}\right)_y = 2$.

d. $\left(\frac{\partial u}{\partial x}\right)_v$ implies that we would like to express u in terms of x and v. From $v = 3x - 4y$, we see that $y = \frac{3x - v}{4}$. Putting this in place of y in the expression $u = 2x - 3y$ leads to

3.2.3(L) continued

$$u = 2x - 3 \left(\frac{3x - v}{4}\right)$$

$$= 2x - \frac{9x + 3v}{4}$$

$$= -\frac{x + 3v}{4}$$

$$= -\frac{1}{4}x + \frac{3}{4}v.$$

Therefore,

$$\left(\frac{\partial u}{\partial x}\right)_v = -\frac{1}{4}$$

which checks with our earlier result that $\left(\frac{\partial x}{\partial u}\right)_v = -4$.

Thus, the problem of whether reciprocals are equal or not, depends
on the semantics of what variables are being considered as being
independent. In defense of the usual textbook-type statement that
$\frac{\partial u}{\partial x}$ is not equal to $\frac{\partial x}{\partial u}$ we should point out that when we are working
in 2-dimensional space, it is usually assumed without further re-
marks that the pair of independent variables are x and y. When we
then make a change of variables, say, u = u(x,y) and v = v(x,y),
where u and v also are independent, we recall that such a change
of variables was to simplify the original problem in one way or
another. In this context, it seems clear that we shall be either
expressing x and y in terms of u and v or u and v in terms of x
and y. There would seldom (though we do not rule out the possi-
bility) be a time when we would want to express, say, u and y in
terms of v and x. For this reason, it is usually understood when
we write $\frac{\partial u}{\partial x}$ that the other independent variable is y and when we
write $\frac{\partial x}{\partial u}$ the other independent variable is v. In this context,
the textbook remark is correct since $\left(\frac{\partial u}{\partial x}\right)_y$ need not equal $\left(\frac{\partial x}{\partial u}\right)_v$.

Note

In this case, we were able to invert the equations algebraically
to obtain the desired results. In many cases, this can only be

3.2.3(L) continued

done implicitly. Had we so desired, we could have started with

$u = 2x - 3y$

$v = 3x - 4y$

and differentiated implicitly with respect to u (meaning that we are assuming the other independent variable is v) to obtain

$$\begin{cases} 1 = 2\left(\frac{\partial x}{\partial u}\right)_v - 3\left(\frac{\partial y}{\partial u}\right)_v \\ \\ 0 = 3\left(\frac{\partial x}{\partial u}\right)_v - 4\left(\frac{\partial y}{\partial u}\right)_v \end{cases}$$

Therefore,

$$\begin{cases} 4 = 8\left(\frac{\partial x}{\partial u}\right)_v - 12\left(\frac{\partial y}{\partial u}\right)_v \\ \\ 0 = -9\left(\frac{\partial x}{\partial u}\right)_v + 12\left(\frac{\partial y}{\partial u}\right)_v \end{cases}$$

or

$$\left(\frac{\partial x}{\partial u}\right)_v = -4.$$

In this particular example, our first approach may have seemed simpler, but the key is that in the second approach, we were never required to solve explicitly for x and y in terms of u and v.
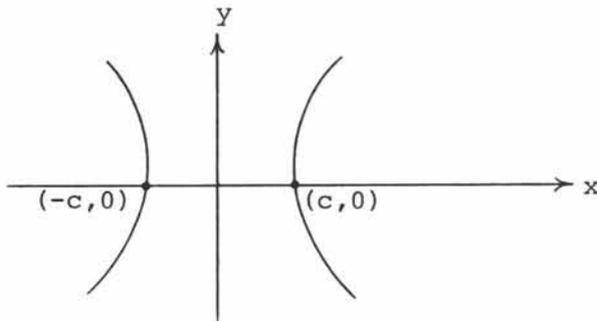
3.2.4

a. Given that $u = x^2 - y^2$ and $v = 2xy$ where x and y are independent variables, means that by specifying a particular value for u, say, $u = c$, merely implies that $c = x^2 - y^2$. Thus, for a given value

3.2.4 continued

of c, $y^2 = x^2 - c$, and we can now let y be chosen at random* simply
by supplying the appropriate value of x.**

Pictorially, $c = x^2 - y^2$ is a hyperbola

(Notice that our diagram
assumes that c > 0. If
c < 0, the role of the x
and y axes are inter-
changed, but this has no
bearing on the following
remarks.)

and we have no way of knowing in advance what y is until someone
supplies us with a value of x. That is, specifying a value for u
still leaves y unspecified. Hence, y and u are a pair of inde-
pendent variables.

b. $u = x^2 - y^2 \rightarrow \left(\dfrac{\partial u}{\partial x}\right)_y = 2x.$

We now wish to show that $\left(\dfrac{\partial x}{\partial u}\right)_y = \dfrac{1}{2x}.$ This means that we must
first express x in terms óf u and y.

From $u = x^2 - y^2$, we obtain, quite simply, that

$$x^2 = u + y^2.$$

Differentiating implicitly with respect to u, <u>holding y constant,</u>
yields

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*We must be careful to observe that since $x^2 = y^2 + c$, we cannot
solve for x as a real number unless $y^2 + c \geqslant 0$ (since $x^2 \geqslant 0$). Thus,
we must be a bit particular when we say "y may be chosen at random."
What we mean is that once the domain of y is determined from the
equation, y may be chosen at random <u>within the domain.</u>

**Notice we must again think in terms of "branches." Namely, a
<u>permissible</u> value of x determines <u>two</u> values of y, i.e.,
$y = \pm\sqrt{x^2 - c}.$

3.2.4 continued

$$2x\left(\frac{\partial x}{\partial u}\right)_y = 1$$

(since $\frac{\partial y^2}{\partial u} = 0$ because y and u are the independent variables).
Therefore,

$$\left(\frac{\partial x}{\partial u}\right)_y = \frac{1}{2x}.$$

c. This seems a bit messier since it seems that we must invert

$$\begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases} \tag{1}$$

Rather than do this algebracially (which can be done - in fact, we
supply it as a note at the end of this exercise), we proceed by
the note at the end of Exercise 3.2.3 and differentiate (1) im-
plicitly with respect to u (holding v constant). In this case,
$\frac{\partial v}{\partial u} = 0$ while x and y are functions of u and v. Thus,

$$\begin{cases} 1 = 2x\left(\frac{\partial x}{\partial u}\right)_v - 2y\left(\frac{\partial y}{\partial u}\right)_v \\ 0 = 2y\left(\frac{\partial x}{\partial u}\right)_v + 2x\left(\frac{\partial y}{\partial u}\right)_v \end{cases}$$

Therefore,

$$\begin{cases} x = 2x^2\left(\frac{\partial x}{\partial u}\right)_v - 2xy\left(\frac{\partial y}{\partial u}\right)_v & (x \neq 0) \\ 0 = 2y^2\left(\frac{\partial x}{\partial u}\right)_v + 2xy\left(\frac{\partial y}{\partial u}\right)_v & (y \neq 0) \end{cases}$$

Therefore,

$$(2x^2 + 2y^2)\left(\frac{\partial x}{\partial u}\right)_v = x$$

3.2.4 continued

Therefore,

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{x}{2x^2 + 2y^2} \qquad (x,y) \neq (0,0). \tag{2}$$

Note

In this case, we can explicitly express x and y in terms of u and v. Namely,

$$u = x^2 - y^2 \rightarrow u^2 = x^4 - 2x^2y^2 + y^4$$

$$v = 2xy \rightarrow v^2 = 4x^2y^2$$

Therefore,

$$u^2 + v^2 = x^4 + 2x^2y^2 + y^4 = (x^2 + y^2)^2$$

Therefore,

$$x^2 + y^2 = \sqrt{u^2 + v^2}$$

(where we take the positive root since $x^2 + y^2$ cannot be negative).
Thus,

$$x^2 + y^2 = \sqrt{u^2 + v^2}$$

$$x^2 - y^2 = u.$$

Therefore,

$$2x^2 = \sqrt{u^2 + v^2} + u \tag{3}$$

Therefore,

3.2.4 continued

$$4x \left(\frac{\partial x}{\partial u} \right)_v = \frac{1}{2} \left(u^2 + v^2 \right)^{-\frac{1}{2}} 2u + 1$$

$$= \frac{u}{\sqrt{u^2 + v^2}} + 1$$

$$\left(\frac{\partial x}{\partial u} \right)_v = \frac{u + \sqrt{u^2 + v^2}}{4x \sqrt{u^2 + v^2}} \tag{4}$$

[If we wish, from (3) we have that $\sqrt{2}x = \sqrt{\sqrt{u^2 + v^2} + u}$ and accordingly, we can express $\left(\frac{\partial x}{\partial u} \right)_v$ in (4) completely in terms of u and v.]

From the facts that $u = x^2 - y^2$ and $\sqrt{u^2 + v^2} = x^2 + y^2$, (4) becomes

$$\left(\frac{\partial x}{\partial u} \right)_v = \frac{(x^2 - y^2) + (x^2 + y^2)}{4x (x^2 + y^2)} = \frac{2x^2}{4x (x^2 + y^2)}$$

$$= \frac{x}{2 (x^2 + y^2)}$$

$$= \frac{x}{2x^2 + 2y^2},$$

which agrees with (2).

3.2.5(L)

One relationship between Polar and Cartesian coordinates takes the form
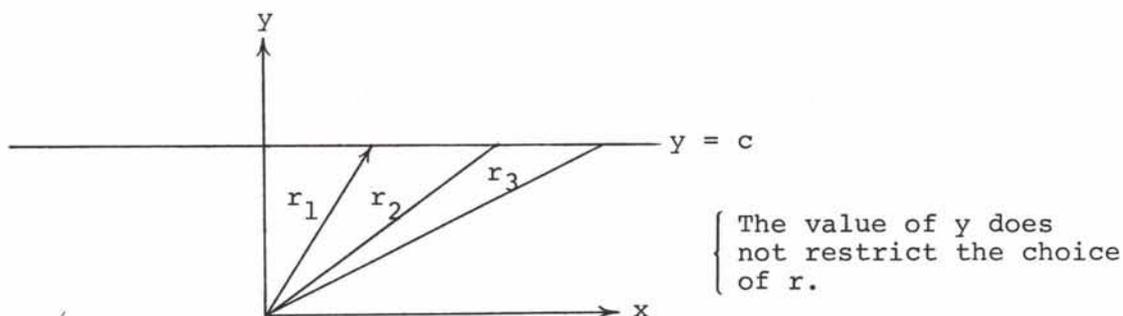
$$\sin \theta = \frac{y}{r}. \tag{1}$$

Suppose we wish to compute $\frac{\partial \theta}{\partial y}$ from (1).

The point here is again to be careful. Granted that y and r are independent (see picture below) the convention is that without a

3.2.5(L) continued

subscript $\frac{\partial \theta}{\partial y}$ means $\left(\frac{\partial \theta}{\partial y}\right)_x$. In this context, x and y are the inde-
pendent variables, while <u>r and θ depend on y and x.</u>

The value of y does
not restrict the choice
of r.

Thus, if we differentiate (1) with respect to y, we must remember
that r is implicitly a function of x and y (in this case,
$r^2 = x^2 + y^2$, but we don't have to know this).

At any rate, we obtain

$$\cos \theta \left(\frac{\partial \theta}{\partial y}\right)_x = \frac{r \frac{\partial y}{\partial y} - y \left(\frac{\partial r}{\partial y}\right)_x}{r^2}$$

$$\left(\frac{\partial \theta}{\partial y}\right)_x = \frac{r - y \left(\frac{\partial r}{\partial y}\right)_x}{r^2 \cos \theta} . \qquad (2)$$

Unless we know how r and y are related, we cannot simplify $\left(\frac{\partial r}{\partial y}\right)_x$
in (2). However, since $r^2 = x^2 + y^2$, we have that

$$2r \left(\frac{\partial r}{\partial y}\right)_x = 2y$$

Therefore,

$$\left(\frac{\partial r}{\partial y}\right)_x = \frac{y}{r}. \qquad (3)$$

3.2.5(L) continued

Putting (3) into (2) yields

$$\left(\frac{\partial \theta}{\partial y}\right)_x = \frac{r - y\left(\frac{y}{r}\right)}{r^2 \cos \theta} = \frac{r^2 - y^2}{r^3 \cos \theta}$$

or, since $\sin \theta = \frac{y}{r}$, $y^2 = r^2 \sin^2\theta$, and, therefore,

$$\left(\frac{\partial \theta}{\partial y}\right)_x = \frac{r^2 - r^2 \sin^2\theta}{r^3 \cos \theta} = \frac{r^2}{r^3}\left(\frac{1 - \sin^2\theta}{\cos \theta}\right) = \frac{\cos \theta}{r}.$$

As a check,

$$\sin \theta = \frac{y}{r} \rightarrow$$

$$\sin \theta = \frac{y}{\sqrt{x^2 + y^2}}.$$

Therefore,

$$\cos \theta \; \frac{\partial \theta}{\partial y} = \frac{\sqrt{x^2 + y^2} - y \; \frac{1}{2}\left(x^2 + y^2\right)^{-\frac{1}{2}} 2y}{x^2 + y^2}$$

$$= \frac{\sqrt{x^2 + y^2} - \dfrac{y^2}{\sqrt{x^2 + y^2}}}{x^2 + y^2}$$

$$= \frac{x^2 + y^2 - y^2}{\left(x^2 + y^2\right)^{\frac{3}{2}}}$$

$$= \frac{x^2}{r^3} = \frac{r^2 \cos^2\theta}{r^3} = \frac{\cos^2\theta}{r}.$$

Therefore,

$$\frac{\partial \theta}{\partial y} = \frac{\cos \theta}{r}.$$

3.2.5(L) continued

As a final note, had we looked at

$$\sin \theta = \frac{y}{r}$$

and written

$$\cos \theta \, \frac{\partial \theta}{\partial y} = \frac{1}{r}$$

this would have been correct <u>only</u> had we interpreted $\frac{\partial \theta}{\partial y}$ as $\left(\frac{\partial \theta}{\partial y}\right)_r$ since we are varying y but treating r as a constant.

<hr>

3.2.6

<hr>

We are assuming that $u = u(x,y)$ and $v = v(x,y)$ are independent and that

$$u^2 = y^2 v. \tag{1}$$

Therefore,

$$2u\left(\frac{\partial u}{\partial y}\right)_x = y^2\left(\frac{\partial v}{\partial y}\right)_x + 2yv$$

$$\left(\frac{\partial u}{\partial y}\right)_x = \frac{y^2\left(\frac{\partial v}{\partial y}\right)_x + 2yv}{2u}. \tag{2}$$

Again, and we are deliberately belaboring the point, had we written $2u \frac{\partial u}{\partial y} = 2yv$, it would have meant that $\frac{\partial u}{\partial y} = \left(\frac{\partial u}{\partial y}\right)_v$.

As a concrete example in which (1) would be obeyed, let $v = x^2$ and $u = xy$. Then clearly $u^2 = y^2 v$. In this event,

$$\left(\frac{\partial u}{\partial y}\right)_x = x, \text{ while } \left(\frac{\partial v}{\partial y}\right)_x = 0.$$
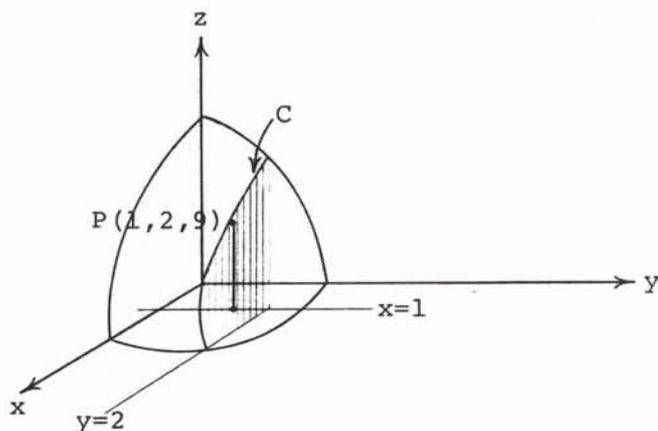
3.2.6 continued

Then (2) becomes

$$x = \frac{y^2(0) + 2yx^2}{2xy}$$

$$= \frac{2yx^2}{2xy} = x$$

which checks. Notice that there are infinitely many ways of choosing u and v so that $u^2 = y^2 v$. Namely, pick v = f(x,y) whereupon $u^2 = y^2 f(x,y)$ or $u = y\sqrt{f(x,y)}$.

3.2.7(L)

If the surface S, $z = x^2 + y^3$ does have a tangent plane at P(1,2,9), then this plane must contain the line tangent to the curve C at P obtained when the plane y = 2 intersects S. As we saw in Exercise 3.2.1(L), the slope of this line is $z_x(1,2) = (2x)_{x=1} = 2$. Pictorially,

(Note : We are making no attempt to draw $z = x^2 + y^3$ at all accurately; an accurate diagram would add nothing to our discussion.)
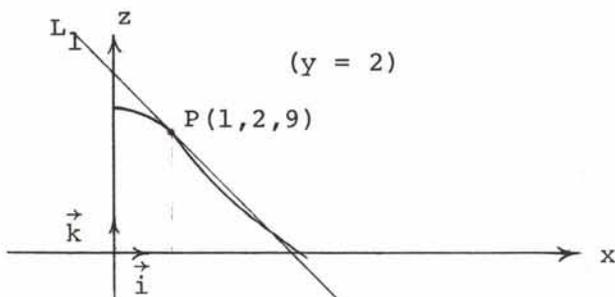
$$\text{slope of } L_1 = \left.\frac{\partial z}{\partial x}\right|_{x=1} = 2$$
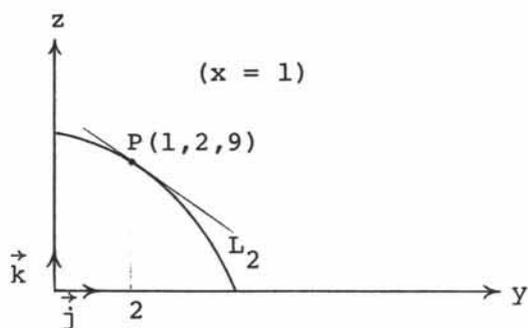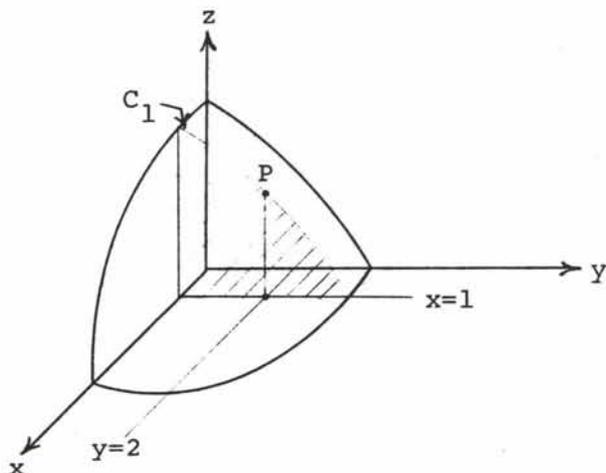
3.2.7(L) continued

Vectorizing L_1, we have that its slope must equal 2 and the slope of $\vec{i} + 2\vec{k}$ is 2.

Thus, we may view L_1 (since all we care about is slope, not magnitude) as the vector

$$\vec{v}_1 = \vec{i} + 2\vec{k}. \tag{1}$$

In a similar way, the line tangent to C_1 at P, where C_1 is the curve obtained when the plane $x = 1$ intersects S, must also lie in the tangent plane. The slope of this line is $z_y(1,2) = 3y^2\Big|_{y=2} = 12$.

Again pictorially,

slope of $L_2 = \dfrac{\partial z}{\partial y}\Big|_{y=2} = 12$

Vectorizing L_2 yields

$$\vec{v}_2 = \vec{j} + 12\vec{k}. \tag{2}$$

3.2.7(L) continued

Since $\vec{v}_1$ and $\vec{v}_2$ must lie in the tangent plane (where, of course, we assume that $\vec{v}_1$ and $\vec{v}_2$ originate at P), their cross product $\vec{N}$ is a vector normal to the plane. Therefore,

$$\vec{N} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 0 & 1 & 12 \end{vmatrix} = -2\vec{i} - 12\vec{j} + \vec{k}.$$
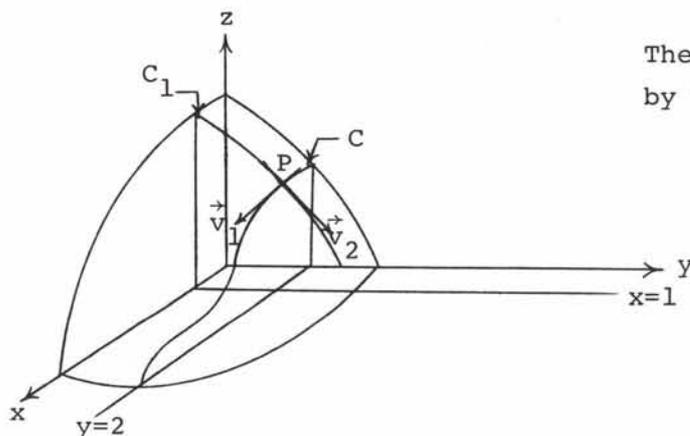
Since P(1,2,9) is a point in the plane and $\vec{N}$ is normal to the plane, the equation of the plane is given by

$$-2(x-1) - 12(y-2) + 1(z-9) = 0$$

or

$$z = 2x + 12y - 17.$$

As a pictorial summary,

The plane is determined by $\vec{v}_1$ and $\vec{v}_2$.

The main point of this exercise in general is that a tangent plane replaces the idea of a tangent line when we deal with graphs of functions of two independent variables.

3.2.7(L) continued

What we have shown in this exercise is that if* the surface S
given by $z = f(x,y)$ has a tangent plane at the point $P\big(x_o, y_o,$
$f(x_o, y_o)\big)$ on S, then this plane must in particular contain two
special tangent lines (among infinitely many others). One is the
line tangent to C at P where C is the curve obtained when the
plane $y = y_o$ intersects S. The slope of this line is $f_x(x_o, y_o)$.
Vectorizing this line leads to

$$\vec{V}_1 = \vec{i} + f_x(x_o, y_o)\ \vec{k}.$$

In a similar way, the line tangent to C_1 at P must also lie in the
tangent plane, where C_1 is the intersection of our surface with
the plane $x = x_o$. The slope of this tangent line is $f_y(x_o, y_o)$ and
hence vectorizing it leads to

$$\vec{V}_2 = \vec{j} + f_y(x_o, y_o)\ \vec{k}.$$

Then $\vec{V}_1 \times \vec{V}_2 = \vec{N}$ is normal to the plane and $P\big(x_o, y_o, f(x_o, y_o)\big)$ is
in the plane, so the equation of the plane is

$$z - z_o = f_x(x_o, y_o)(x - x_o) + f_y(x_o, y_o)(y - y_o)$$

where

$$z_o = f(x_o, y_o).$$

Since z is measured to the tangent plane, this equation may be
written in the more suggestive form

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*Notice that in this exercise we keep talking about what happens
if there is a tangent plane. The idea is that we cannot, as yet,
be sure that the surface has such a tangent plane (where tangent
plane is as defined in the text). In the next unit, we shall dis-
cuss conditions under which we can be sure that the surface does
possess a tangent plane (i.e., when the surface is "sufficiently
smooth" in a neighborhood of the point). For now, we have art-
fully dodged this issue by simply asking what the tangent plane
would look like if it exists. It happens in this particular exer-
cise, that the tangent plane does indeed exist.

3.2.7(L) continued

$$\Delta z_{tan} = \frac{\partial z}{\partial x} \Bigg|_{(x_o, y_o)} \Delta x + \frac{\partial z}{\partial y} \Bigg|_{(x_o, y_o)} \Delta y. \tag{3}$$

Notice that $\frac{\partial z}{\partial x} \Delta x$ suggests the change in z due to x, while $\frac{\partial z}{\partial y} \Delta y$ suggests the change in z due to y. Since x and y are independent, $\frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y$ suggests the total change in z. The reason that Δz_{tan} appears rather than Δz is that in the tangent plane

$$\frac{\partial z}{\partial x} \Bigg|_{(x_o, y_o)} \quad \text{and} \quad \frac{\partial z}{\partial y} \Bigg|_{(x_o, y_o)} \quad \text{are constants.}$$ (In other words, just as

with calculus of a single variable, $\frac{dy}{dx} \Delta x$ is not Δy but Δy_{tan} since in general $\frac{dy}{dx}$ is a variable quantity, and is constant only along the tangent line.)

In any event, after it has been derived once (to convince yourself it's correct), equation (3), in terms of the above discussion, should be easy to memorize.

3.2.8

We have

$$z = x^3 y^2 + x^5 + y^7$$

Therefore,

$$\frac{\partial z}{\partial x} = 3x^2 y^2 + 5x^4$$

Therefore,

$$\frac{\partial z}{\partial x} \Bigg|_{(1,1)} = 8$$

$$\frac{\partial z}{\partial y} = 2x^3 y + 7y^6$$

3.2.8 continued

Therefore,

$$\frac{\partial z}{\partial y}\Bigg]_{(1,1)} = 9$$

Therefore,

$$\Delta z_{tan} = 8\Delta x + 9\Delta y$$

i.e.,

$$z - 3 = 8(x - 1) + 9(y - 1)$$

or

$$z = 8x + 9y - 14.$$

3.2.9(L)

a. There is no reason why the equation of a surface must have the
form $z = f(x,y)$. There will be times when the best we can do is
express z as an implicit function of x and y, in which case the
surface has the equation $h(x,y,z) = 0$. There will be other times,
such as in this exercise, when the equation will be given in the
form $x = g(y,z)$. What we are asking in this exercise is what the
equation of the plane tangent to this surface at the point
$P\left(g(y_o,z_o),y_o,z_o\right)$ will be, in terms of derivatives with respect
to y and z (since the form of the equation makes it easier to
differentiate with respect to these two variables). We probably
expect that $\Delta x_{tan} = \frac{\partial x}{\partial y}\Delta y + \frac{\partial x}{\partial z}\Delta z$.

To see that this is true "from scratch," notice that the form of
our equation makes it advisable from a graphical point of view to
let the x-axis denote the height (rather than letting the z-axis
do this). Remembering the orientation of the axes, we have
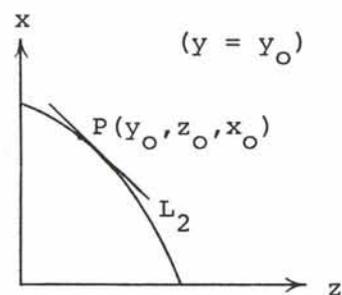
3.2.9(L) continued
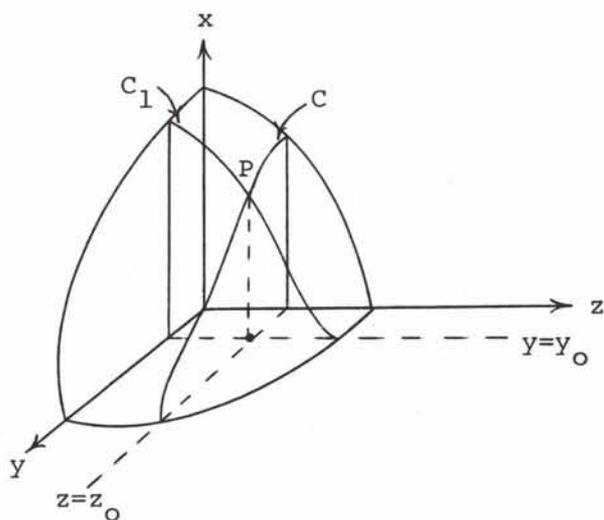
(i.e., rotating x into y must turn a right handed screw in the direction of z)

(Notice that in this orientation, points are more naturally labeled (y,z,x), even though any consistent scheme suffices.)
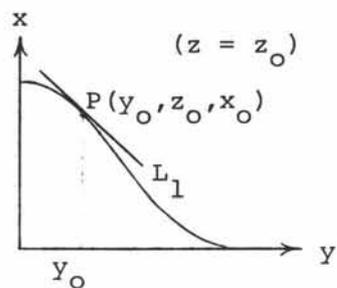
slope of $L_1 = x_y(y_0,z_0)$

Therefore, $\vec{V}_1 = \vec{j} + x_y(y_0,z_0)\,\vec{i}$

slope of $L_2 = x_z(y_0,z_0)$

Therefore, $\vec{V}_2 = \vec{k} + x_z(y_0,z_0)\,\vec{i}$

Therefore,

$$\vec{V}_1 \times \vec{V}_2 = \begin{vmatrix} \vec{j} & \vec{k} & \vec{i} \\ 1 & 0 & x_y(y_0,z_0) \\ 0 & 1 & x_z(y_0,z_0) \end{vmatrix}$$

$$= \left(-\,x_y(y_0,z_0),\; -\,x_z(y_0,z_0),\; 1\right).$$

3.2.9(L) continued

Therefore,

$$- x_y(y_o, z_o)(y - y_o) - x_z(y_o, z_o)(z - z_o) + 1(x - x_o) = 0$$

is the equation of the tangent plane.

Again there was no need to use the orientation (y, z, x) other than to conform to our picture. Using the picture allowed us to mimic more simply the previous case of $z = f(x, y)$. For example, if we feel more at home with the usual (x, y, z) orientation, all we need say is

$$\vec{N} = \left(1, -x_y(y_o, z_o), -x_z(y_o, z_o)\right)$$

$$= \vec{i} - x_y(y_o, z_o) \vec{j} - x_z(y_o, z_o) \vec{k}.$$

b. Given $x = e^{3y-z}$ at $(1, 2, 6)$, we have

$$\frac{\partial x}{\partial z} = -e^{3y-z} \quad \text{and} \quad \frac{\partial x}{\partial y} = 3e^{3y-z}$$

Therefore,

$$x_z(1, 2, 6) = -e^{3(2)-6} = -e^{o} = -1$$

$$x_y(1, 2, 6) = 3e^{3(2)-6} = 3e^{o} = 3.$$

Hence, from (a), the tangent plane is

$$x - 1 = x_z(1, 2, 6)\Delta z + x_y(1, 2, 6)\Delta y$$

$$= -(z - 6) + 3(y - 2)$$

or

$$x - 1 = -z + 6 + 3y - 6$$

$$x - 3y + z = 1.$$

3.2.9(L) continued

c. Here we can check (b) directly. Namely,

$x = e^{3y-z} \rightarrow$

$x = e^{3y} e^{-z} \rightarrow$

$e^z = \dfrac{e^{3y}}{x}$

Therefore,

$$\ln(e^z) = \ln\left(\dfrac{e^{3y}}{x}\right) = \ln e^{3y} - \ln x$$

or

$z = 3y - \ln x$

Therefore,

$z_x = -\dfrac{1}{x}, \ z_y = 3$

$z_x(1,2) = -1, \ z_y(1,2) = 3$

Therefore,

$\Delta z_{tan} = z_x(1,2)\Delta x + z_y(1,2)\Delta y$

$z - 6 = -(x - 1) + 3(y - 2)$

Therefore,

$x - 3y + z = 1$

which checks with (b).

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