Unexpected Behaviors in Higher-Order Positive Feedback Loops

Prepared for the
MIT System Dynamics in Education Project
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ABSTRACT

Positive feedback loops typically show the behavior of exponential growth. However, simple unbounded growth is not the only behavior a single loop is able to exhibit. Other possible behaviors include: exponential decay, damped oscillation, sustained oscillation, and expanding oscillation. Each of these additional behavior modes is unstable, and each will turn to exponential growth if perturbed out of the stable mode. This paper demonstrates and explores these unexpected behaviors in simple positive feedback loops. The loops used for experimentation are all elementary, with direct positive influences and no minor loops. The loops vary from first to fourth order.

1. INTRODUCTION

We naturally expect positive feedback to grow exponentially. It is the behavior we experience daily. Our savings accounts gather interest, as do our debts from loans, and typically at an alarming rate. The world population is exploding about us, and the AIDS epidemic quickly spreads within it. The level of technology doubles every few years. Unchecked nuclear explosions power the sun, which gives us the energy we live by. In all of these examples, the rule is simple: the more you've got, the faster you get it. This behavior is a part of any system whose rate of advancement is directly dependent on the level of its current state—what is commonly called positive feedback. Thus we expect loop structures that have positive influences between its connections to quickly grow without bound—exponentially.

However, simple unbounded growth is not the only behavior a single loop is able to exhibit. Other possible behaviors include: exponential decay, damped oscillation, sustained oscillation, and expanding oscillation. These are behaviors that we expect to see in negative feedback loops; they are signs of stability. As we shall see, in positive feedback loops each of these behavior modes is unstable, and each will turn to exponential growth if perturbed out of the uncharacteristic mode.

To help gain a better understanding of this special kind of behavior, we will experiment with simple loops only. Each level in a loop will have a direct positive influence on the rate of the next level. There will be no other inputs to the rates. In addition, our investigation will start with a second-order loop, and we will increase complexity one level at a time. To explore the behavior of each loop, we will change the initial conditions of the level variables, but we will not alter the model structure. Although changing the structure leads to a wealth of interesting investigations, there will be no changes to the rate equations. I recommend that the reader, while proceeding through the paper, build each model independently and experiment with it freely.
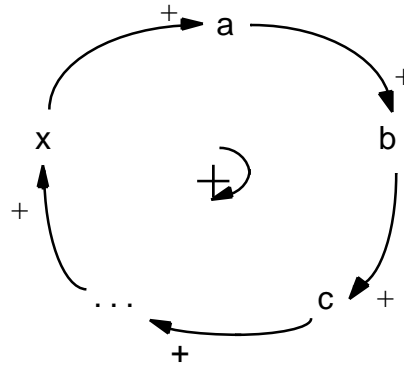
2. UNEXPECTED BEHAVIORS

2.1 First-Order Loop
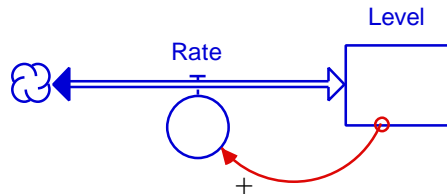
Figure 1: A first-order positive feedback loop

We are familiar with the simple first-order positive feedback loop (Figure 1). In the most simple case Rate = Level. In that case, if Level is positive, and thus Rate is positive, then the rate increases the level, and the level in turn increases the rate, further increasing the level, and so on. This process yields exponential growth, the result of standard positive feedback. Likewise, if the level and the rate start off negative, then the rate decreases the level, making the level more negative, making the rate more negative, which further decreases the level, and so on—again we have positive feedback, now in the direction towards very negative numbers.

In both cases there is the reinforcing pattern. In a specific model the loop may have many additional factors affecting it, depending on the system being modeled (birth fraction, interest rate, and learning ability are a few possibilities). However, it will always exhibit the behavior of exponential growth. The same, however, cannot be said of higher-order positive feedback loops.

Figure 2: Positive Feedback

In higher-order positive feedback loops, exponential growth is not always exhibited, and unexpected behavior modes do appear. This may seem surprising, especially after looking at a causal loop representation of a positive feedback loop (Figure 2). If a increases b and b increases c and ... and x increases a, then shouldn't everything increase (or decrease to very negative numbers) together? Shouldn't everything grow together, faster and faster, exponentially? Usually, yes, but not always.

For example, look again at the first-order model above. If the level is initially 0, then it will stay forever 0. This is because the rate is also 0, as specified in the rate equation. Thus no change takes place. Even though the model is in equilibrium, it is in an unstable balance. If the value of the level is changed the slightest bit in either direction, then the behavior will slip into exponential growth. This is what is meant by an unstable equilibrium. The higher-order models we will see have similarly unstable behavior modes.
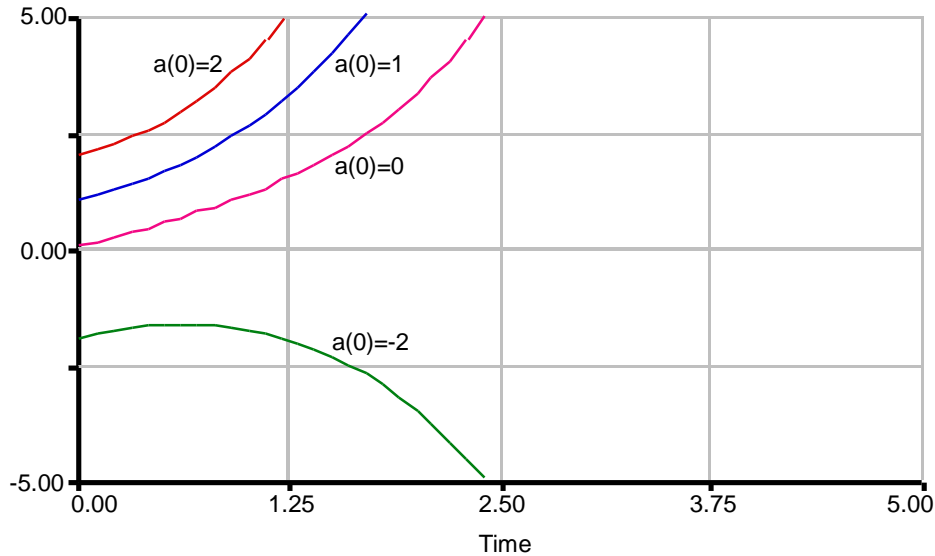
2.2 Second-Order Loop

We now examine the simplest second-order positive feedback loop possible (Figure 3). The rate of level a, da, is equal to the value of level b. Likewise, the rate of level b, db, is equal to the value of level a. (Note: the system equations for all models can be found in Appendix B.)

Once again, the model structure will not be altered. We will only change the initial values of a and b.

First note that when $a = b = 0$, the rate of change for both level a and b is 0; thus the system is in equilibrium, much the same as with the first-order model. Now let $a(0) =$ $b(0) = 1$ (this notation simply means that we are setting the initial condition, when $time=0$, of both levels to 1). We have moved the system out of equilibrium. It can be intuitively seen now to explode exponentially—standard positive feedback. Level a increases at a rate equal to level b. Because initially $a=b$, b increases at the same rate. In fact, a and b push each other identically; they behave as one, growing faster and faster together.
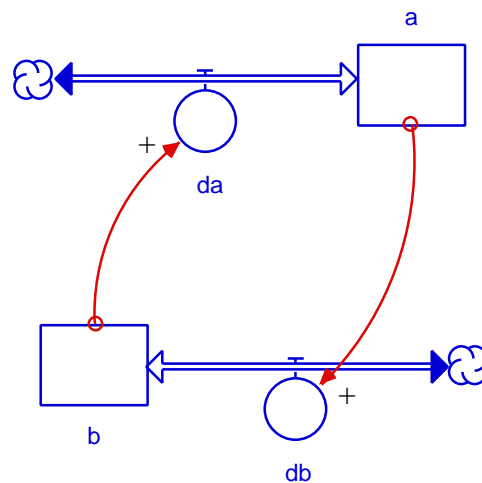
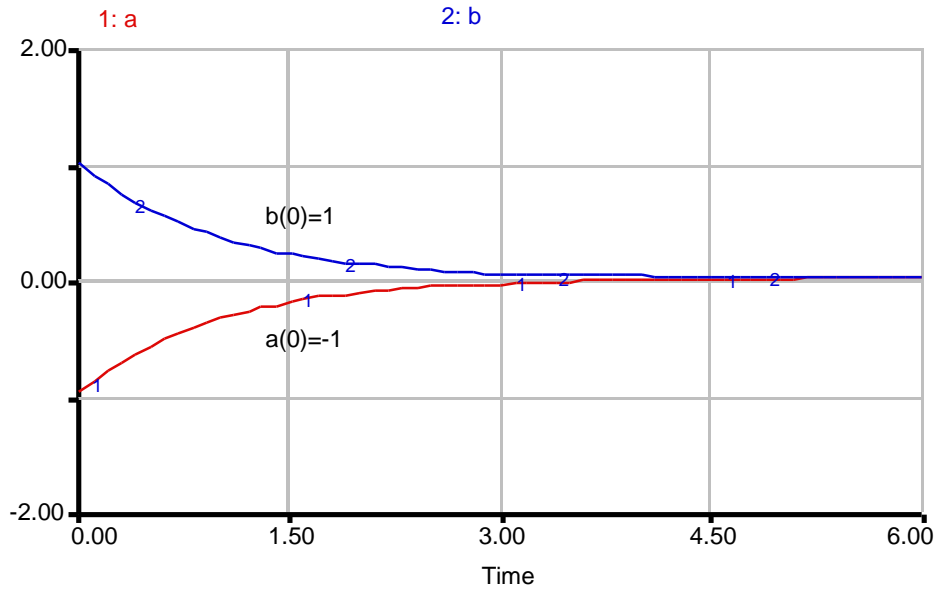**Figure 3: Second-order positive feedback loop**

Figure 4: Exponential behavior from four different initial conditions (second-order positive feedback loop). b(0) = 1 in all cases.

Leave $b(0) = 1$ and try $a(0) = 2$, then $a(0) = 0$. In both cases the level that is initially greater stays greater, and the levels again escape to positive infinity. Next, let $a(0) = -2$. This time a is sufficiently negative to reverse the direction of escape. a consistently decreases b by more than b increases a; eventually b drops below 0, and both a and b plummet. Figure 4 shows the time series of a for the above four cases.

All of these show the expected exponential behavior, some up and some down. But there is a balance between them. Let $a(0) = -1$. Now b is decreased by a at the same rate a is increased by b. The two ride in a perfect balance, asymptotically approaching 0 (Figure 5). This behavior would occur for any initial conditions $a(0) = -b(0)$.   This behavior is usually seen in first-order negative feedback, which shows an approach to stable equilibrium. However, this is **not** a stable behavior mode. The slightest change in either variable destabilizes the approach and returns both to an exponential growth. This is seen in Figure 6 where we let $a(0) = -0.99$, a 1% disturbance.

Although exponential growth is the dominant behavior, it is possible to elicit asymptotic approach to (unstable) equilibrium in a purely positive second-order loop.

1: a   **2: b**

**Figure 5: Convergent behavior from initial conditions a(0) = -1, b(0) = 1
(second-order positive feedback loop).**

1: a   **2: b**
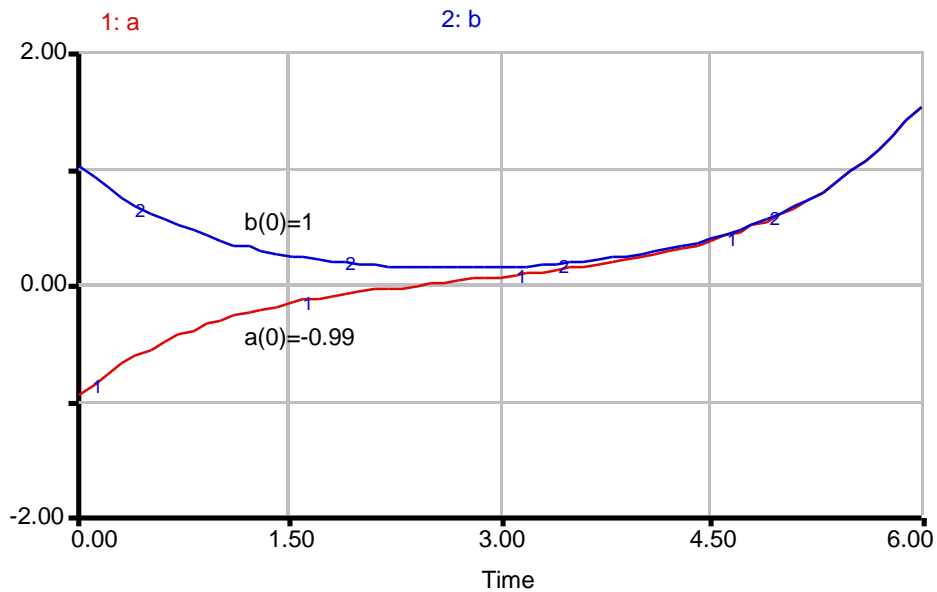
**Figure 6: Unstable convergence perturbed to exponential growth
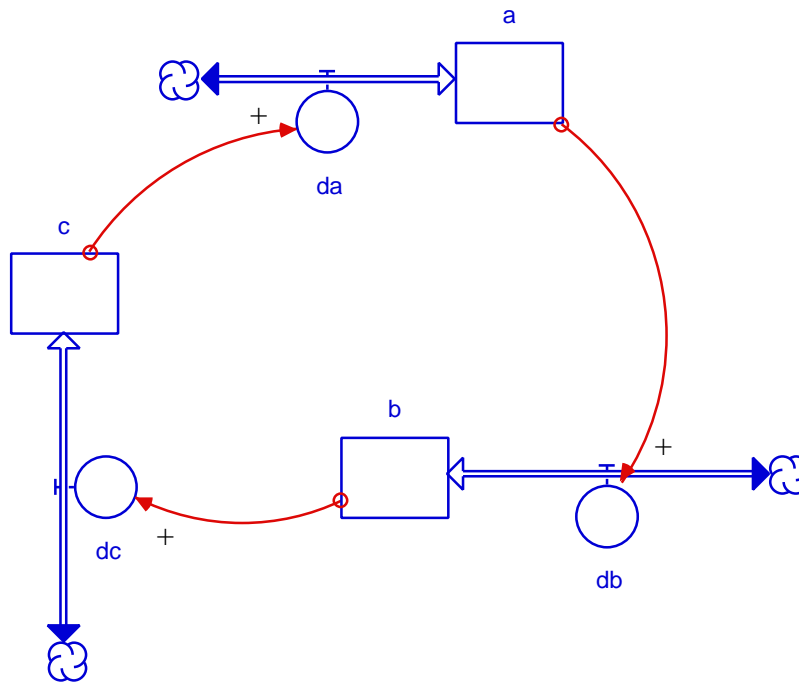(second-order positive feedback loop).**

Independent Exploration:

Make a comparative scatter graph of a vs. b, and try many initial values of a and b. This is done in the Graph Pad Dialog box. First, select Scatter under Graph Type. Then double click a and b under Allowable to be Selected, as normally done. To plot

many initial values on the same graph, you must first check Comparative before identifying the Selected variables. What pattern emerges?

Introduce time constants as an additional input to the rate equations (i.e. $da = b/ta$, $db = a/tb$, where ta and tb are the time constants). These time constants are present in the loops already described above; they are simply equal to 1 and so have no effect. How does changing the time constants affect the system? How about negative time constants (and can you imagine what such a thing might mean)?

2.3 Third-Order Loop

The third-order system being examined is, like the second-order system, the simplest possible. Each level feeds the rate of the next with no other input (Figure 7).

Figure 7: Third-order positive feedback loop

This system can also produce exponential growth from many initial conditions (E.g. $a(0) = b(0) = c(0) = 1$), but we will jump directly to other behavior modes. Let

a(0) = 1, b(0) = -1, and c(0) = 0. This produces the surprising behavior in Figure 8. Here we see, instead of exponential growth or even decay, damped periodic oscillations!

Here, as in the second-order system, we have found a balanced state. The levels, in this balance, can not decide whether to escape up or down, and so they do neither. But again, this apparently stable approach to equilibrium—for when a=b=c=0 the system is in equilibrium—is not truly stable at all. Change one value only slightly, by a scant 1/1000, and the balance collapses as in Figure 9.

Where does this periodic behavior come from? There is no negative feedback to drive it directly. There are no exogenous inputs. There are, however, three delays. These are not delays in the traditional sense, where an intervening level may change a step increase inflow into a smoothed outflow (using additional negative feedback structure). The delays are present inherently within the levels. To see this, notice what happens when a rate suddenly changes. If we are looking only at the level changed by this rate, the surge is not immediately felt—at least not strongly. After some time passes by, we may notice that the level is changing in a new way, due to a new rate value. However, the information passing through the rate and then the level still has this inherent delay. It takes time for qualitative behavior to propagate through.

Thus, while b is increasing from the start, the effect does not have a strong effect on a until approximately time = 2, in the previous example. The delays—and the balance of the levels around 0—incite the oscillations. The oscillations occur in the third-order system and not the second-order system because the higher order system has more room for dynamic behavior; there are not enough delays present in the second-order system. So what might we see in systems of even higher orders?

Independent Exploration:

Again, create comparative scatter graphs, this time of <u>a vs. b</u>, <u>a vs. c</u>, and <u>b vs. c</u>. In this way you can begin to see a three dimensional picture of the behavior (in the space of a, b, and c). The three graphs give you perspectives that we might call front, side, and top. What do you see now, and how does it compare to what you saw in the second-order system?

The balance for convergent behavior in the third-order system occurs when $a(0) + b(0) + c(0) = 0$. Can you discover why?
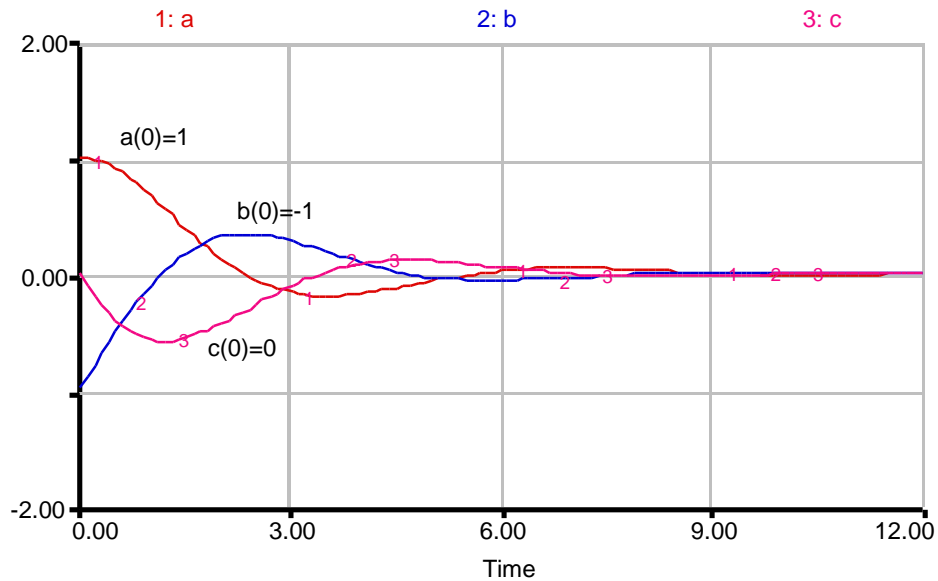
Figure 8: Convergent periodic oscillations
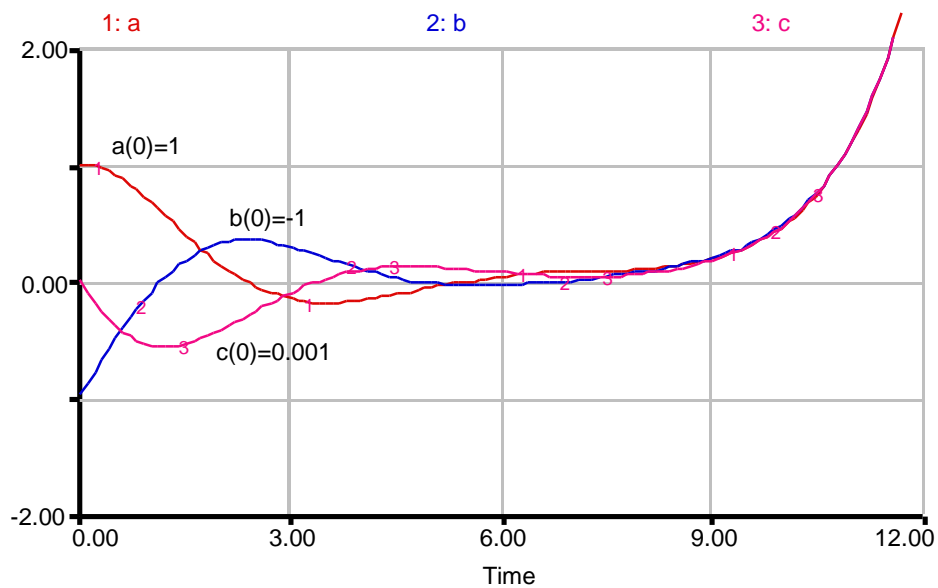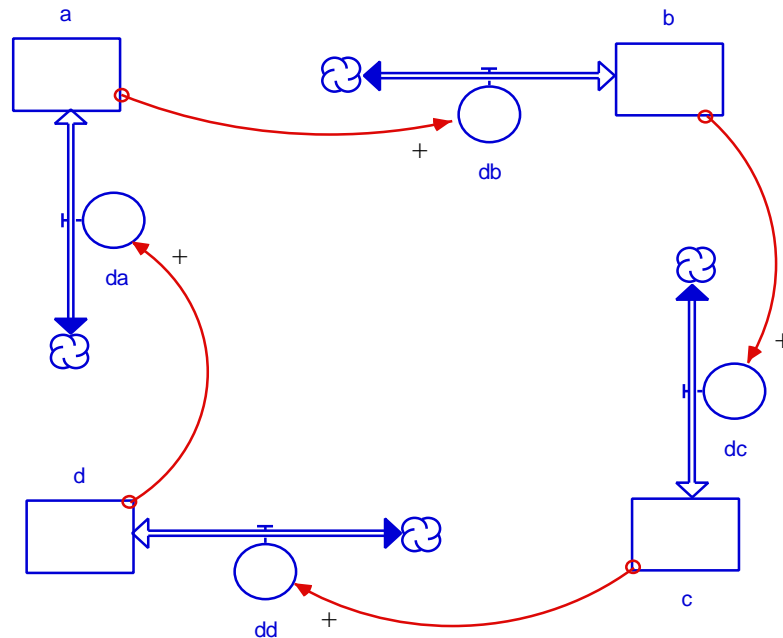
(third-order positive feedback loop).

Figure 9: Convergent periodic oscillations destabilizing to exponential growth

(third-order positive feedback loop)

2.4 Fourth-Order Loop

The fourth-order system is identical to the previous systems except for that it adds one more level to the chain (Figure 10).

What surprising behavior can we obtain within the freedom of four level variables? As it happens, three distinct modes appear. The expected divergent exponential ($a(0)$ = $b(0)$ = $c(0)$ = $d(0)$ = 1) and convergent decay ($a(0)$ = $c(0)$=1, $b(0)$ = $d(0)$ = -1) we have seen already, but—as unbelievable as it may seem—we can now also have sustained oscillations! With only positive feedback between the individual variables—and with the initial conditions properly balanced—the delays between distant levels is great enough to create sustained oscillations. In Figure 11 $a(0)$ = 1, $b(0)$ = -1, and $c(0)$ = $d(0)$ = 0. (You may get expanding oscillations when you simulate this. If so, it is due only to DT error, and it can be rectified by using a different integration method or a smaller DT. I used DT=0.01; Runge-Kutta at DT=0.1 works just as well.)

As is the case in the other systems, this peculiar behavior mode is not stable. A 1/1000 perturbation—changing $c(0)$=0.001—creates the destabilization in Figure 12.
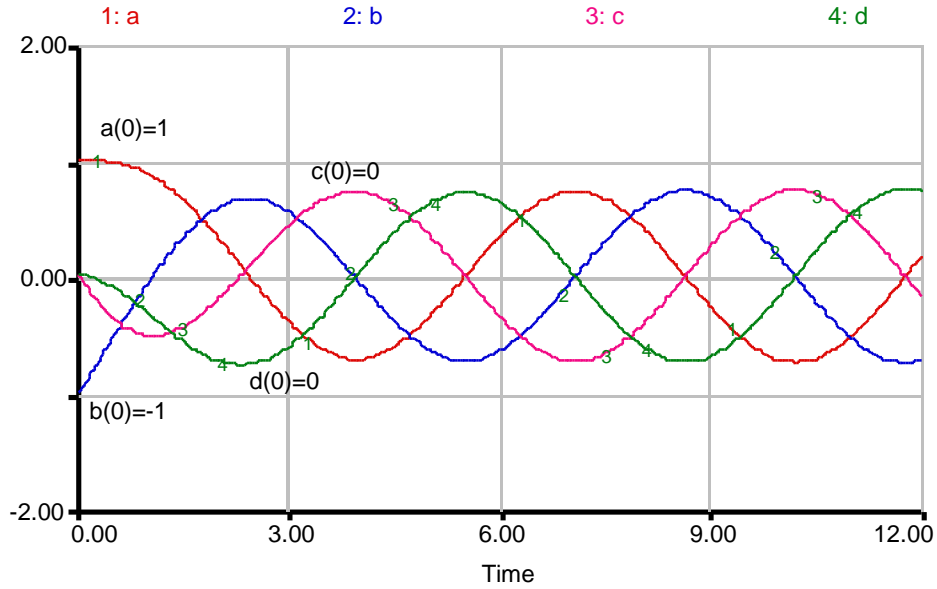

Figure 10: Fourth-order positive feedback loop

Fourth-order feedback has the flexibility and inherent delay to foster sustained oscillations, as well as convergence to an unstable equilibrium. This system might be viewed as taking a two-level oscillating system and "hiding" it within the second-order system seen earlier; their behaviors together form the behavior we see here.

Independent Exploration:

Look closely at the time-series for each of the feedback loops examined. In the second and third-order systems, the convergent behavior mode died out when it was perturbed into destabilization (given a little push, it blew up). The damped oscillations turned into (nearly) pure exponential growth. After destabilization (as in Figure 12), do the sustained oscillations disappear in the fourth-order system as well? (You may find the geometries constructed from the scatter graphs useful in exploring this question.)

The fifth-order system offers still greater behavior. Explore and see what you can find. Under special initial conditions, it will show damped oscillations, but I'll warn you, they are quite difficult to find (it took me half a year)!

**Figure 11: Sustained oscillations within a
fourth-order positive feedback loop**
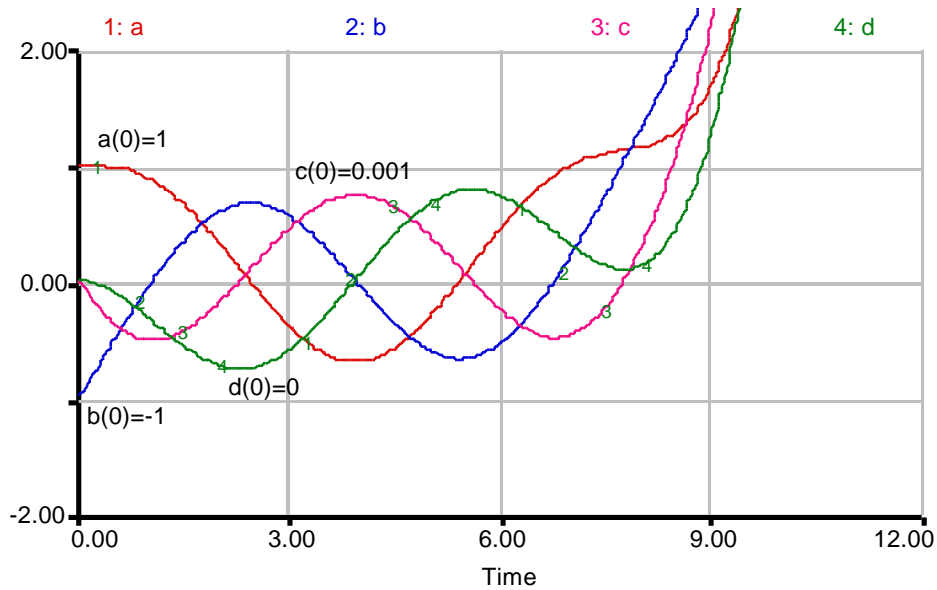
**Figure 12: Sustained oscillations destabilizing to exponential growth
(fourth-order positive feedback loop)**

3. SUMMARY

Although positive feedback loops are usually associated with exponential growth, that is not the only behavior possible within them. Convergence and oscillation can also be found. We looked at four different positive feedback loops.

The first order loop only behaves exponentially. All initial conditions will reinforce the system state, and push it out to infinity (be it up or down). The only exception is the degenerate behavior at the origin. There it stays fixed at one point, but note that this special behavior, like in the other systems, is unstable. Exponential growth is still the stable behavior.

The second order loop showed that exponential growth was not the only possibility for positive loops. While in perfect balance, the two levels settled to unstable equilibrium. Notice that the fixed behavior of the origin is still present (and unstable). The higher—order positive feedback loops contain many repeated behaviors.

The third order loop brought us an interesting twist: oscillations! As in the second order loop, the levels had to be in a special balance, but this time their decent to equilibrium was done with a rhythmic dance.

The fourth order loop, the final loop examined, went a step further than the third and produced stable oscillations. This result clearly demonstrated the surprising character of positive feedback. A feedback loop that has, at each individual step, only positive influences can produce behavior that is undeniably negative in appearance.

We have found great behavioral diversity even within the simplest of positive loops. The truly stable behavior in each case is exponential growth. It is stable in that small perturbations do not change the behavior greatly, and yet it is unstable in that its behavior looks most unsettled. On the other hand, the other behaviors we have seen are unstable because small perturbations do drastically change their behavior, even though they all give the appearance of stability. We can almost always find unusual unstable behavior disguised as what truly would be stable in other models.

As it happens, the behaviors we have seen here demonstrate all the possibilities that models of this simple nature can exhibit. Remember that most models of the real world are not nearly so constrained or simplified as what we have seen in this paper. Most positive feedback loops in real world models will show (for the most part) only

exponential growth. But still keep in mind that unexpected behaviors are possible, even likely, and they are bound to be quite complex.

4. APPENDIX A: SOME SOLUTIONS TO EXPLORATIONS

4.1 Second-order System

Make a comparative scatter graph of <u>a vs. b</u>, and try many initial values of a and b; what pattern emerges?

This geometric pattern (shown in Figure 13.) is called a saddle. To see why it is given this name, imagine letting a marble roll off an actual horse saddle. It moves along the surface, falling down the sides where one's legs would normally go. If you were able to place the marble just right, you could balance it in the center of the saddle, and it would not fall in either direction. If you were even more careful, you could place the marble near the top of the saddle, still centered left and right, and it would roll down to the center, but not fall to either side. If the saddle were sticky, then the marble would also slow to a stop at the center, just as we have in our saddle here. The two orbits with initial conditions (-3, 3) and (2, -2) are in a balanced path along the saddle. They exponentially approach the origin, while orbits to either side quickly zoom off to infinity.
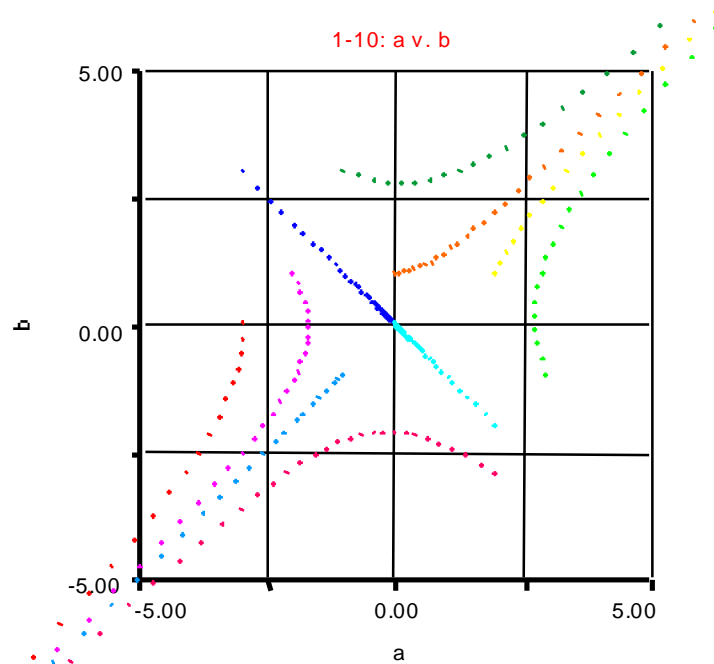
Figure 13: Paths on a saddle from 10 different initial conditions

4.2 Third-order System

What do you see in the new scatter plots, and how does it compare to what you saw in the second-order system?

This pattern is significantly more complicated to look at than the saddle. The problem is, it is a three-dimensional pattern. Figure 14 shows what it looks like from one perspective. The best way for you to see it is probably to watch it as the simulation runs, flipping between perspectives.
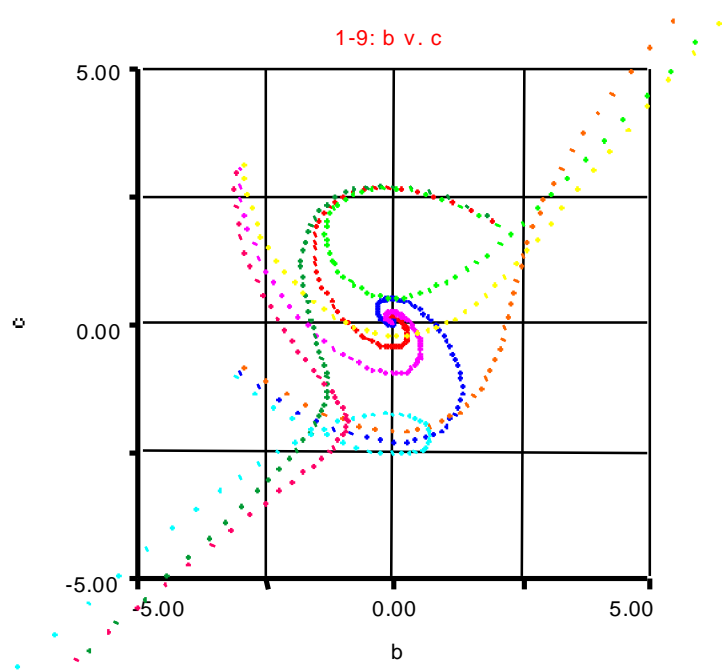
Figure 14: Paths in the third-order system; 3 paths are decaying; 6 are expanding

In the center you can see the decaying oscillations, they form the spiral pattern. Shooting off to the upper right and lower left are orbits perturbed into exponential growth. The spiral lies in a plane, and the exponential jets point perpendicular to this plane. If a point starts on the plane, it will stay on it, spiraling in to the origin. But if the point is even slightly off the plane, then it will be pulled away into the jet on its side of the plane.

If you could look at the plane edge on, the paths would look much like the paths seen in the saddle, only intertwining. This spiraling geometry is a kind of extension of the saddle. Now you might think of it as a pair vortices put end to end. Water rushes down a whirlpool or shoots up in a spout. The saddle is a little like a vertical cross section of the vortices. Perhaps a simpler way to look at it is just as the combination of decaying oscillations (spirals) and exponential growth (jets), each in its own direction.

4.3 Fourth-order System

After destabilization, do the sustained oscillations disappear in the fourth-order system as well?

No, the oscillations continue, even while rocketing off to infinity. The way to picture this is to use the geometries just shown for the second and third-order systems. The pattern now is even more complicated, and now it is in **four** dimensions! For this reason, I won't try to show it to you here, although I applaud all those who try to make sense out of the scatter graphs.

The way to think about it is to picture the orbits as traveling around a cylinder. At the plane cutting through the cylinder and the origin, the orbits go round and round in a circle. This is the sustained oscillations. Above and below this plane, the orbit fly away, but they continue to spiral around the cylinder. So, even though they become more and more drowned out by the exponential growth, the oscillations do not disappear.

The way to check it is to see what happens to the differences between the level variables. Plot a time series of a-b, for example. This should oscillate even while exploding exponentially.

4.4 Fifth-order System

Can you find the initial condition (for the fifth-order system) that results in decayed oscillations?

Here it is. Try it out, and look at nearby points too.

a=1   c=cos(2p/5)   e= -cos(p/5)
b= -cos(p/5)   d=cos(2p/5)

5. APPENDIX B: MODEL EQUATIONS

5.1 First-Order Loop

a(t) = a(t - dt) + (da) * dt
INIT a = 1
da = a

5.2 Second-Order Loop

a(t) = a(t - dt) + (da) * dt
INIT a = 1
da = b

b(t) = b(t - dt) + (db) * dt
INIT b = -1
db = a

5.3 Third-Order Loop

a(t) = a(t - dt) + (da) * dt
INIT a = 1
da = c

b(t) = b(t - dt) + (db) * dt
INIT b = -1
db = a

c(t) = c(t - dt) + (dc) * dt
INIT c = 0
dc = b

5.4 Fourth-Order Loop

a(t) = a(t - dt) + (da) * dt
INIT a = 1
da = d

b(t) = b(t - dt) + (db) * dt
INIT b = -1
db = a

c(t) = c(t - dt) + (dc) * dt
INIT c = 0
dc = b

d(t) = d(t - dt) + (dd) * dt
INIT d = 0
dd = c

5.5 Fifth-Order Loop

a(t) = a(t - dt) + (da) * dt
INIT a = 1
da = e

b(t) = b(t - dt) + (db) * dt
INIT b = -cos(p/5)
db = a

c(t) = c(t - dt) + (dc) * dt
INIT c = cos(2p/5)
dc = b

d(t) = d(t - dt) + (dd) * dt
INIT d = cos(2p/5)
dd = c

e(t) = e(t - dt) + (de) * dt
INIT e = -cos(p/5)
de = d

Vensim Examples:

Unexpected Behaviors in Higher-Order Positive Feedback Loops

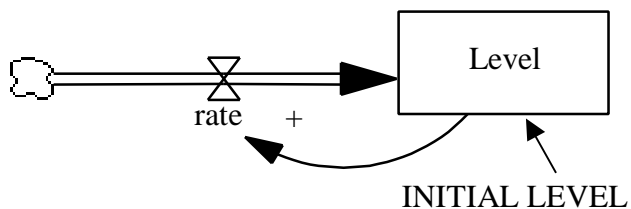Aaron Diamond

October 2001

3.1 First-Order Loop

Figure 15: Vensim Equivalent of Figure 1: A first-order positive feedback loop

Documentation for first-order positive feedback loop

(1)   FINAL TIME = 6

Units: none

The final time for the simulation.

(2)   INITIAL LEVEL=1

Units: **undefined**

(3)   INITIAL TIME = 0

Units: none

The initial time for the simulation.

(4)   Level= INTEG (Rate, INITIAL LEVEL)

Units: **undefined**

(5)   rate=Level

       Units: **undefined**

(6)   SAVEPER = TIME STEP

       Units: none

       The frequency with which output is stored.

(7)   TIME STEP = 0.0625

       Units: none

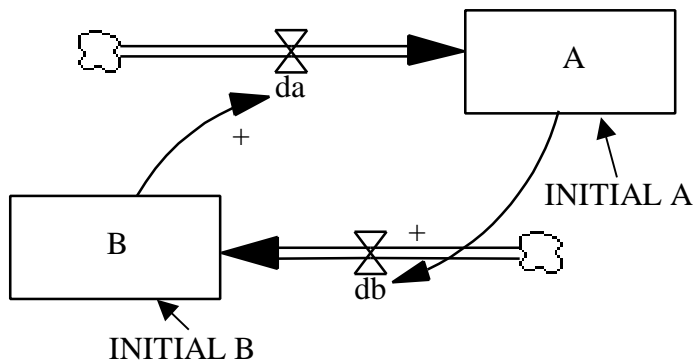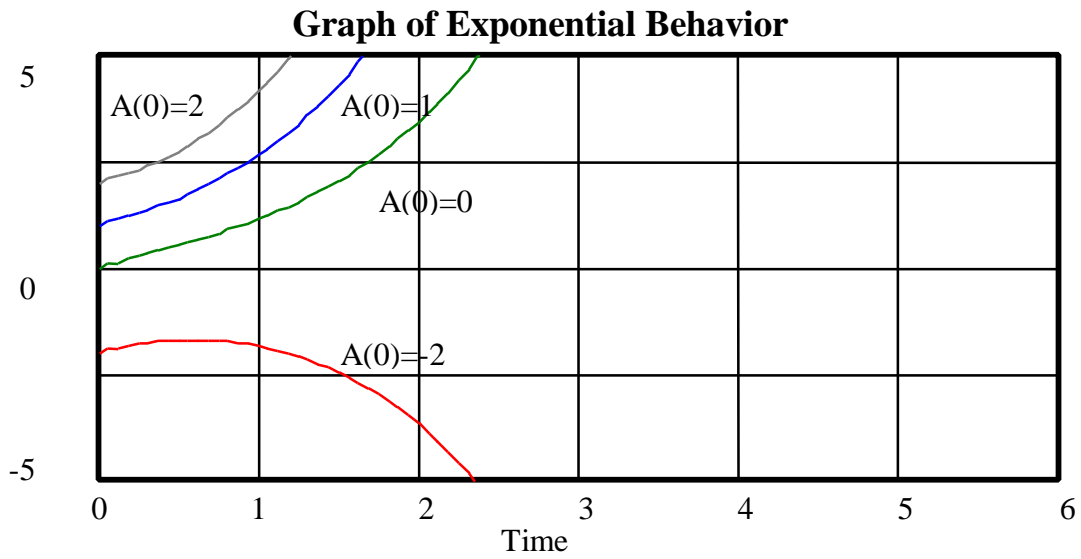   The time step for the simulation.

3.2 Second-Order Loop

Figure 16: Vensim Equivalent of Figure 3: Second-order positive feedback loop

Documentation for second-order positive feedback loop

(1)   A= INTEG (da, INITIAL A)

       Units: **undefined**

(2)   B= INTEG (db, INITIAL B)

       Units: **undefined**

(3)   da=B

Units: **undefined**

(4)   db=A

Units: **undefined**

(5)   FINAL TIME = 6

Units: none

The final time for the simulation.

(6)   INITIAL A=1

Units: **undefined**

(7)   INITIAL B=1

Units: **undefined**

(8)   INITIAL TIME = 0

Units: none

The initial time for the simulation.

(9)   SAVEPER = TIME STEP

Units: none

The frequency with which output is stored.

(10)   TIME STEP = 0.0625

Units: none

The time step for the simulation.

Graph of Exponential Behavior

Figure 17: Vensim Equivalent of Figure 4: Exponential behavior from four different
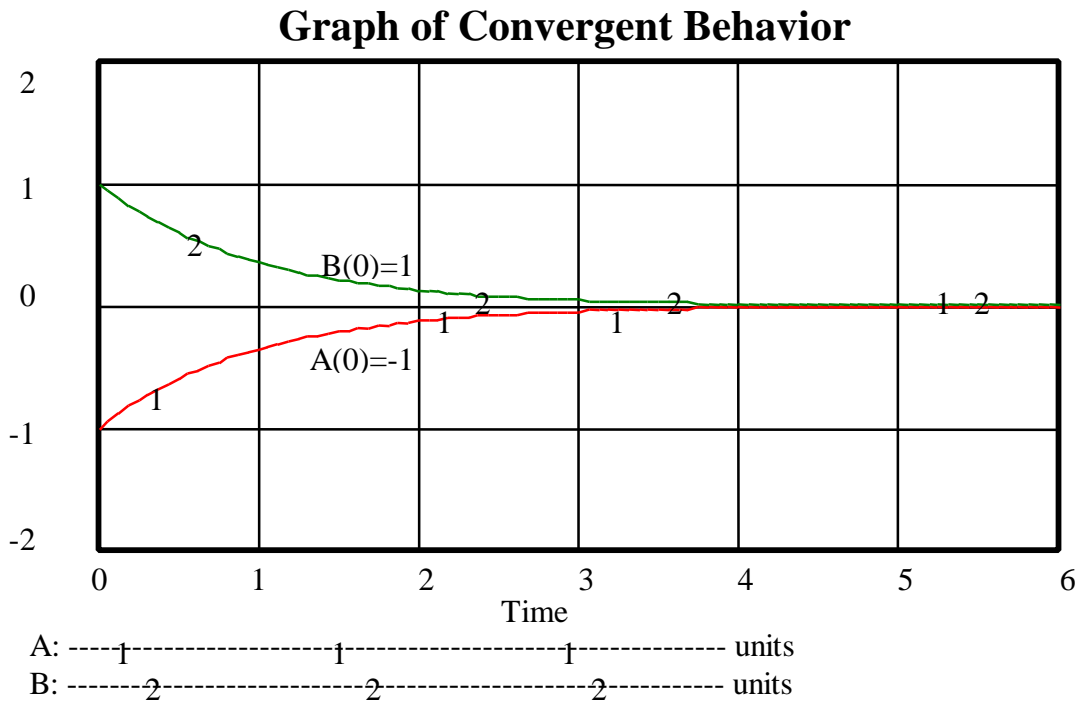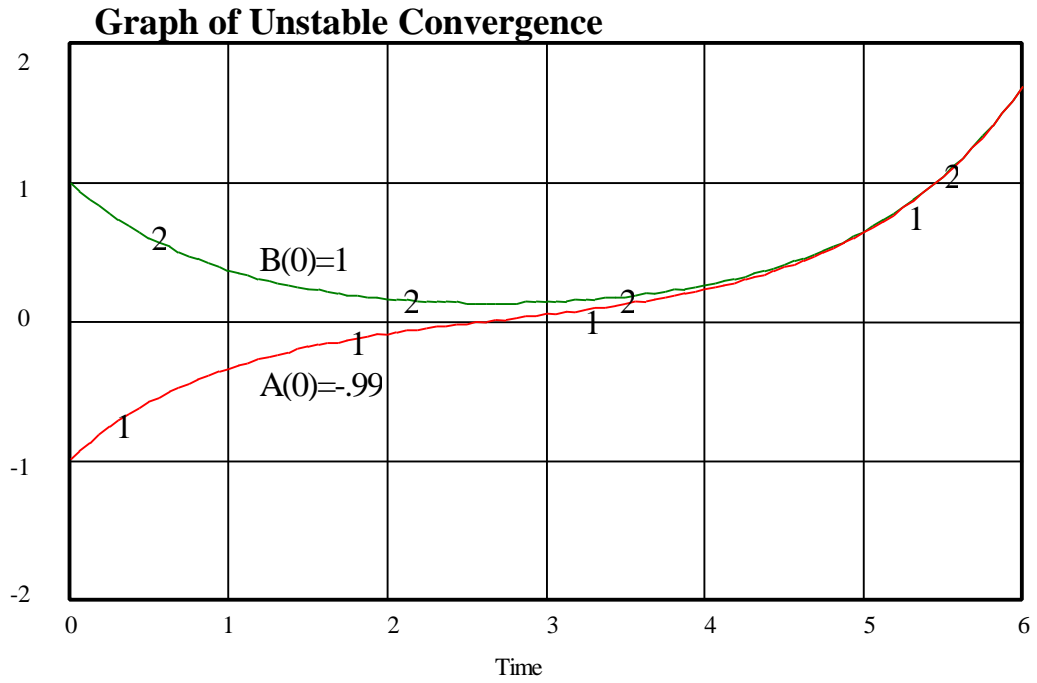initial conditions (second-order positive feedback loop). B(0)=1 in all cases.

Graph of Convergent Behavior

Figure 18: Vensim Equivalent of Figure 5:Convergent behavior from initial conditions
A(0)=-1, B(0)=1 (second order positive loop).

Graph of Unstable Convergence

A: ------1----------------------1--------------------1---------------- units

B: --------2----------------------2--------------------2------------- units

Figure 19: Vensim Equivalent of Figure 6: Unstable convergence perturbed to exponential growth (second-order positive feedback loop).
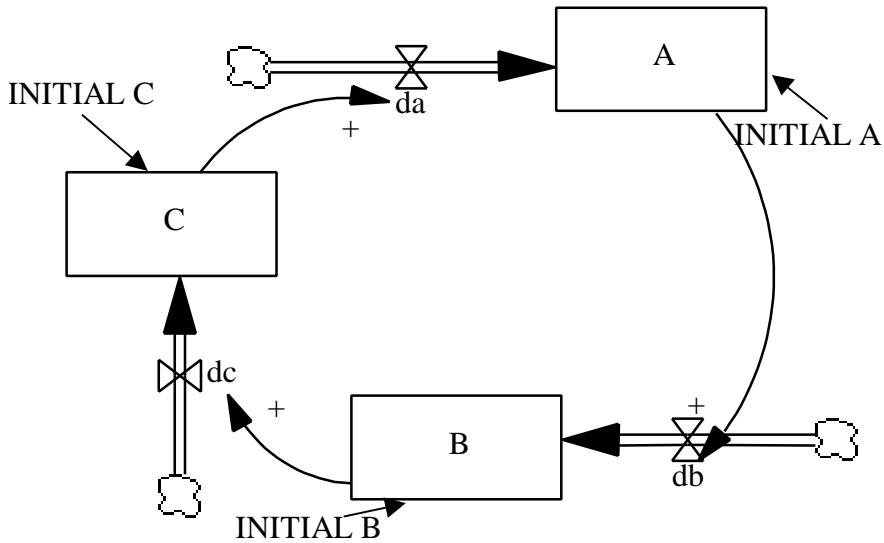
3.3 Third-Order Loop

Figure 20: Vensim Equivalent of Figure 7: Third-order positive feedback loop.

Documentation for third-order positive feedback loop

(01)   A= INTEG (da, INITIAL A)

Units: **undefined**

(02)   B= INTEG (db, INITIAL B)

Units: **undefined**

(03)   C= INTEG (dc, INITIAL C)

Units: **undefined**

(04)   da=C

Units: **undefined**

(05)   db=A

Units: **undefined**

(06)   dc=B

Units: **undefined**

(07)   FINAL TIME = 12

Units: none

The final time for the simulation.

(8)   INITIAL A=1

Units: **undefined**

(9)   INITIAL B=-1

Units: **undefined**

(10)   INITIAL C=0

Units: **undefined**

(11)   INITIAL TIME = 0

Units: none

The initial time for the simulation.

(12)   SAVEPER =TIME STEP

Units: none

The frequency with which output is stored.

(13)   TIME STEP = 0.0625

Units: none

The time step for the simulation.
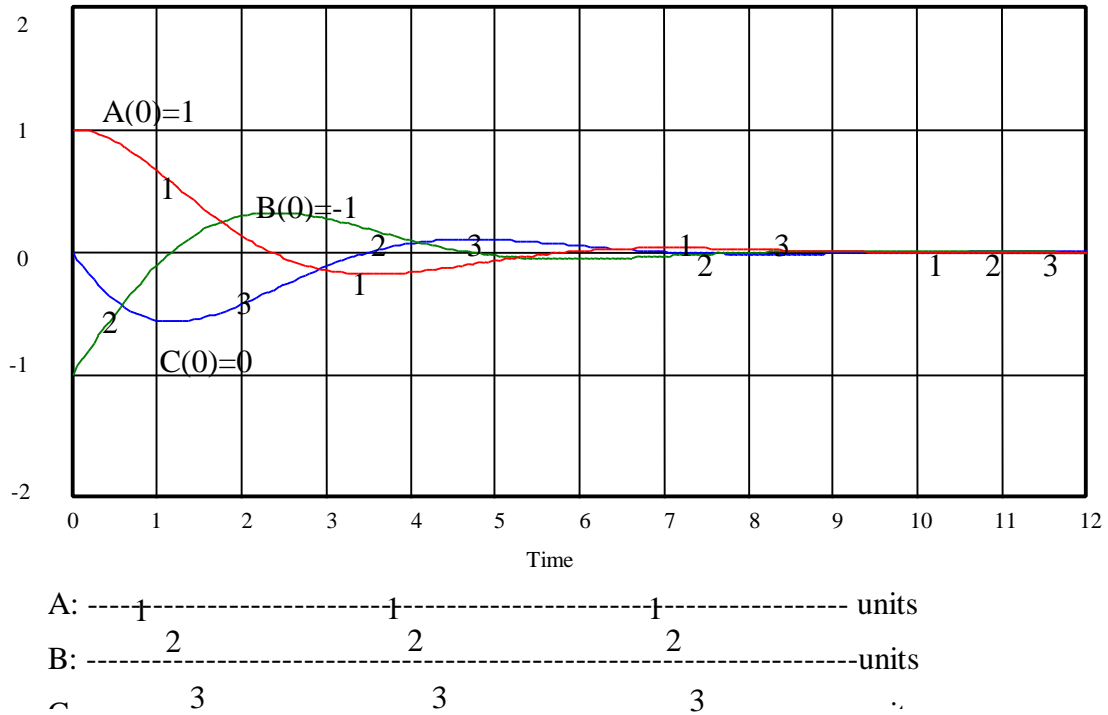
Graph of Convergent Periodic Oscillations

Figure 21: Vensim Equivalent of Figure 8: Convergent periodic oscillations (third-order positive feedback loop).

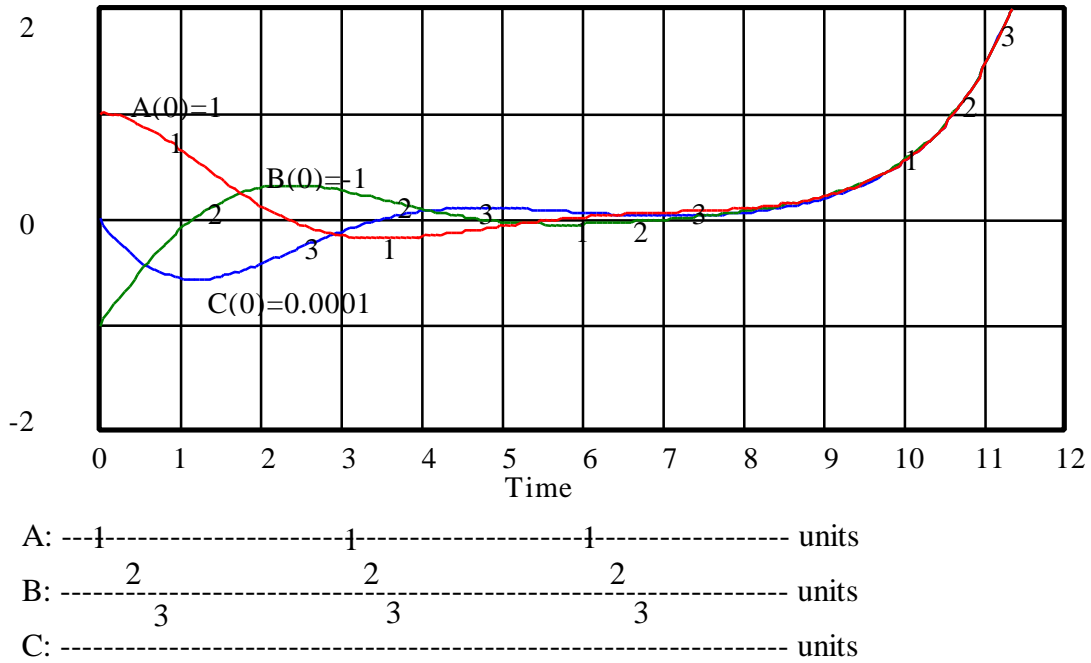**Graph of Destabilizing Convergent Periodic Oscillations**

Figure 22: Vensim Equivalent of Figure 9: Convergent periodic oscillations destabilizing to exponential growth (third-order positive feedback loop).
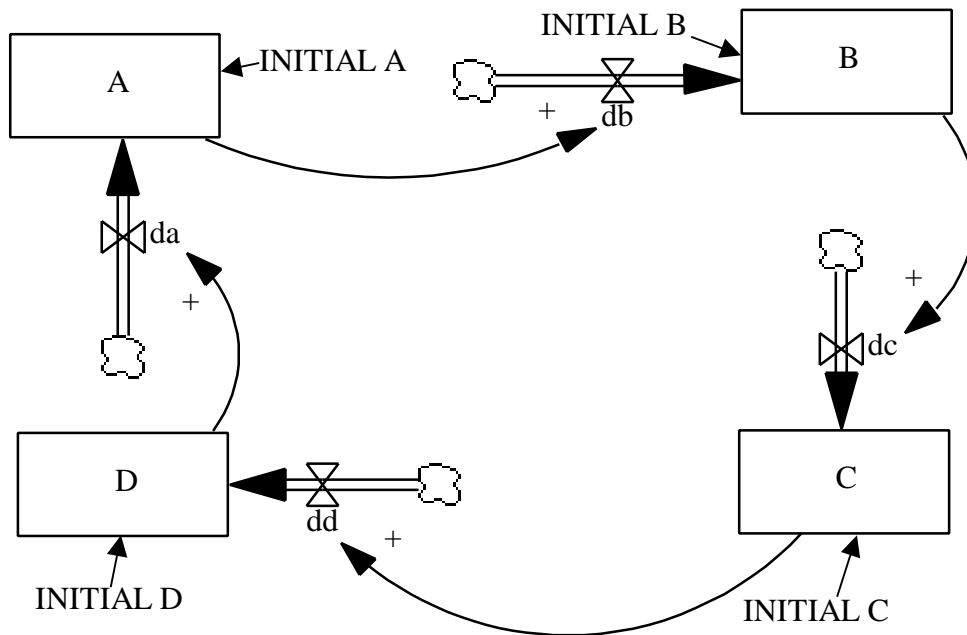
3.4 Fourth-Order Loop

Figure 23: Vensim Equivalent of Figure 10: Fourth-order positive feedback.

Documentation for fourth order feedback loop

(01)   A= INTEG (da, INITIAL A)

Units: **undefined**

(02)   B= INTEG (db, INITIAL B)

Units: **undefined**

(03)   C= INTEG (dc, INITIAL C)

Units: **undefined**

(04)   D= INTEG (dd, INITIAL D)

Units: **undefined**

(05)   da=D

Units: **undefined**

(06)   db=A

Units: **undefined**

(07)   dc=B

Units: **undefined**

(08)   dd=C

Units: **undefined**

(09)   FINAL TIME = 12

Units: none

The final time for the simulation.

(10)   INITIAL A=1

Units: **undefined**

(11)   INITIAL B=-1

Units: **undefined**

(12)   INITIAL C=0

Units: **undefined**

(13)   INITIAL D=0

Units: **undefined**

(14)   INITIAL TIME = 0

Units: none

The initial time for the simulation.

(15)   SAVEPER = TIME STEP

Units: none

The frequency with which output is stored.

(16)   TIME STEP = 0.015625
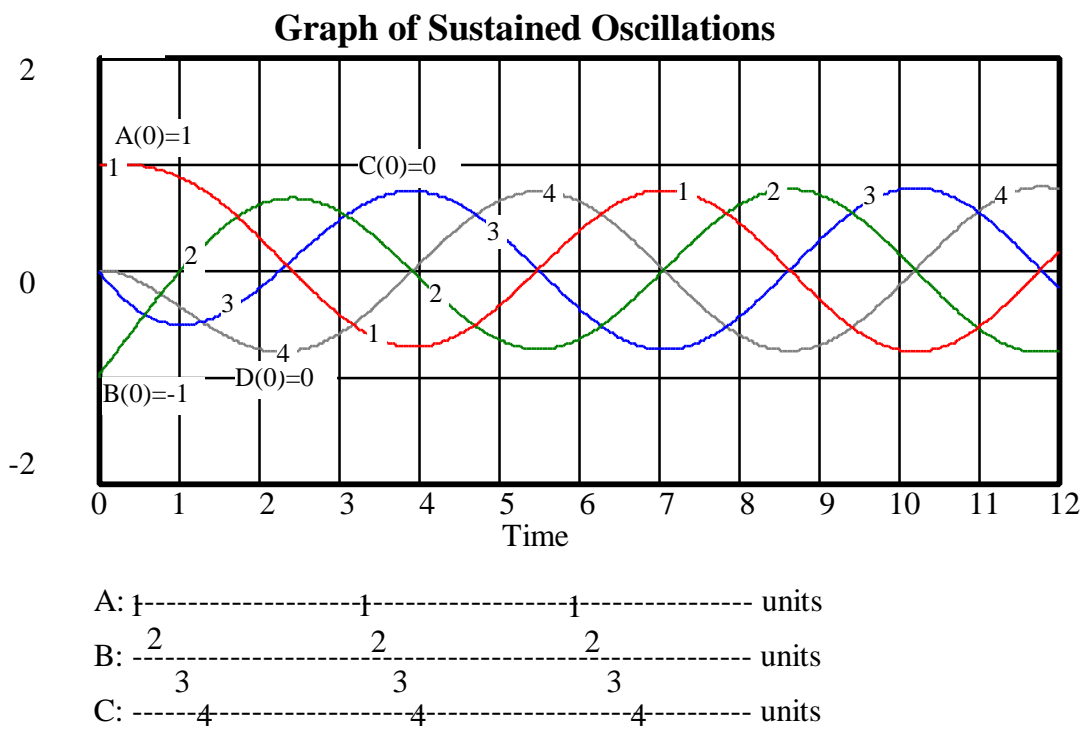
Units: none

The time step for the simulation.

Figure 24: Vensim Equivalent of Figure 11: Sustained oscillations within a fourth-order positive feedback loop.

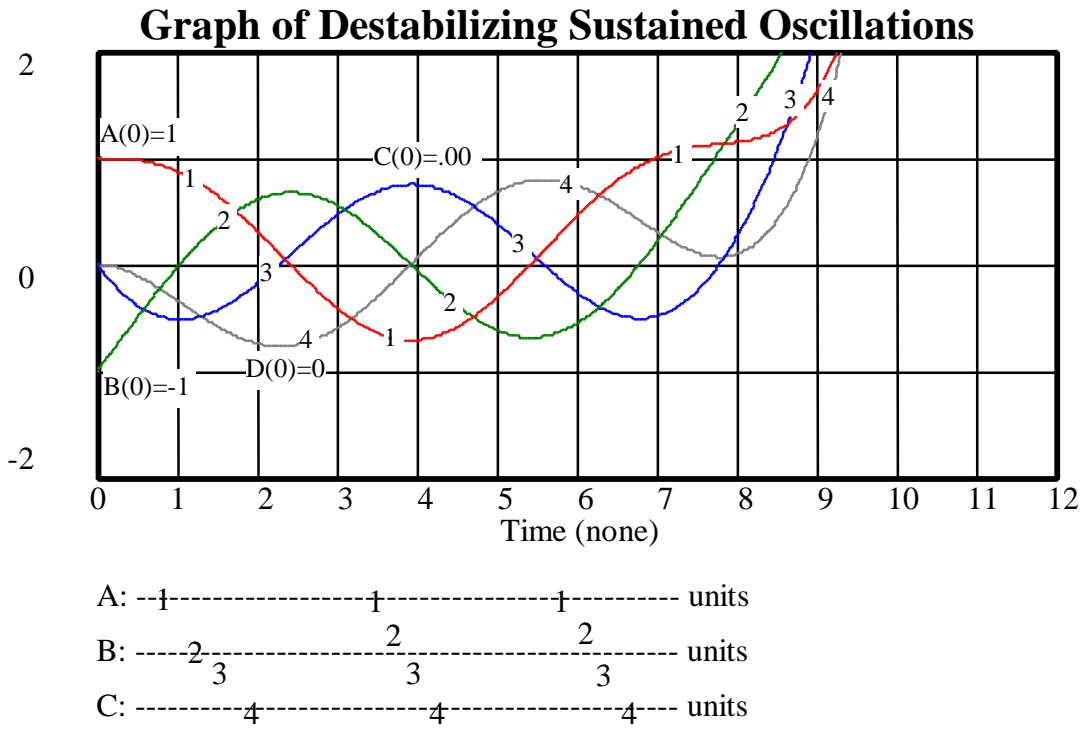## Graph of Destabilizing Sustained Oscillations

Figure 25: Vensim Equivalent of Figure 12: Sustained oscillations destabilizing to exponential growth (fourth-order positive feedback loop).