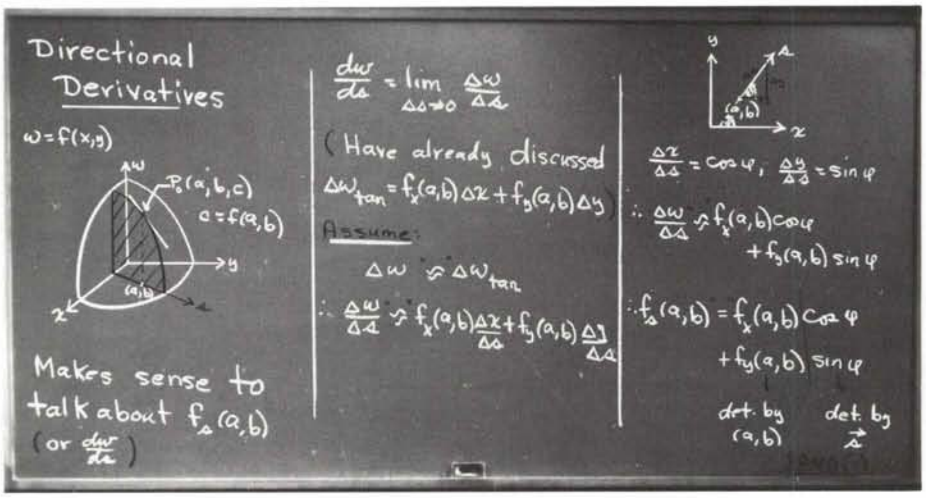
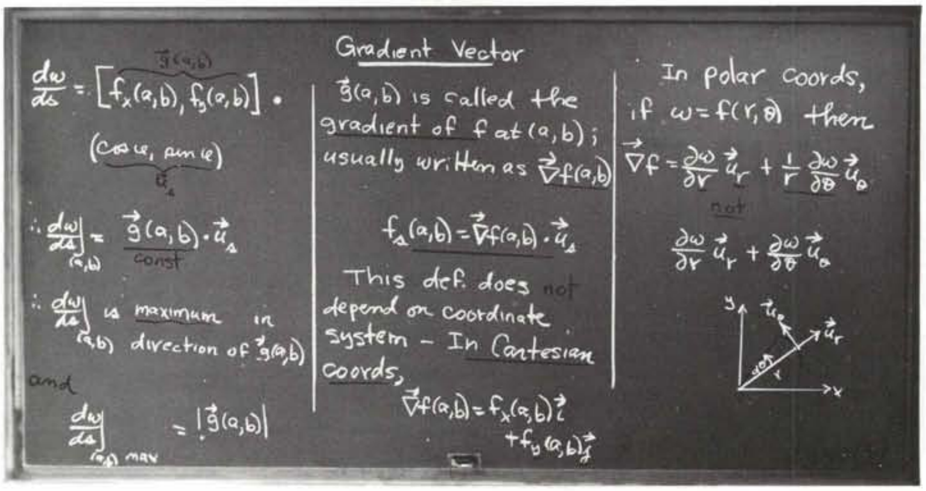
Unit 3: Differentiability and the Gradient

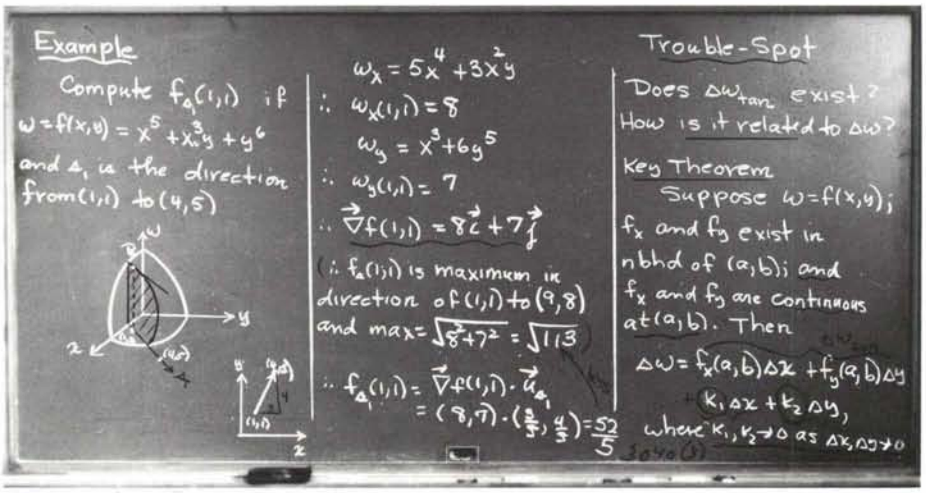
1. Lecture 3.030



a.



b.



c.

2. Read Thomas, Sections 15.4, 15.5, and 15.6.

3. Exercises:

3.3.1(L)

Define g by

$$g(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Show that $g_x(0,0)$ and $g_y(0,0)$ both exist, yet g is not continuous at (0,0). Does this tell us anything about the behavior of either g_x or g_y at (0,0)? Explain.

3.3.2

a. Let $w = f(x,y) = x^2+x^3y + y^3$. Express $f(x + \Delta x, y + \Delta y)$ in the form

$$f(x+\Delta x,y+\Delta y) = f(x,y)+(f_x(x,y)\Delta x+f_y(x,y)\Delta y+k_1\Delta x+k_2\Delta y)$$

where k_1 and $k_2 \to 0$ as Δx and $\Delta y \to 0$.

b. Use (a) to find an approximation for

$$(1.001)^2 + (1.001)^3(1.002) + (1.002)^3$$

3.3.3(L)

Let $w = f(x,y) = x^5+x^3y^2+y^4$

a. Find the value of $\frac{dw}{ds}$ at (1,2) if s is the direction of the line from (1,2) to (3,4). [Do not use the concept of $\vec{\nabla} f$ here.]

b. Derive the same result using $\vec{\nabla} f(1,2)$.

c. Use $\vec{\nabla} f(1,2)$ to compute $\frac{dw}{ds}$ at (1,2) if s is the line which passes through (1,2) and (4,6).

d. What is the maximum value that $\frac{dw}{ds}$ can have at (1,2), and in what direction does this maximum take place?

3.3.4

Let $w = f(x,y) = e^x \cos y + x^2 y$

a. Compute $\vec{\nabla} f(\ln 2, \frac{\pi}{2})$.

b. Compute $\frac{dw}{ds}$ at $(\ln 2, \frac{\pi}{2})$ in the direction $\vec{i}$.

c. Compute $\frac{dw}{ds}$ at $(\ln 2, \frac{\pi}{2})$ in the direction $\vec{j}$.

d. In what direction is $\frac{dw}{ds}$ at $(\ln 2, \frac{\pi}{2})$ maximum and what is this maximum value?

3.3.5(L)

a. Given that $w = f(r,\theta) = r \sin\theta$, compute $\frac{\partial w}{\partial r}\vec{u}_r + \frac{\partial w}{\partial \theta}\vec{u}_\theta$ in terms of x, y, $\vec{i}$, and $\vec{j}$.

b. Rewrite w in terms of x and y and compute $\frac{\partial w}{\partial x}\vec{i} + \frac{\partial w}{\partial y}\vec{j}$ and compare this with (a) to show that $\frac{\partial w}{\partial r}\vec{u}_r + \frac{\partial w}{\partial \theta}\vec{u}_\theta$ is <u>not</u> $\vec{\nabla} w$.

3.3.6(L)

Determine the normal to the curve $y = x^2+4$ at any point by considering $\vec{\nabla} f$ where $f(x,y) = y-x^2$.

3.3.7(L)

Find a vector normal to the surface S at the point (1,1,1) if the Cartesian equation of S is

$$x^5 + y^4 z^3 + xyz^5 = 3 \quad .$$

3.3.8

Find the equation of the plane which is tangent to

$x^4 + y^6 z + xyz^5 = 3$

at (1,1,1).

3.3.9 (Optional)

(The main aim of this exercise is to give additional insight into understanding why if $f(x_1,\ldots,x_n)$ that $f_{x_1},\ldots,$ and f_{x_n} must be continuous (as well as merely exist) at $\underline{a} = (a_1,\ldots,a_n)$ if we are to be able to consider that f is differentiable at $\underline{x} = \underline{a}$. The key structural idea is that the basic formula in 1-dimensional space

$$f(a + \Delta x) - f(a) = f'(a)\,\Delta x + k\Delta x, \quad \text{where } \lim_{\Delta x\to 0} k = 0$$

has a rather nice n-dimensional analog.

Part (a) of this exercise has us simply rewrite a previously obtained result in vector form so that we get an idea of what is happening, and part (b) has us generalize this result into 4-dimensional space.

This exercise bridges the gap between the fairly intuitive approach of the text and our approach in the next (optional) unit wherein we shall try to define differentiability in terms of preserving the structure of the 1-dimensional derivative, without reference to those special derivatives $f_{x_1},\ldots,f_{x_n}$.)

a. Letting $\underline{x} = (x_1,x_2)$, we have already seen that if f_{x_1} and f_{x_2} exist and are continuous at $\underline{a} = (a_1,a_2)$ then

$$\Delta f \; [= f(a_1+\Delta x_1,\ a_2+\Delta x_2) - f(a_1,a_2)]$$

$$= f_{x_1}(a_1,a_2)\Delta x_1 + f_{x_2}(a_1,a_2)\Delta x_2 + k_1\Delta x_1 + k_2\Delta x_2$$

where k_1 and k_2 approach 0 as Δx_1 and Δx_2 approach 0. Rewrite this result in vector notation.

b. Show that the result of part (a) can be extended into n-dimensional space, using as a specific illustration the 4-dimensional case. That is, assume $f:E^4\to E$, or, in other words, using n-tuple notation, f is a function of the four independent variables, $x_1,x_2,x_3,$ and x_4, so that we have $f(x_1,x_2,x_3,x_4)$.