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Sloan School of Management

15.450-Fall 2010
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Problem Set 2

1. (To be solved in a group) This problem illustrates how to extend a static model like CAPM to a dynamic setting, allowing us to price risky streams of cash flows.

   Time is discrete, $t = 0, 1, 2, ...$ Consider a project with free cash flows C_t following an AR(1) process:

   $$C_t - \bar{C} = \theta(C_{t-1} - \bar{C}) + \sigma\varepsilon_t, \quad \theta \in (0,1) \qquad (1)$$

   where ε_t are IID, $\mathcal{N}(0,1)$ shocks. Assume that the project generates an infinite stream of free cash flows according to the above distribution.

   Next, suppose that the state-price density in the economy is given by

   $$\pi_t = \pi_{t-1} \exp\left(-a - b(r_t^m - \overline{r_m})\right), \quad a, b > 0$$

   where r_t^m are market returns, distributed identically over time, $r_t^m \sim \mathcal{N}(\overline{r_m}, (\sigma_m)^2)$. Assume that the correlation between ε_t and r_t^m is constant and equal to $\rho > 0$.

   (a) Derive the one-period risk-free interest rate in this economy.

   (b) Derive the T-period interest rate in this economy.

   (c) Consider a stock in this economy, with IID returns $r_t^j \sim \mathcal{N}(\mu_j, (\sigma_j)^2)$, $\text{corr}(r_t^j, r_t^m) = \rho_{jm}$. Derive the expected excess return on this stock, discuss how your answer depends on the parameters. Relate your finding to the CAPM formula for expected returns, in particular, relate the expected return of stock j to its market beta.

   (d) Derive the process for free cash flows C_t in (1) under the risk-neutral probability distribution corresponding to the SPD π_t. Show that it is still and AR(1) process, but with different parameters. (Hint: compare this problem to our analysis of futures prices).

   (e) Compute the market value of the project. If this project was tradable, what would be the conditional expected return on this project? Discuss the role of parameter θ: how does the risk of the project depend on persistence of cash flows?

   (f) This question is qualitative. What would be the naive definition of cash flow beta of the project in (1)? Relate the cash flow beta to your answer in item (1e) and discuss the potential problems with the naive application of CAPM to pricing risky cash flows.

2. (To be solved individually) Consider the process X_t satisfying an SDE

$$dX_t = -\theta(X_t - \overline{X})\,dt + \sigma X_t^{\gamma}\,dZ_t$$

Using the Ito's formula, derive an SDE on

(a) $Y_t = 1/(1 + X_t)$;

(b) $Y_t = \exp(X_t)$;

(c) $Y_t = \ln(X_t)$.

3. (To be solved individually) Solve the SDE

$$dX_t = -\theta(X_t - \overline{X})\,dt + \sigma\,dZ_t$$

Compute conditional mean and variance

$$\mathrm{E}_t[X_{t+s}] \quad \text{and} \quad \mathrm{Var}_t[X_{t+s}]$$

4. (To be solved in a group) Consider a square-root process

$$dX_t = -\theta(X_t - \overline{X})\,dt + \sigma\sqrt{X_t}\,dZ_t$$

Your objective is to compute the conditional expectation

$$\mathrm{E}_t[X_T|X_t], \quad T > t$$

(a) Assume that the conditional expectation can be expressed as a function

$$\mathrm{E}_t[X_T|X_t] = f(t, X_t)$$

Characterize $f(t, X)$ as a solution of a PDE with appropriate boundary conditions.

(b) Find the solution, assuming the functional form

$$f(t, X) = a_0(t) + a_1(t)X$$

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