

Mid Term Solution

Mr. Smith's Garden

FORMULATION

Decision variables (5 points): there are decision variables for vegetables planted, for tomatoes sold, for tomatoes frozen, for soup made and sold. All variables are non negative.

Define

$X_1, X_2, X_3, X_4, X_5, X_6$ – number of plants to grow, for beefsteak tomato, better boy tomato, ... green pepper, respectively. (units are number of plants)

T_7, T_8, \dots, T_{20} – pounds of tomatoes to sell in weeks 7, 8, ... 20.

F_7, F_8, \dots, F_{20} - pounds of tomatoes to freeze in weeks 7, 8, ... 20.

S_9, S_8, \dots, S_{16} – pounds of soup to make in week 9, 10, ... 16.

Y – pounds of soup to sell.

Constraints (20 points): there are resource constraints, balance constraints for tomatoes, and supply constraints for soup.

Resource constraints

There are only 2000 square feet for planting:

$$9 X_1 + 8 X_2 + \dots + 3 X_6 \leq 2000.$$

There are only 25 hours of labor each week:

$$0.1 X_1 + 0.1 X_2 + \dots + 0.05 X_6 \leq 25.$$

Tomato balance constraints

In each week, tomatoes produced equals tomatoes consumed by Mr. Smith, or frozen, or sold.

$$\begin{array}{ll} 1.5 X_3 - T_i - F_i = 10 & \text{for } i = 7, 8, 9 \\ 2 X_2 + 1.5 X_3 - T_i - F_i = 10 & \text{for } i = 10 \\ 2 X_1 + 2 X_2 + 1.5 X_3 - T_i - F_i = 10 & \text{for } i = 11, 12 \dots 16 \\ 2 X_1 + 2 X_2 - T_i - F_i = 10 & \text{for } i = 17 \end{array}$$

$$2 X1 - T_i - F_i = 10 \quad \text{for } i = 18, 19, 20$$

There is also a demand constraint as Mr. Smith needs 250 pounds of frozen tomatoes.

$$F7 + F8 + \dots + F20 \geq 250.$$

Soup supply constraints

Each week the amount of soup made is limited by the ingredients, that need be supplied according to the recipe. One can deduce that soup can only be made in weeks 9, 10, ... 16, due to the requirement to use fresh ingredients. The amount made cannot exceed the available supply of ingredients. Since one eggplant bush yields one pound of vegetable each week and one pound of eggplant is needed to produce four pounds of soup, then one eggplant bush is needed for every four pounds of soup that we make. Similarly we need one zucchini plant to make twelve pounds of soup, and one green pepper plant for every two pounds of soup we make.

$$\begin{aligned} 4 X4 - S_i &\geq 0 \\ 12 X5 - S_i &\geq 0 \\ 2 X6 - S_i &\geq 0 \end{aligned} \quad \text{for } i = 9, 10, \dots 16.$$

We need 200 pounds of soup for winter and can sell up to 2000 pounds of soup:

$$S9 + S10 + \dots + S16 - Y \geq 200,$$

and $Y \leq 2000$.

Objective function (10 points): The objective is to maximize revenue net of operating costs.

Mr. Smith gets revenue from selling tomatoes according to the declining price schedule, and from selling soup a \$5 per pound:

$$\text{Revenue}(\$) = 4 T7 + \dots + 3.2 T11 + \dots + 2.2 T16 + \dots 1.4 T20 + 5 Y$$

Mr. Smith has a labor cost for each plant. For instance, for each Beefsteak plant, he requires 0.1 hours of labor each week for 20 weeks, at a cost of \$8 per hour. Thus the labor cost is \$16 per Beefsteak plant for the season.

$$\text{Cost}(\$) = 16 X1 + 16 X2 + 24 X3 + 8 X4 + 8 X5 + 8 X6$$

The objective is to maximize Revenue – Cost.

SOLUTION

(15 points)

The optimal solution is given by

	Number of plants
Beefsteak tomato	67.5
Better Boy tomato	19.3
Rutgers tomato	6.7
Eggplant	68.8
Zucchini	22.9
Green Pepper	137.5

	Tomatoes sold	Tomatoes frozen	Soup Made
Week 7			
Week 8			
Week 9			275
Week 10	38.5		275
Week 11	173.5		275
Week 12	173.5		275
Week13	173.5		275
Week 14	173.5		275
Week 15	173.5		275
Week 16	173.5		275
Week 17	163.5		
Week 18	125		
Week 19		125	
Week 20		125	

At the end of the season 2000 pounds of soup are sold. The net revenue to Mr. Smith from this plan is \$10112.83.

Extra Labor (10 points): Mr. Smith should not seek extra labor as the optimal plan only requires 21 hours now, and there is five hours of slack. The shadow price of additional labor is zero.

More Space (15 points): As shown below, the shadow price for more space is \$3.40 per square foot. This shadow price is valid for an increase of 309 square feet. Thus, you can't answer the question without re-running the solver.

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$C\$9	garden area	2000	3.40	2000	309.17	154.17

When one re-runs the problem with 310 more square feet of space (total equals 2310), the shadow price for space drops to \$3.20 per square foot, and this is valid between 309 and 367 additional square feet.

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$C\$9	garden area	2310	3.2	2310	57.08	0.83

When one re-runs the problem with 368 more square feet of space, the shadow price drops to zero, as the space constraint is no longer binding; the solution remains unchanged for any amount of additional space beyond 368 square feet.

With 367 more square feet of space, the profit is \$11349.33, an increase of \$1236.67. This is the most that Mr. Smith would pay for 500 square feet of space. Indeed, he would be willing to pay as much as \$3.40 per square foot for the first 309 square feet, and then \$3.20 per square foot for the next 58 square feet. But he would not pay anything for incremental space beyond 367 square feet.

He will use the additional land to modify the mix of Beefsteak tomatoes and Better Boy tomatoes. The number of Rutgers tomatoes, eggplant, zucchini and green peppers remain the same. The table below shows how the plant mix changes as the size of the garden expands.

	2000 square feet	2309 square feet	2367 square feet
Beef Steak	67.5	67.5	125.4
Better Boy	19.3	57.9	0

As one initially adds space, the extra space is used to plant more Better Boy tomatoes, and ultimately to sell more tomatoes. This is because the revenue per square foot for Better Boy is higher than for Beef steak (once we have satisfied the requirement for frozen tomatoes).

However, after we add 309 square feet, then the labor constraint is binding. As we add additional space, we actually reduce the number of Better Boy plants and increase the number of Beef Steak plants. For each additional square foot we substitute one Beef

Steak for one Better Boy, since each require the same amount of labor but Beef steak requires one more square foot than Better Boy. Profit goes up since a single Beef Steak plant generates more revenue than a single Better Boy plant. This substitution stops after we add 58 more square feet since the number of Better Boy plants is now zero.

Tomato Purchases (15 points): The following table shows the non-zero reduced costs for the decision variable for selling tomatoes for the optimal solution. In all of the other weeks the reduced cost is zero.

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
\$N\$17	week 9 sell tom	0.00	-0.87	3.60	0.87	1E+30
\$N\$27	week 19 sell tom	0.00	-0.05	1.60	0.05	1E+30
\$N\$28	week 20 sell tom	0.00	-0.25	1.40	0.25	1E+30

These reduced costs suggest that Mr. Smith might want to buy tomatoes in weeks 9, 19, and 20. We can also look at the shadow prices for the tomato balance constraints, which indicate the impact on profit from having to supply one more pound of tomatoes. These shadow prices exceed the market price for tomatoes in weeks 9, 19 and 20, which suggests that Mr. Smith would want to buy tomatoes in these weeks.

To confirm this intuition, we can re-solve the LP – where we add a set of variables to denote buying tomatoes in each week. The tomato balance constraints are now modified, e.g., for weeks 11, 12...16:

$$2 X_1 + 2 X_2 + 1.5 X_3 - T_i - F_i + B_i = 10 \quad \text{for } i = 11, 12 \dots 16$$

where B_i is the amount bought in week i .

The garden plan is now

	beefsteak	betterboy	rutgers	eggplant	zucchini	green pepper
	0	101.04	0	68.75	22.92	137.5

With a profit of 10187.

Furthermore, Mr Smith buys 10 pounds of tomatoes in weeks 7, 8, 9, 18, and 19, and then buys 260 pounds in week 20. It is cheaper for Mr. Smith to buy tomatoes in these weeks, rather than plant Rutgers plants for the early weeks or plant Beef Steaks for the last weeks of the season.

Soup Sale (10 points) From the sensitivity report, we get the reduced cost for the upper bound on the amount of soup sold:

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
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\$L\$6	soup sold	2000.00	2.33	5.00	1E+30	2.33
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This tells us that if we could sell an incremental pound of soup for \$5, then our total profit increases by \$2.33. So if we were to sell the incremental pound of soup for \$3, then our profit increases by \$0.33. So the proposed contract has potential to be of interest.

The above sensitivity information does not tell us the range over which the reduced cost is valid. So we need to re-solve the LP. When we do, we get a new garden plan:

beefsteak	betterboy	rutgers	eggplant	zucchini	green pepper
67.5	5.73	6.67	75	25	150

With a profit of \$10177.83. We take the contract for 200 additional pounds of soup. Consequently we increase the amount of the garden used for eggplant, zucchini, and green peppers so as to make 200 more pounds of soup. We make 300 pounds of soup in each week from week 9 to week 16.

We still plant 67.5 Beef Steak and 6.67 Rutgers – so as to satisfy Mr. Smith’s requirements. We reduce the number of Better Boy plants, and thus reduce the amount of tomatoes that we sell.