Stocking Retail Assortments Under Dynamic Consumer Substitution

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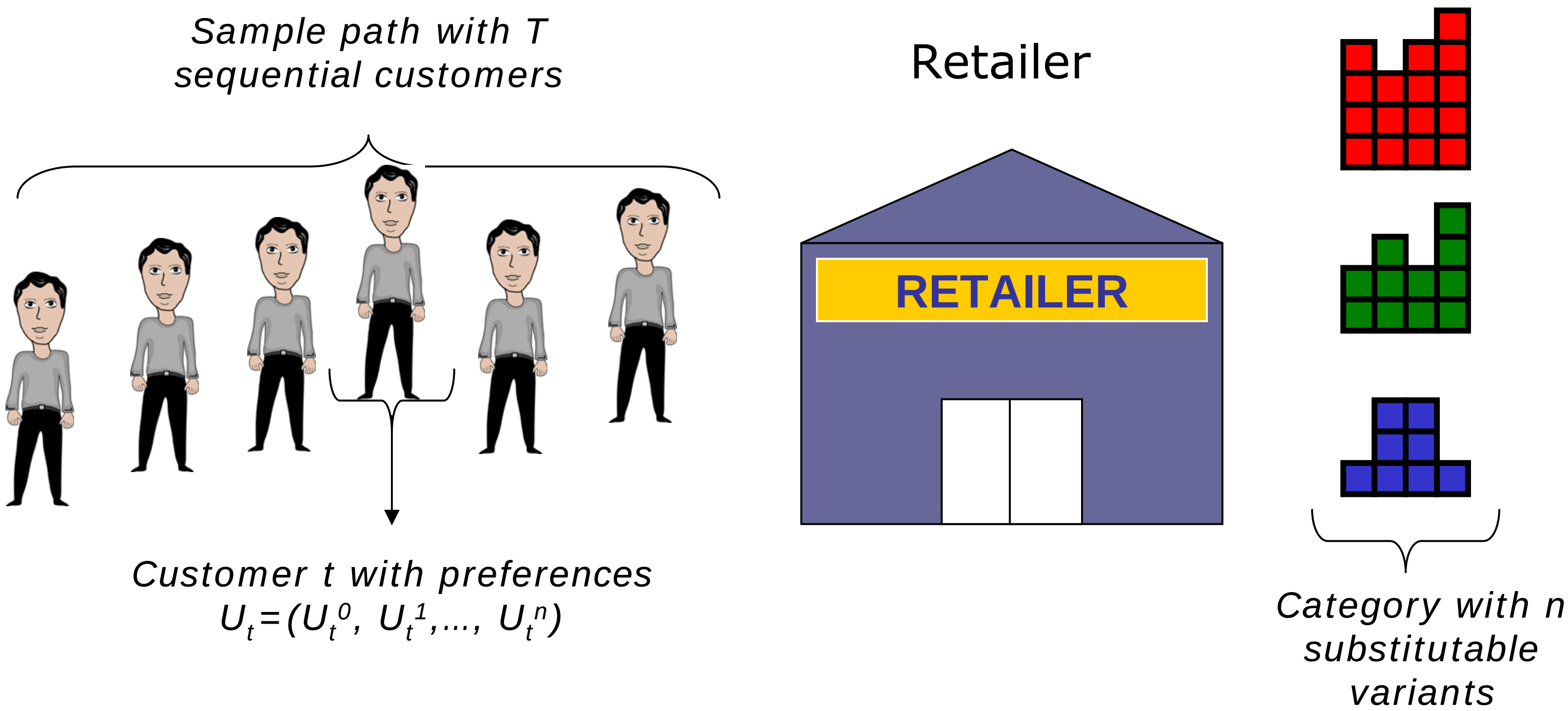
This summary presentation is based on: Mahajan, Siddharth, and Garrett van Ryzin. "Stocking Retail Assortments Under Dynamic Consumer Substitution." *Operations Research* 49, no. 3 (2001).

Motivation



Sample path with T sequential customers
Retailer
RETAILER
Customer t with preferences $U_t=(U_t^0, U_t^1,\dots, U_t^n)$
Category with n substitutable variants

Motivation

- Retail consumers might substitute if their initial choice is out of stock
  - Retailer's inventory decisions should account for substitution effect

- Consumers' final choice depends on what he/she sees available "on the shelf".
  - In most previous models demand is independent of inventory levels.

- Contribution of this paper:
  - Determination of initial inventory levels (single-period) taking into account dynamic substitution effects

Outline

- Brief Literature Review
- Model Formulation
- Structural Properties
  - Component-wise Sales Function
  - Total Profit Function
  - Continuous Model
- Optimizing Assortment Inventories
- Numerical Experiments
- Price and Scale Effects
- Conclusions

Brief Literature Review

- Demand models with *static* substitution:
  - Smith and Agrawal (2000) / van Ryzin and Mahajan (1999):
    1. First choice is independent of stock levels
    2. If first choice is out of stock, the sale is lost (no second choice)
    3. Consequently, demand is independent of inventory levels
- Papers that model dynamic effects of stock-outs on consumer behavior
  - Anupindi et al. (1997): concerned solely with estimation problems (no inventory decisions)
  - Noonan (1995): only one substitution attempt

Model Formulation

- Notation:

  (See "Model Formulation" on page 336 of the Mahajan and van Ryzin paper)

Model Description

- Assumptions:
  - –Sequence of customers is finite w.p.1
  - –Each customer makes a unique choice w.p.1

- Some special cases:
  - -Multinomial Logit (MNL): $U_t^j = u^j + \xi_t^j$
  - -Markovian Second Choice:

$$q^j = P(U_t^{[1]} = U_t^j)$$

$$P(U_t^{[2]} = U_t^k \mid U_t^{[1]} = U_t^j) = p_j^k \qquad U_t^0 > U_t^{[3]} > \ldots > U_t^{[n]}$$

  - -Universal Backup: all customers have an identical second choice
  - -Lancaster demand: attribute space [0,1] and customer t has a random "ideal point" L_t, then $U_t^j = a - b\left\| L_t - l^j \right\|$

Model Formulation

- Profit function:
  - $\eta^j(x,\omega)$ = sales of variant j given x and ω.
  - Individual profit: $\pi^j(x ,\omega) = p^j \eta^j(x,\omega) - c^j x^j$.
  - Total profit: $\pi(x ,\omega) = \Sigma\, \pi^j(x ,\omega)$.

- Retailer's objective: $\max_{x \geq 0} E[\pi(x,\omega)]$

- Recursive formulation:
  - System function: $f(x_t, U_t) = x_t - e^{d(x_t, U_t)} = x_{t+1}$
  - Sales-to-go:

$$\eta_t^j(x_t, \omega) = x_t^j - f^j(x_t, U_t) + \eta_{t+1}^j(x_{t+1}, \omega)$$

  - Border conditions: $\eta_{T+1}^j(x_{T+1}, \omega) = 0 \qquad x_1 = x$

Structural Properties

- Lemma 1:
  - $x \geq y \Longrightarrow x_t \geq y_t$ for all sample paths.

- Decreasing Differences:
  - $h: S \times \Theta \longrightarrow \Re$ satisfies decreasing differences in (z, θ) if $h(z', \theta) - h(z, \theta) \geq h(z', \theta') - h(z, \theta')$ for all $z' \geq z$, $\theta' \geq \theta$.

  - Lemma: if $\max\{h(z, \theta): z \in S\}$ has at least one solution for every $\theta \in \Theta$, then the largest maximizer $z^*(\theta)$ is nonincreasing in θ.

Structural Properties

- Theorem 1:

  The function $\eta^j(z \cdot e^j + \theta, \omega)$ satisfies:

  (a) Concavity in z for all ω.

  (b) Decreasing differences in (z, θ) for all ω.

- Corollary 1:

  (a) A base-stock level is optimal for maximizing the component-wise profits.

  (b) The component-wise optimal base-stock level for j is nonincreasing in x_i $(i \neq j)$.

$\Longrightarrow$ Usual newsboy problem

Structural Properties

- Let:
  - $T(x, \omega) = \sum \eta^j(x, \omega)$ = total sales
  - $H^j(z, \omega) = T(z \cdot e^j + \theta, \omega)$

- If all variants have identical price and cost, then:

  $$\pi(x, \omega) = p \cdot T(x, \omega) - c \cdot \sum x^j$$

- Theorem 2:

  There exists initial inventory levels x and sample paths ω for which:

  (a) $T(x, \omega)$ is not component-wise concave in x.

  (b) $H^j(z, \omega)$ does not satisfy decreasing differences in (z, θ).

Structural Properties

- Counterexample for (a):
  - (See the first two tables on page 339 of the Mahajan and van Ryzin paper)

- Continuous model:
  - Customer t requires a quantity Q_t of fluid with distribution $F_t(\cdot)$
  - Redefine sample paths: $\omega = \{(U_t, Q_t): t=1,\ldots,T\}$

- Theorem 3:

  There exist sample paths on which $\pi(x, \omega)$ is not quasi-concave

Optimizing Assortment Inventories

- Lemma 3:

  If the purchase quantities Q_t are bounded continuous random variables then $\nabla E[\eta(x,\omega)] = E[\nabla\eta(x,\omega)]$

- Calculating $\nabla\eta(x,\omega)$:

  –(See the equations and explanation in section 4.1, page 341 of the Mahajan and van Ryzin paper)

Optimizing Assortment Inventories

- Sample Path Gradient Algorithm
  - (See the steps in section 4.2, page 341 of the Mahajan and van Ryzin paper)

- Theorem 4:

  If $F_t(\cdot)$ is Lipschitz for all t then any limit of y_k is a stationary point

Numerical Experiments

- Heuristic policies (with T~Poisson)

Let $q_j(S)$ be the probability of choosing variant j from S

1. Independent Newsboy: demand for each variant is independent of stock on hand.

$$x_I^j = \lambda q^j(S) + z^j \sqrt{2\lambda q^j(S)} \qquad j \in S$$

2. Pooled Newsboy: customers freely substitute among all available variants.

$$q(S) = \sum_{j \in S} q^j(S) \qquad x(S) = \lambda q(S) + z\sqrt{2\lambda q(S)}$$

$$x_P^j = x(S) \frac{q^j(S)}{q(S)}$$

Numerical Experiments

- Assumptions:
  - -T~Poisson(30).
  - -Q_t~exp(1).

- Example 1:
  - –Assume MNL utilities:

$$\begin{cases} q^j(S) = P(U_t^j = \max\{U_t^i : i \in S\}) = \dfrac{\upsilon^j}{\sum_{i \in S} \upsilon^i + \upsilon^0} \\ \text{where } \upsilon^j = \begin{cases} e^{u_j/\mu} & j \in S \\ e^{u_0/\mu} & j = 0 \end{cases} \end{cases}$$

  - –Equal cost and prices.
  - –Result from van Ryzin and Mahajan: assume $v_1 \geq v_2 \geq \ldots \geq v_n$, the optimal assortment is a set $A_k=\{1,2,\ldots,k\}$ with $k=1,\ldots,n$.
  - –Simulation: profit within ±1% with 95% confidence.
  - –Several starting points tested.

Numerical Experiments

- Performance Comparison for Example 1.
  - -Independent newsboy is biased: underestimates popular items, over estimates less popular items.

(See Table 2 and Figure 1 on pages 343-4 of the Mahajan and van Ryzin paper.)

Numerical Experiments

- Example 2:
  - –Two variants: variant 1 less popular but with high margin, and variant 2 more popular but lower margin.
  - –Sample Path Gradient policy induces customers to “upgrade” to the high-margin variant.

(See Table 3 and Figure 4 on page 345 of the Mahajan and van Ryzin paper.)

Numerical Experiments

- Example 3:
  - Lancaster demand model:

$$U_t = a - b\ L_t - l^j$$

  - $L_t \sim U[0,1]$, a=0.2, b=1

(See Table 4, Figures 5, and Figure 6 on pages 345-6 of the Mahajan and van Ryzin paper.)

Price and Scale Effects

- Measure of "evenness":
  - y is more "fashionable" than z, if z majorizes y:

$$\sum_{i=1}^{n} y^{[i]} = \sum_{i=1}^{n} z^{[i]}$$

$$\sum_{i=1}^{k} y[i] \le \sum_{i=1}^{k} z^{[i]} \quad k = 1, \ldots, n-1$$

- Observations:

1) If v is more fashionable than w, then w is more profitable

2) Price or volume increase $\Longrightarrow$ higher variety is offered

Conclusions

- Concluding remarks
  - General choice model.
  - Improvement upon the existing literature.
  - Interesting numerical experiments with valuable insights.

- Comments
  - Assortment or inventory problem?
  - Assumes full information.
  - Single replenishment: price decision might be relevant.
  - Industry evidence (field study).