Permutations, Combinations, Partitions

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Review of last class

- What is Bayes' rule?

Review of last class

- What is the total probability theorem?

Review of last class

- What does “A is independent from B” mean?

Review of last class

- How do we test for independence?

Last class catchup

- If we have probability, and **conditional probability**…
- We can have independence, and **conditional independence** too

Conditional Independence

- Definition:

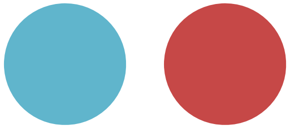
  $$P(A \cap B \mid C) = P(A|C)P(B|C)$$

  given C, A and B are independent

- Another way to write this:

  $$P(A \mid B \cap C) = P(A|C)$$

Example: Biased Coin Toss



- We have two coins: blue and red
- We choose one of the coins at random (probability = 1/2), and toss it twice
- Tosses are independent from each other given a coin
- The blue coin lands a head 99% of the time
- The red coin lands a head 1% of the time

Events: $H1$ = 1st toss is a head
$H2$ = 2nd toss is a head

Example: Biased Coin Toss

- Tosses are independent from each other GIVEN the choice of coin

conditional independence

Problem #4: Biased Coin Toss

- What if you don't know what coin it is? Are the tosses still independent?

Last Class - Summary

- Bayes' rule
- Independence
- Conditional Independence

- Things are not always what they seem! But with these tools you can calculate the probabilities accurately

Counting in Probability

- Where have we seen this?
  - When sample space is finite and made up of equally likely outcomes
  - $P(A) = \dfrac{\#\text{ elements in } A}{\#\text{ elements in } \Omega}$
- But counting can be more challenging…

Divide & Conquer

- Use the tree to visualize stages
- Stage 1 has n_1 possible choices, stage 2 has n_2 possible choices, etc…

Divide & Conquer

- All branches of the tree must have the same number of choices for the same stage

The Counting Principle

- An experiment with m stages has

$$n_1 n_2 \ldots n_m \text{ results,}$$

where n_1 = # choices in the 1^{st} stage,
n_2 = # choices in the 2^{nd} stage,
...
n_m = # choices in the m^{th} stage

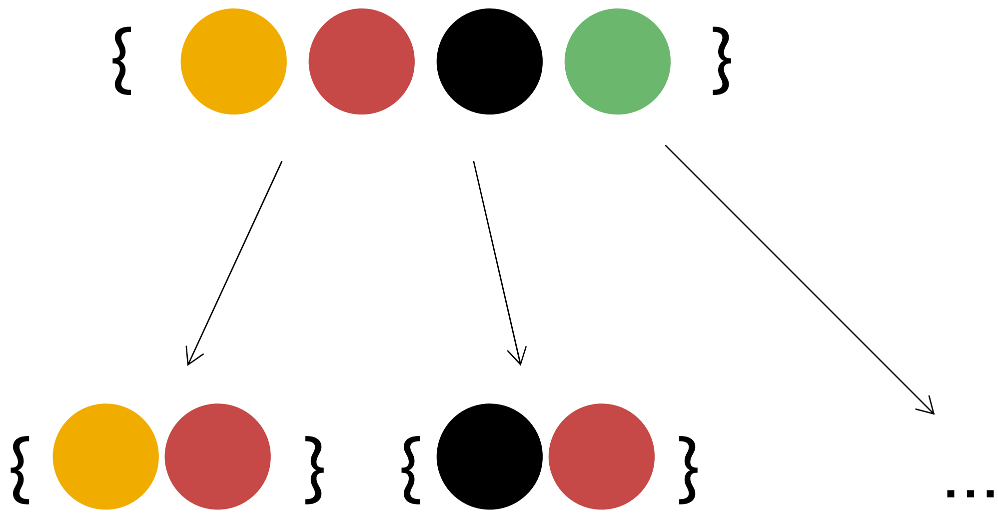
k-permutations

- How many ways can we pick k objects out of n distinct objects and arrange them in a sequence?

- Restriction: $k \leq n$

Example: M&M's

- Pick 4 colors of M&Ms to be your universal set
- How many 2-color sequences can you make?



Deriving a formula

- At each stage, how many possible choices are there? [Use the counting principle]

Formula for *k*-permutations

- Start with *n* distinct objects
- Arrange *k* of these objects into a sequence

\# of possible sequences:

$$= \frac{n!}{(n-k)!}$$

Special case: k=n

- Formula reduces to: n!

- This makes sense – at every stage we lose a choice: (n)(n-1)(n-2)…(1)

Combinations

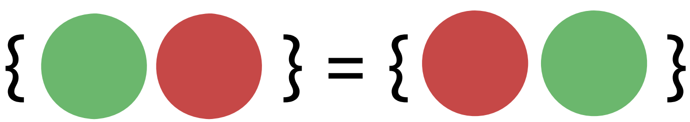
- Start with *n* distinct objects
- Pick *k* to form a set

- How is this different from permutations?
  - Order does NOT matter
  - Forming a subset, not a sequence

Example: M&M's

- Pick 4 colors as the universal set
- How many 2-color combinations can you create?

Remember that for combinations,



Deriving a formula

- Permutations =
  - 1. Selecting a combination of *k* items
  - 2. Ordering the items
- How many ways can you order a combination of *k* items?

Deriving a formula

(# k-permutations) =

(# ways to order k elements) x (# of combinations of size k)

Formula for combinations

- Start with *n* distinct objects
- Arrange *k* of these objects into a set

# of possible combinations:

$$= \frac{n!}{k!\,(n-k)!}$$

Another way to write combinations

- "n" choose "k"

$$\binom{n}{k}$$

- Side note: this is also known as the "binomial coefficient," used for polynomial expansion of the binomial power [outside of class scope]

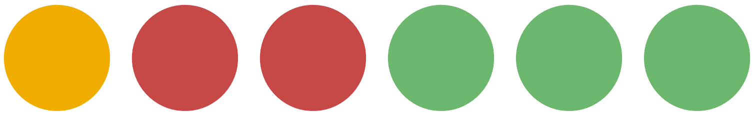
Partitions

- We have a set with n elements
- Partition of this set has r subsets
- The ith subset has n_i elements

- How many ways can we form these subsets from the n elements?

Example: M&Ms

- 6 total M&Ms
  - 1 of one color
  - 2 of one color
  - 3 of one color



- How many ways can you arrange them in a sequence?

Example: M&M's

- One perspective
  - 6 slots = 3 subsets (size 1, size 2, size 3)
  - Each subset corresponds to a color
- At each stage, we calculate the number of ways to form each subset

______ ______ ______ ______ ______ ______

Example: M&M's

- Stage #1: Place the first color
- 6 possible slots
- Need to fill 1 slot

combinations: $\binom{6}{1}$

Example: M&M's

- Stage #2: Place the second color
- 5 possible slots
- Need to fill 2 slots

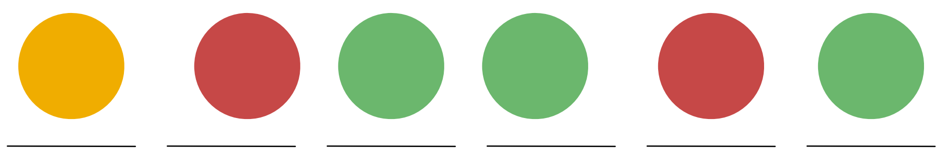
combinations: $\binom{5}{2}$

Notice how it does not matter which M&M we place in which slot – this implies order does not matter → use combinations

Example: M&M's

- Stage #3: Place the third color
- 3 possible slots
- Need to fill 3 slots

combinations: $\binom{3}{3}$



Deriving a formula for partitions

- Solution to our example:

$$\binom{6}{1}\binom{5}{2}\binom{3}{3}$$

- Generalized form?

Formula for partitions

- Start with n-element set (no order)
- In this set, there are r disjoint subsets
- The ith subset contains n_i elements

- How many ways can we form the subsets?

$$\frac{n!}{n_1!n_2!\ldots n_r!}$$

Problem Revisited

- A class has 4 boys and 12 girls. They are randomly divided into 4 groups of 4. What's the probability that each group has 1 boy?

- Use counting methods (partitions) this time

Summary

- The Counting Principle
- Permutations
- Combinations
- Partitions