Gomory Cuts and a little more

Hi, Mita and I are here to introduce a tutorial on cutting planes.

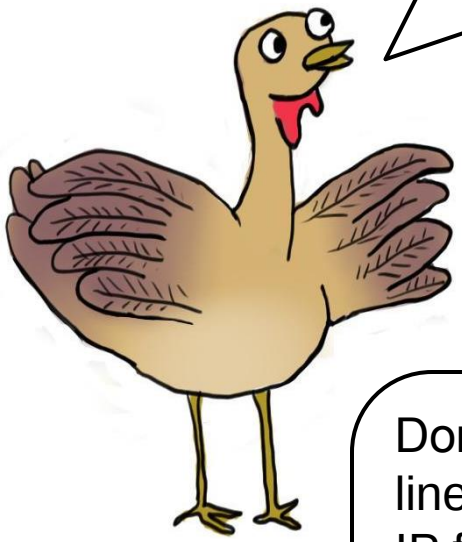
Amit

Cutting planes are a useful technique that, in conjunction with branch and bound, enable us to solve integer programs more quickly. Our friends from the course will tell you more…

Mita

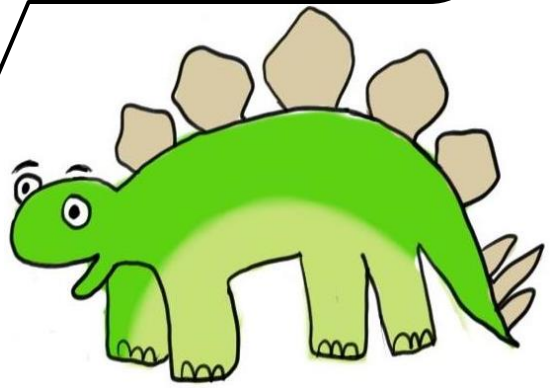
This tutorial was developed by Zach Leung in April 2012.

I heard professor Orlin say in lecture that “cutting planes” are a useful technique to enable us to solve IPs more quickly. What are these “cutting planes?”



Tom

Is “cutting planes” something that terrorists do? It sure sounds dangerous to me. After what happened to the dinosaurs, I've become extremely risk-averse.



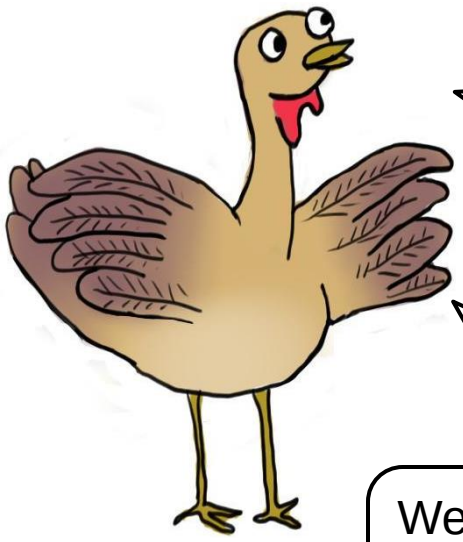
Stan

Don’t be alarmed, Stan. Cutting planes are linear inequalities that allow us to improve IP formulations, by cutting down the feasible region. This makes it more likely that the LP relaxation finds an integer optimal solution, as well as improving the upper bound (for maximization problems).

Ella

Ella, your explanation is too abstract for me. Could you please go through an example for me?

Of course your horse! *That's what friends are for*. da da da da da, du du du du du du du, Well you came in loving me…

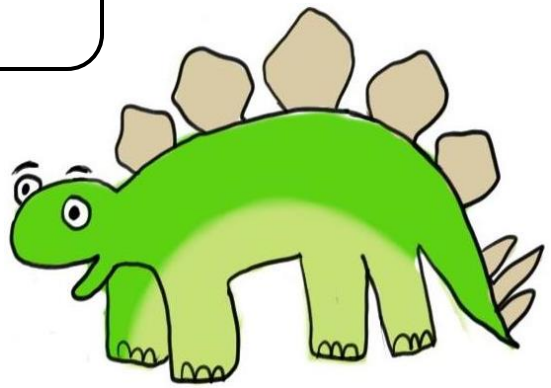


Hey, isn't that the song *That's what friends are for* by Dianne Warwick and friends?

Tom, you're right. Er, how did you know? I thought it was a song before your time.

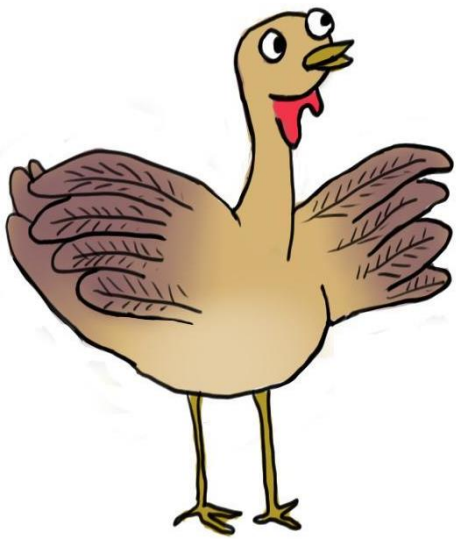
Well, you sang that during the last tutorial on branch and bound. I have been paying attention, you know.

This repeated joke suggests to me that the writer of this tutorial is running out of fresh jokes. By the way, I didn't get the joke last time or this time.**



** Stan's comment was added by Professor Orlin during editing.

Suppose that we are trying to solve the IP problem on the right. What should I do?



IP **max** $x_1 + x_2$

subject to $-5x_1 + 4x_2 \leq 0$

$5x_1 + 2x_2 \leq 15$

$x_1, x_2 \geq 0$ x_1, x_2 **integral**

The starting point is to solve the linear programming relaxation of the problem, which is shown below. Basically, we "relax" the integrality constraints, i.e. we "forget" about them.

LP(1) **max** $x_1 + x_2$

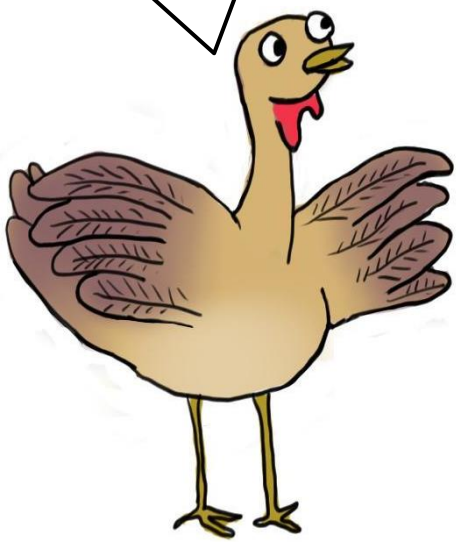
subject to $-5x_1 + 4x_2 \leq 0$

$5x_1 + 2x_2 \leq 15$

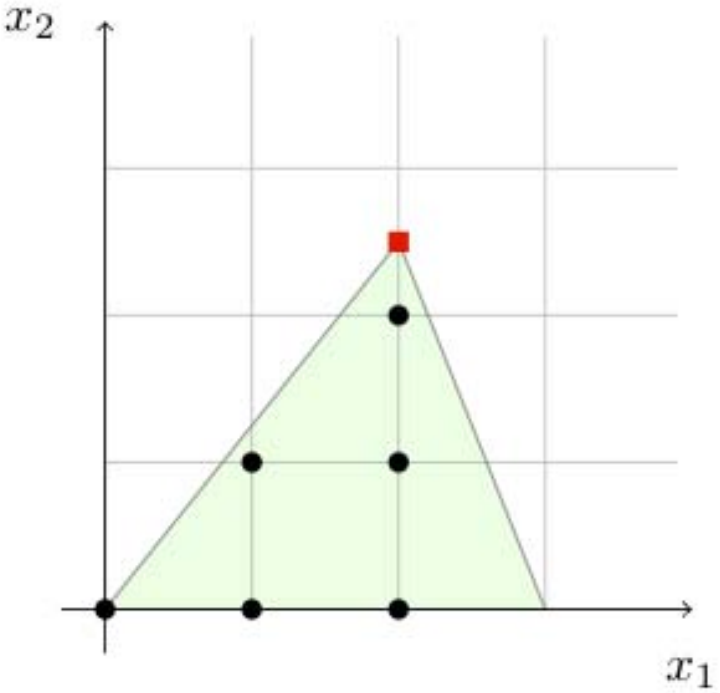
$x_1, x_2 \geq 0$

Acknowledgements: This example is taken from Professor Forbes Lewis from the University of Kentucky

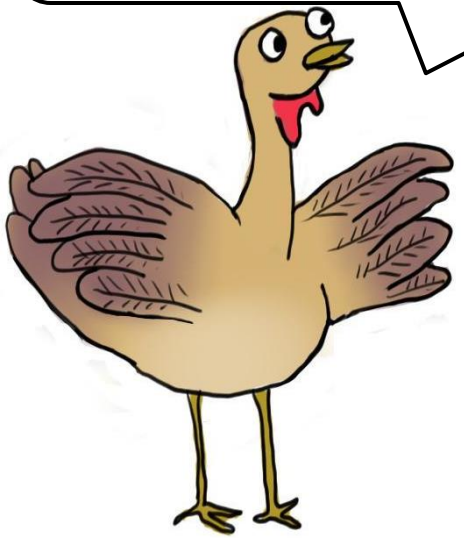
Ella, could you use graphs in your explanation? I find that seeing graphs helps me to understand what is going on.



Tom, that's an excellent idea! The graph below shows the feasible region, represented by the green shaded triangle, and the feasible integer solutions, represented by the black dots. The optimal solution is shown as a red square.

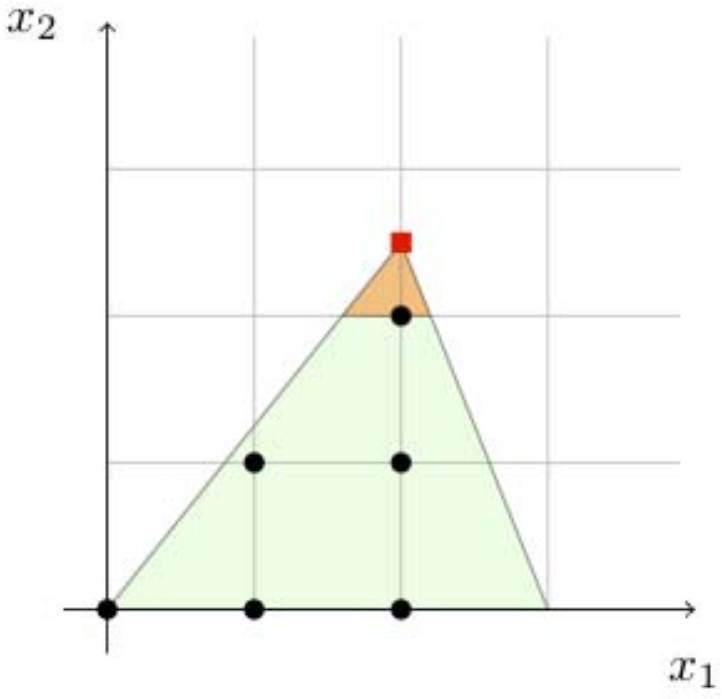


I notice that the optimal solution to the LP is not integer. Rats!

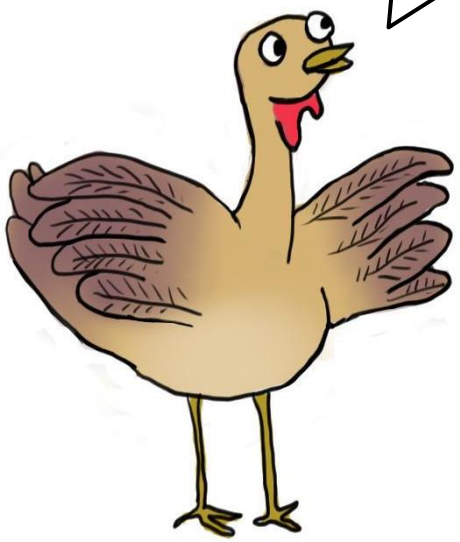


Don't worry Tom, I eat rats for breakfast! We can use a *Gomory cut* to reduce the feasible region. In the later part of the tutorial, we will derive the Gomory cut. But for now, you can take my word for it that the Gomory cut is $x_2 \leq 2$.

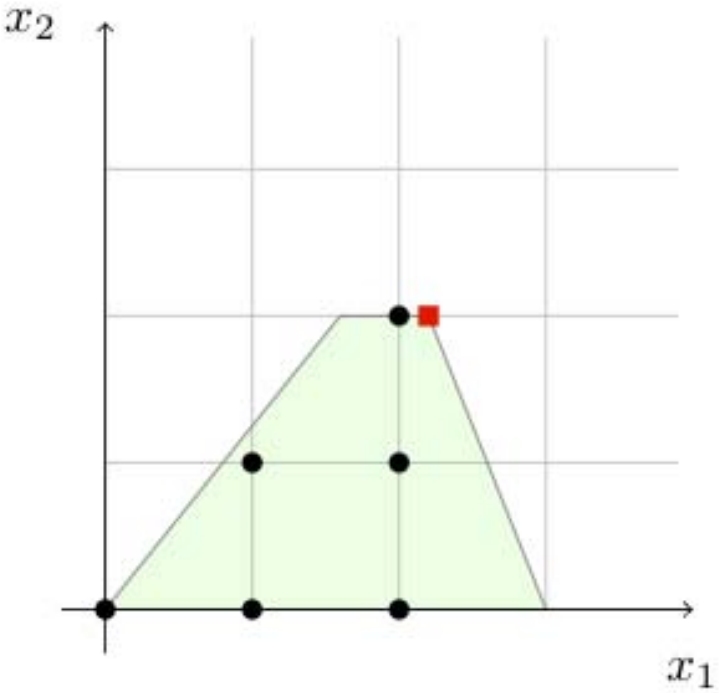
The region that has been cut off is shown in orange. I notice that this is a *valid cut* because:
(1) The linear inequality has cut the feasible region so that the original LP optimal solution is now outside the feasible region;
and (2) all the feasible integer solutions are still in the feasible region. Very cool!



By adding the new Gomory cut to the original LP, we now have a new LP formulation shown below.



The feasible region, and the optimal solution is shown below as well. Notice that the feasible region has shrunk to more closely surround the integer feasible solutions.



LP(2) **max** $x_1 + x_2$

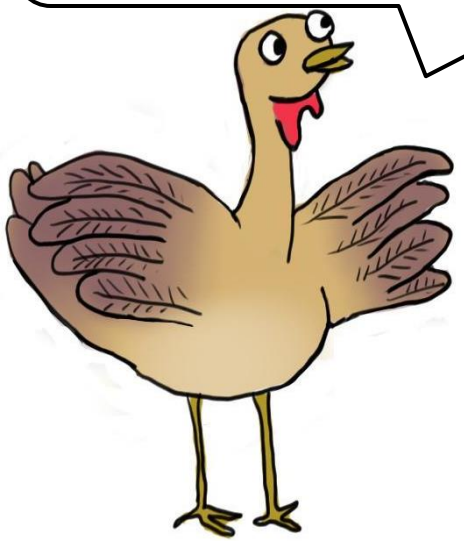
subject to $-5x_1 + 4x_2 \leq 0$

$5x_1 + 2x_2 \leq 15$

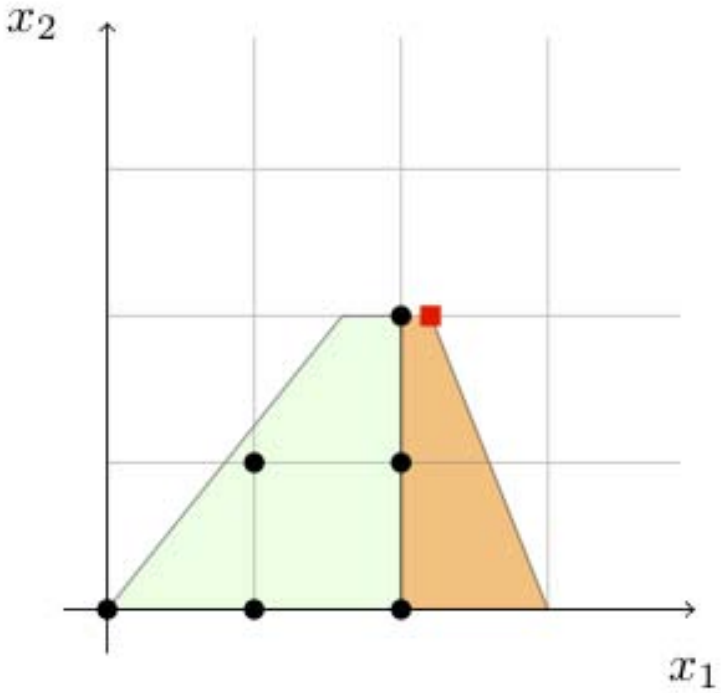
$x_2 \leq 2$

$x_1, x_2 \geq 0$

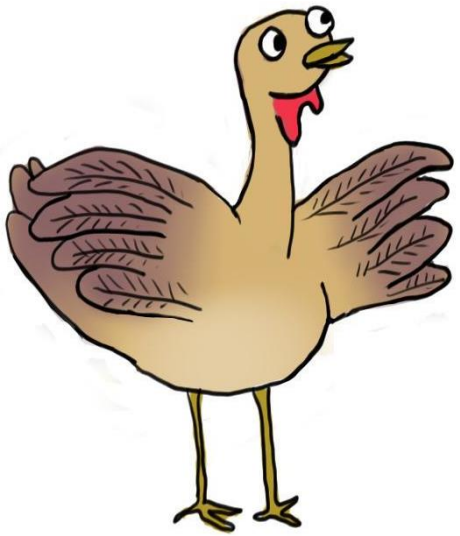
The optimal solution is still not integral. What a bummer!



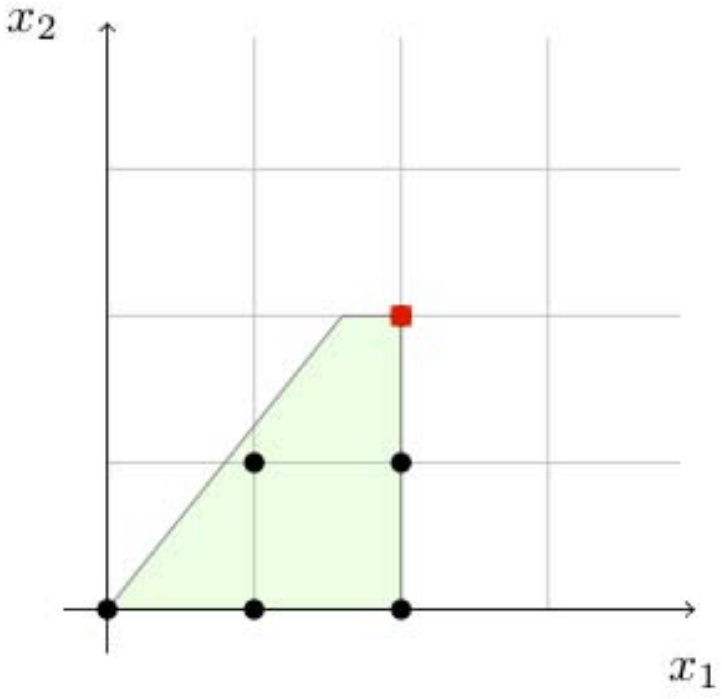
Don’t give up Tom, we’re getting close. We can add yet another Gomory cut, this time the cut is $x_1 \leq 2$.



OK, if I add in the new cut, I get the LP shown below. And the optimal solution is integral! Booyah!



See, I told you we were close!



LP(2) **max** $x_1 + x_2$

subject to $-5x_1 + 4x_2 \leq 0$

$5x_1 + 2x_2 \leq 15$

$x_2 \leq 2$

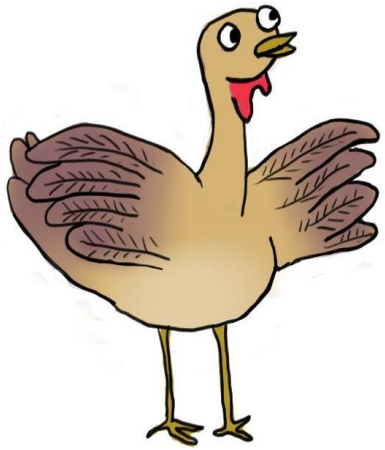
$x_1 \leq 2$

$x_1, x_2 \geq 0$

I now understand why cuts are useful to help us solve IPs. But how did you derive the Gomory cuts in the previous example?

Example 1. $1.6x - y = 1.8$

$x \geq 0, \quad y \geq 0 \qquad x, y$ integer



Tom, that's a great question. I don't know how we would do these tutorials if you didn't come up with all your great questions. Let's look first at the example that is right over my head.

Step 1: Notice that we can round down the left hand side coefficients, and we obtain the inequality.

$$x - y \leq 1.8$$

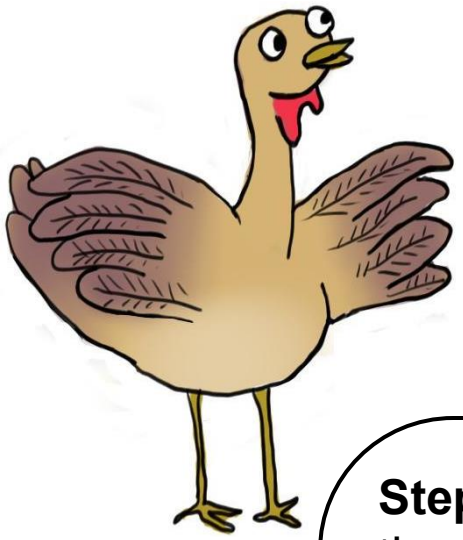
Step 2: Notice that the *lhs* is integer valued because x and y are integers. Therefore

$$x - y \leq 1$$

Step 3: Subtract the constraint of Step 2 from the original constraint. Thus

$$0.6x \geq 0.8$$

I like your explanation. How about another example?



Example 2. **2.3 x - .6 y = 3.9**

x ≥ 0, y ≥ 0 **x, y integer**

It's done in much the same way.

Step 1: Round down the positive coefficients of the *lhs* of the constraint. Also round down the negative coefficients. e.g., treat "**2.3 x - .6 y = 3.9" as "2.3 x + -.6 y = 3.9".**

2x - y ≤ 3.9

Step 2: Round down the rhs.

2x ≤ 3

Step 3: Subtract the constraint in Step 2 from the original constraint.

0.3 x + 0.4 y ≥ 0.9

I've tried to give another explanation for how the steps to work out a Gomory cut are derived. But if you're the type of person who just wants to know the steps, here they are as found in the lecture. If we just apply the steps, we get the answer very easily.

Example 1. $1.6x - y = 1.8$

$x \geq 0, \quad y \geq 0 \qquad x, y$ integer

1. Round down coefficients of the LHS and replace the equality sign by a "≤" sign.

2. Round down the RHS.

3. Subtract the constraint obtained in Step 2 from the original constraint.

$$0.6x \geq 0.8$$

I've tried to give another explanation for how the steps to work out a Gomory cut are derived. But if you're the type of person who just wants to know the steps, here they are as found in the lecture.

Example 2. $2.3x - .6y = 3.9$

$x \geq 0, \quad y \geq 0 \qquad x, y$ integer

1. Replace each coefficient by its fractional part,

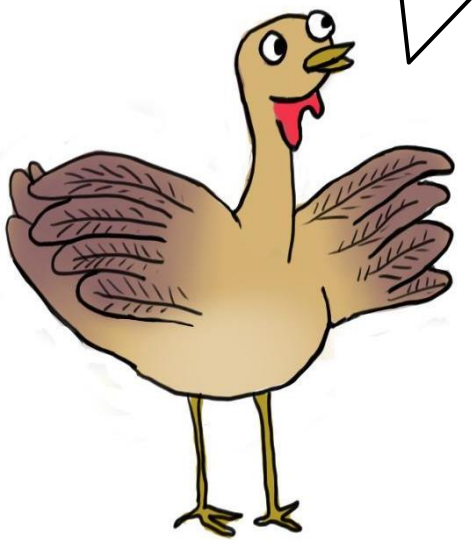
$0 \leq \text{FractionalPart} < 1$

2. Replace the RHS by its fractional part

3. Replace the equality by a " ≥ "

$.3x + .4y \geq .9$

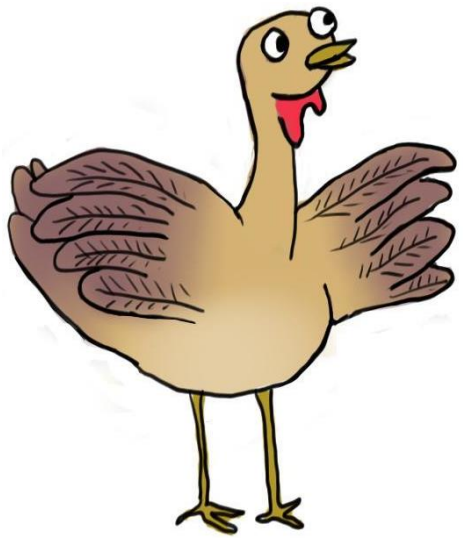
OK, I understand how you derive the Gomory cuts for the two examples in the lecture, but how about for our original cutting plane example?



To derive the Gomory cuts, first you work out the simplex tableau. Here is the optimal tableau for LP(1).
As you can see, the first constraint row (not the z-row) has the rhs = 5/2 which is non-integral, and so we apply a Gomory cut based on that constraint.

x_1	x_2	y_1	y_2	b_i
0	0	-1/10	-3/10	-9/2
0	1	1/6	1/6	5/2
1	0	-1/15	-1/15	2

Ella, aren't you going to show me how to work out the Gomory cuts? Please, pretty pretty please?



Tom, you should figure it out on your own. Whatever you do, don't click if you are using PowerPoint. Otherwise, the answer will be revealed.

By the way, the tableau below is the optimal tableau for LP (2). This time, I'll really let you work out the details.

x_1	x_2	y_1	Y_2	Y_3	b_i
0	0	0	-1/5	-4/5	-21/5
0	1	0	0	1	2
1	0	0	1/5	-2/5	11/5
0	0	1	1	-6	3

That's it for this tutorial.
We hope it was helpful.

Bye for now, but
I'm sure we'll
meet again soon!

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15.053 Optimization Methods in Management Science

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