HERBERT GROSS: Hi, today we start a new segment of our course. It could be entitled 'Techniques of Integration'. And a subtitle could be the difference between 'knowing what to do and knowing how to do it'. You see, essentially in this chapter, it shall be our aim to develop various recipes to evaluate the inverse derivative.

So in fact, let's call it that. Some basic recipes. The idea is something like this. Let's start with an easier problem. Let's suppose we start with a differentiable function 'G'. Suppose, for example, that 'G' prime is equal to 'f'. Then in terms of the language of the indefinite integral, or the inverse derivative, what we have is that the indefinite integral of 'f of x' with respect to 'x' is then 'G of x' plus 'c'.

Now, you see, the harder problem is the inverse of this. What we're going to be talking about is not do you find 'f' once 'G' is given, but techniques for finding 'G' once 'f' is given. And as we've seen on several occasions already, that given 'f', it is not that easy, in general, to find a function whose derivative is 'f'. In fact, in many cases, we can't do it as explicitly as we would like to no matter how sharp we are.

For example, oh, I've just reviewed this so that you can think of this in one block. Given 'f' to require 'G' may not exist in familiar form. I don't want to pin down what familiar means any more than that right now.

For example, the illustration that we've used several times already in the course, suppose 'f of x' is 'e to the minus 'x squared''. And we want a function whose derivative is 'e to the minus 'x squared''. In terms of areas, the second fundamental theorem of integral calculus, we can construct such a function in a rather straightforward way, even though it may be difficult to name it in terms of familiar functions.

Namely, what we do is we take the curve 'y' equals 'e to the minus 't squared'' in the yt-plane. Compute the area bounded above by that curve, below by the t-axis, on the left by the y-axis, and on the right by the line 't' equals 'x'. And the area under the curve is a function of 'x'. And what we've already seen is that in this particular case, 'A prime of x' is precisely 'e to the minus 'x squared''. In other words, this area function, which is a function of 'x', does turn out to be the function whose derivative with respect to 'x' is 'e to the minus 'x squared''.

Now you see, that will not be primarily what we're interested in, in this particular phase of our course. What we're interested in this phase of the course is given various classifications for the function 'f', to find recipes that will give us 'G' in familiar form, where 'G' is the function whose derivative is 'f'. Summarize for you to look at here.

The objective of this block of material is to find recipes for finding 'G of x' for various types of 'f of x'.

Now the best way to illustrate what that means is to actually pick a few examples. And by the way, what you'll notice in today's lecture is, roughly speaking, we do nothing at all that's new. The new stuff will begin with out next lecture essentially. But for now what we'll really be doing is giving us an excuse to pull together all of the recipes that we've developed so far. All of the recipes either called indefinite integrals or inverse derivatives.

For example, the first class of functions which we could handle were things of the form integral ''u to the n' du'. Recall that very early in our course, we found out that the integral of ''u to the n' du' was ''u to the 'n + 1'' over 'n plus 1' plus a constant if 'n' was unequal to minus 1. And then, in our last block of material we learned to handle the case where 'n' was equal to minus 1. Namely, when 'n' is minus 1, integral ''u to the minus 1' du'. Namely, integral 'du' over 'u' is natural log absolute value of 'u' plus a constant.

Now this then, is one of our basic building blocks. What we're saying is, if an indefinite integral can be written in this form, we already have a recipe that allows us to evaluate the integral.

By the way, let us also point out as was mentioned again, several times in the past, that 'u' is the generic name for a variable over here. It is not important in this recipe that we use 'u'. What is important is that whatever is being raised to the n-th power inside the integrand must be precisely the thing that's following the 'd' over here.

For example, instead of 'u', suppose I used 'sine x' over here. In other words, suppose I had 'sine x' to the n-th power 'd sine x'. See, what's being raised to the n-th power is the same thing as what's following the 'd'. That's all I really care about. In other words, if this had been the integrand, the recipe would have said, this is 1 over 'n + 1' times 'sine x' to the 'n + 1' power. Plus a constant if 'n' is not equal to minus 1. And it's natural log absolute value of 'sine x' plus a constant if 'n' is equal to minus 1. Again, the important thing is not the 'u', but the fact that what's being raised to the n-th power is the same as what's following the 'd' over here. By the way, we can write this in more familiar form.

You see, namely, notice that the differential of 'sine x' is 'cosine x dx'. And consequently, in this form what our recipe says is that integral 'sine x' to the n-th power times 'cosine x dx' is given by this. In other words, it's sine to the 'n + 1' power 'x over 'n + 1'' plus a constant if 'n' is unequal to minus 1. And it's the natural log absolute value of 'sine x' plus 'c' if 'n' is equal to minus 1.

Again, notice this looks more complicated than this. But to get our problem, to get into the form ''u to the n' du', notice that the natural substitution here is to let 'u' equal 'sine x'. In which case, 'du' would be 'cosine x dx'. In other words, the extra factor of 'cosine x' here actually is to our advantage over here. That when we make the substitution, it gives us the form that we want. With the factor missing, we're in a little bit of trouble.

By the way, this is what makes this a rather exciting topic. That there seems to be no really canned ways of finding indefinite integrals. That we have recipes. Frequently, the recipe that we have doesn't cover the situation that we have. In fact, I know a lot of times in teaching a course like this students say, why do we have to learn these recipes and techniques? Why can't we just look them up in the tables? And the answer quite simply, is that in many applications, the integral that we have doesn't appear in that form in the table. That frequently what we must do is a tremendous amount of algebraic manipulation and the like to even get the given integral to resemble one that we can find in the table.

Oh, we're starting off on a fairly straightforward level here. Nothing really too tricky. But at least enough so that we get an idea of what kind of gimmicks are involved.

For example, suppose we had 'cosine x' to the seventh power 'dx'. You see the idea here is that when you have cosine raised to a power, it would have been nice if a 'sine x' factor had been in here. Because the differential of 'cosine x' is minus 'sine x dx'. And what I'm driving at here is that we often have to have a certain kind of ingenuity that allows us to reduce a new integrand to a more familiar one. And, as I say, there are many techniques done in the textbook. There'll be plenty of exercises again, that will allow you to experiment on this. All I want to do is hit a few highlights and create the mood as to what's going on.

For example, you see frequently what happens is, if we have 'cosine x' raised to an odd power, a very convenient way of handling this is to split off one of the factors of 'cosine x' separately. In other words, write this as 'cosine x' to the sixth power times 'cosine x dx'. The idea being that cosine squared can be written as '1 minus 'sine squared''. In other words, 'cosine x' to the sixth power is really 'cosine squared x' cubed. But 'cosine squared x' is '1 minus 'sine squared x''. In other words, notice again the role of the trigonometric identities.

I take this integrand, write it this way, and now you see if I use the binomial theorem and expand this, notice that my typical term will have 'sine x' to a power multiplied by a 'cosine x'. And that's precisely the kind of a term that I handled in the previous case. And again, I'm not going to bore you with the details here. That is the easiest part of the problem. The hard part is getting to understand why one would want to use this particular kind of form.

Another variation is, what happens if the cosine is raised to an even power rather than to an odd power? How do we handle that? Again, the same kind of use of trigonometric identities.

For example, it would not be to our advantage in this case, to replace cosine squared by 1 minus sine squared. Because then we would wind up with the same kind of integrands that we started with, only using powers of sine instead of powers of cosine. You see, the idea is that if we have a power of sine, we want a cosine factor to accompany it. And inversely, if we have a power of cosine, we'd like a factor of sine to accompany it.

Why? So that we can get the integral into the form ''u to the n' du'.

In this particular case, a very common device is to invoke the identity that cosine squared of an angle is 1 plus cosine twice the angle over 2. You see again, this is where all of these particular identities come into play. It's not a case of saying, let's learn these identities so that we can recite them. It's a case of getting into a situation where you want a certain answer, can't handle it conveniently in the form that you're in, and hope that you can pick off a synonym that allows you to tackle the problem more successfully. That's where this becomes a matter of insight, lucky guessing, skill, whatever you want to call it. It's one thing to transform an integral into an equivalent one, and another thing to have that equivalent integral be something that you can handle.

Well, at any rate, for example in a problem of this type, a tendency is to write cosine sixth theta as cosine squared cubed. Cosine squared theta is 1 plus cosine 2 theta over 2. To cube this we can take out the factor of 1/8. You see, we use the binomial theorem to expand 1 plus cosine 2 theta cubed. And notice that of the terms that we get, this term we can handle. This term we can handle, it's just cosine to the first power. Which involves a sine term when we integrate. And here's a cosine cubed. But we've just seen how you can handle something to an odd power, so that reduces this term to a type that we can handle. And then we see what? That we have another term in here, which is cosine squared 2 theta. That's an even power of cosine. Again, we do what? We write cosine squared 2 theta as 1 plus cosine of twice this angle. That's 1 plus cosine 4 theta over 2. In other words, we keep reducing these things, always hoping to break them down into collections of problems that we solved before.

As I said, these are things that we'll drill on in the exercises and I think it will become more meaningful there.

Here, as I say, the main aim is to create the mood that what we're trying to do is to reduce unfamiliar problems to equivalent familiar ones to which we already know the answer.

Now again, in the way of review, sometimes an integrand occurs in the form of the sum or difference of two squares. We've already handled this in terms of the inverse circular functions and the inverse hyperbolic functions. It's covered for the first time in our textbook at this particular stage of the course. And I think it's a tough enough concept so that it's worth emphasizing at this stage. So I just separate this particular type out. How do we handle sums and differences of squares?

And what I intend to show is, is that by a suitable trigonometric function, either circular trigonometric or hyperbolic trigonometric, we can always reduce that kind of a problem to the type that we can handle. Again, to keep computation at a minimum, I have not tried to pick a rather complicated situation. In fact, I've picked ones where you can very easily look them up in the table if that was the main aim of the exercise. You see here, it's not so much that I want to get this answer, as much as it is that I want to emphasize the technique that one uses.

The idea is something like this. Here we see the difference of two squares. That's the square root of a squared minus x squared. What this should do for us is to suggest a right triangle whose hypotenuse is a and one of whose sides is 'x'. That's the diagram I've drawn over here. This is my reference triangle. As I say, we've done this before, but I would like to reinforce this at this particular stage. Especially now that we're more familiar with both the inverse circular and inverse hyperbolic functions.

At any rate, we use this reference triangle. From this triangle, the easiest relationship to pick off involving 'x' is that sine theta is 'x over a', or 'a 'sine theta'' equals 'x'. From which it follows that ''a' cosine theta 'd theta'' is 'dx'. The square root of ''a squared' minus 'x squared'' over 'a' is cosine theta. Therefore, the square root of ''a squared' minus 'x squared'' is 'a cosine theta'.

If we now make this substitution in this particular integral, we wind up with integral 'dx' over the square root of ''a squared' minus 'x squared'' is equal to integral 'd theta'. Which is theta plus 'c'. And since theta is the number whose sine is 'x over a', we have that the answer to this problem is inverse 'sine 'x over a'' plus a constant. Again, just a straightforward review, but a rather important technique.

In our textbook, this is put in a separate section, and the technique is called trigonometric substitution. And I just wanted to show you what motivates the trigonometric substitution. The harder part as I mentioned in our previous lecture, is how you motivate trigonometric substitutions when it's the hyperbolic functions that are involved.

You see, what I'd like to do now is instead of dealing with the square root of ''a squared' minus 'x squared'', let me just reverse the order of the terms. And now let me take integral 'dx' over the 'square root of ''x squared' minus 'a squared'''.

Notice that the difference between ''x squared' minus 'a squared'' and ''a squared' minus 'x squared'' is just a factor of minus 1. And the fact that that minus 1 is under the square root sign, really means in a certain manner of speaking that you've multiplied by i, the square root of minus 1, to transform the integral that we just had into this particular form here. And in terms of our lecture on the hyperbolic functions that should, in some way, give us a preview of things to come in the sense that the hyperbolic functions are very strongly related to the circular functions.

My technique for doing this-- and again, this is highly subjective. It's very simple. I learnt the circular functions long before I learnt the hyperbolic functions. Consequently, I feel very much at home with the circular functions. As a result, whenever I see the sum or difference of two squares, I always think in terms of a triangle. I make a circular trigonometric substitution.

If that works for me, fine. I haven't lost anything. If it doesn't work for me, it gives me the hint that I can't seem to get in my own mind without this hint as to which hyperbolic function I should use. Let me show you this in terms of this particular example.

If I saw this problem from scratch and didn't have any previous knowledge of how to handle this I'd say, this seems like I should think of a right triangle whose hypotenuse is 'x' and one of whose sides is 'a'. So I think of this particular reference triangle.

Now, thinking of this triangle, and in fact, even look at this triangle, I very easily pick off that 'x' is equal to 'a cosecant theta'. See, 'x over a' is 1 over sine theta. That's cosecant theta. Therefore and remembering how to differentiate the cosecant, I take the differential of both sides and I find that minus 'a cosecant theta cotangent theta 'd theta'' is 'dx'.

Again, from this diagram I see that the square root of ''x squared' minus 'a squared'' is-- well, let's see. The square root of ''x squared' minus 'a squared'' over 'a' would be cotangent theta. So the square root of ''x squared' minus 'a squared'' itself is 'a cotangent theta'.

Making the substitution in the integral 'dx' over the square root of ''x squared' minus 'a squared'', notice that the 'a's cancel, the cotangent's cancel, and I'm left with 'cosecant theta 'd theta'' and a minus sign inside my integral. In other words, somehow or other, this circular trigonometric function replaces the integral 'dx' over the 'square root of ''x squared' minus 'a squared''' by integral minus 'cosecant theta 'd theta''.

And the point I'd like to emphasize at this particular stage is that we don't say that the substitution has failed. The substitution has been made. We have replaced an integral involving 'x' by one involving theta. It's just that in the same way that it's difficult to integrate secant theta-- in fact, that will be later in this block of material we'll do that. But the point is that secant and cosecant are rather difficult functions to integrate. In fact, at this stage of our development, we do not know a function whose derivative is cosecant theta. Consequently, we don't know one whose derivative is minus cosecant theta. We seemed to have arrived at an impasse here. We have successfully made the substitution, but the key point is that the new integral is no easier for us to handle than the old integral.

Now, here's how I visualize the hyperbolic substitutions. We did this in the last lecture. I think it's tough enough, so I'd like to do it a second time and give you a chance of seeing how easy this is once you see the structural form.

Notice that the substitution I made here was 'x' equals 'a cosecant theta'. The cosecant was the key step here. The reference triangle, knowing how to draw this geometrically, allowed me not to consciously have to think of the trigonometric identities. But what I really did in evaluating this problem, if I left the diagram out and wanted to do it analytically, the identity that I made use of was the one that said cosecant squared theta minus cotangent squared theta is identically 1. In other words, I would like an identity for the hyperbolic functions that has the structure that the difference of two squares is identically 1.

The easiest one I can think of is the one that says cosh squared theta minus sinh squared theta is 1. You see, structurally, these two are equivalent. In other words, sinh and cosh are related hyperbolically, the same way that cotangent and cosecant are in terms of circular functions.

The identification is this. Since in the original problem I tried 'x' equals 'a cosecant theta', and since the identification here is that cosecant and cosh are matched up, what I try next-- see once this has failed, I very quickly now say, OK, what I'll do is instead of 'x' equals 'a cosecant theta', I'll try 'x' equals 'a cosh theta'. That's exactly what I did over here. 'a cosh theta' equals 'x'. From which it follows that 'dx' is 'a sinh theta 'd theta''. The square root of ''x squared' minus 'a squared'' is just ''a squared' cosh squared theta'. That's 'x squared'. Minus 'a squared'. That can be written as the square root of 'a squared' times the quantity 'cosh squared theta minus 1'.

Now, the interesting point is the 'cosh squared theta minus 1' turns out to be sinh squared theta. In fact, if we want to just look back here for a moment, recall that's exactly what we have over here. That cosh squared theta is 1. Well, cosh squared theta is 1 plus sinh squared theta. cosh squared theta minus 1 is sinh squared theta.

And you might, say wasn't that terrific? What a lucky break it was that this complicated expression just happened to be sinh squared theta. And now taking the square root I get a sinh theta. The thing I'd like to point out is that the element of luck was removed from this at the instant that I recognized the fact that cosh and sinh were related by the same identity that cosecant and cotangent were. In other words, I rigged this thing so that when I finally had to take this particular difference, I had to wind up with an easy square root to extract. You see, I had the right structure to begin with.

At any rate, to make a long story short here, 'dx' is 'a sinh theta 'd theta''. The square root of ''x squared' minus 'a squared'' is 'a sinh theta'. Therefore, 'dx' over the square root of ''x squared' minus 'a squared''. Well, look how nicely this works out. The a sinh theta cancels from both the numerator and the denominator. And it turns out that the integral that I'm looking for, 'dx' over the square root of ''x squared' minus 'a squared'' is just theta plus a constant.

But now, recalling that a cosh theta equals 'x'. In other words, cosh theta is 'x over a'. And therefore, theta is 'inverse cosh 'x over a'', I arrive at the result that this integral is 'inverse cosh 'x over a'' plus a constant. In other words, what I hope that these two little examples show is that when I have the sum or difference of two squares, the trigonometric substitutions will usually give me a crack at getting an equivalent integral that will be easier to handle.

What may happen is that the circular functions won't help me, but the hyperbolic ones will. Or vice versa, and I hope that you see how these are so delicately connected.

Well, there is one final type of technique that I would like to show for today's lesson that's related to what we've already done. And with that, we'll conclude today's lesson and go into some more elaborate recipes next time. But this is a take-off on our old high school construction of completing a square.

In other words, let's suppose we have an expression of the form 'ax squared' plus 'bx' plus 'c'. And this may sound old hat to you. You may remember this as looking pretty much like the development of the quadratic equation. The idea works something like this.

First of all, I can factor an a out of here. I'm assuming that a is not 0. In fact, if a were 0, I really wouldn't have a square term in here. I factor out an 'a' and I'm left with 'a' times 'x squared' plus ''b over a'x' plus 'c over a'.

Now it turns out that whenever you have something of the form 'x squared' plus something times 'x', to convert that into a perfect square. And again, we'll have ample opportunity to practice this in the exercises. You take half the coefficient of 'x' and square it.

Half the coefficient of 'x' here is 'b over 2a'. If I square that it's 'b squared' over ''4a' squared'.

Now to keep the identity intact here, if I add in 'b squared' over ''4a' squared', I must subtract that. In other words, I just add and subtract this term that will make this a perfect square. In fact, what is 'x squared' plus ''b over a'x' plus ''b squared' over '4a' squared''? You'll notice that this is just ''x' plus 'b over 2a' squared'. 'x squared'. Just multiply this thing out. You see how this particular thing works. In other words, what I've done is I can write this as 'a' times a perfect square.

And by the way, what's left over here if I multiply these two terms through by 'a'? This is 'c' minus 'b squared over 4a'. In other words, recognizing that this is ''x' plus 'b over 2a' squared'.

And multiplying these two terms through by 'a', I wind up with the fact that ''ax' squared' plus 'bx' plus 'c' can always be written in this particular form. This is called completing the square. And notice that this is now a perfect square and this is some constant.

And the question that comes up is, what does this have to do with integral calculus? In other words, with finding antiderivatives. And again, as is so often the case, and granted that as this course-- especially this chapter gets more complicated, the degree of sophistication will become much greater. But the basic idea will always remain the same. Somehow or other, we will always try to reduce a new situation to a more familiar one.

In other words, let's suppose now we don't have an integrand that's the sum and difference of two squares, but we have an integrand that involves a quadratic expression. Say, for example, we have 'dx' over ''ax' squared' plus 'bx' plus 'c'. Well, you see, using hindsight rather than foresight, I already took care of this problem before we started. Namely, we just finished completing the square over here. Namely, what is ''ax' squared' plus 'bx' plus 'c'?

Notice it can be written in what form? Let's come back here and take a look. It's 'a' times ''x' plus 'b over 2a' squared' plus some constant. I don't care what it happens to be here. I'll just call that constant 'c sub 1'. And by the way, notice depending upon the relative sizes of 'a', 'b', and 'c', this constant can be either positive or negative. Or in fact, even 0.

So to handle that what I'll say is I can always write this denominator in the form 'a' times ''x' plus 'b over 2a' squared' plus or minus some constant. Where I'm assuming now that the constant is positive. That's why I put the plus or minus sign in. It's plus some positive constant or else it's minus some positive constant.

Now what I can do is factor out an a from the denominator. If I factor out an a from the denominator here, I have 1 over 'a' times a certain integral. What integral? 'dx' over 'x plus 'b over 2a' squared' plus or minus still some constant. In other words, I had 'c1' here. I factored out on 'a'. 'c2' was actually 'c1 over a'. I just factored out an 'a' from this thing here. But it doesn't make any difference. The important point is I now have this written in this particular form.

Now if I assume that 'c2' is a positive constant, notice that any positive number is a perfect square. Namely, it's the square of its square root. And again, as I say with the exercises, this will become clearer when we work with specific numbers. But the idea here is what? This is some perfect square, which I'll call 'k'. And in other words, this integrand can now be written in the form '1 over a', 'dx' over 'x plus 'b over 2a' squared' plus or minus 'k squared'.

Well, again, what am I leading up to? If I now make the substitution that 'u' equals 'x plus 'b over 2a'', notice that my denominator simply becomes 'u squared' plus or minus 'k squared'. In other words, my denominator now has the form of either the sum or difference of two squares. On the other hand, if 'u' is 'x plus 'b over 2a'', what is 'du'?

Remember, 'b' and 'a' are constants. 'b over 2a' is a constant. The derivative of a constant is 0. So 'du' is equal to 'dx'. In other words, notice that if I now make this substitution in here, outside I have a '1 over a', inside I have 'du' over ''u squared' plus or minus 'k squared''.

In other words, when I have a quadratic I can write that in the form the sum and/or difference of two squares, simply by completing the square. Simply by completing square. In other words, I can solve the problem of the quadratic by converting it back into an integral, the type of which I've solved before. And this, by and large, becomes the technique that we will use throughout this chapter.

Well, at any rate, I won't go into that in any more detail right now. We will pick up additional techniques in our next lecture. And until the next lecture, goodbye.

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