

**Optimization Methods in
Operations Research &
Management Science
15.053/058**

**Sensitivity Analysis in 2
Dimensions**

Hi, welcome to a tutorial on sensitivity analysis for linear programs with two variables.



Amit, an MIT Beaver

It's good to see you again.



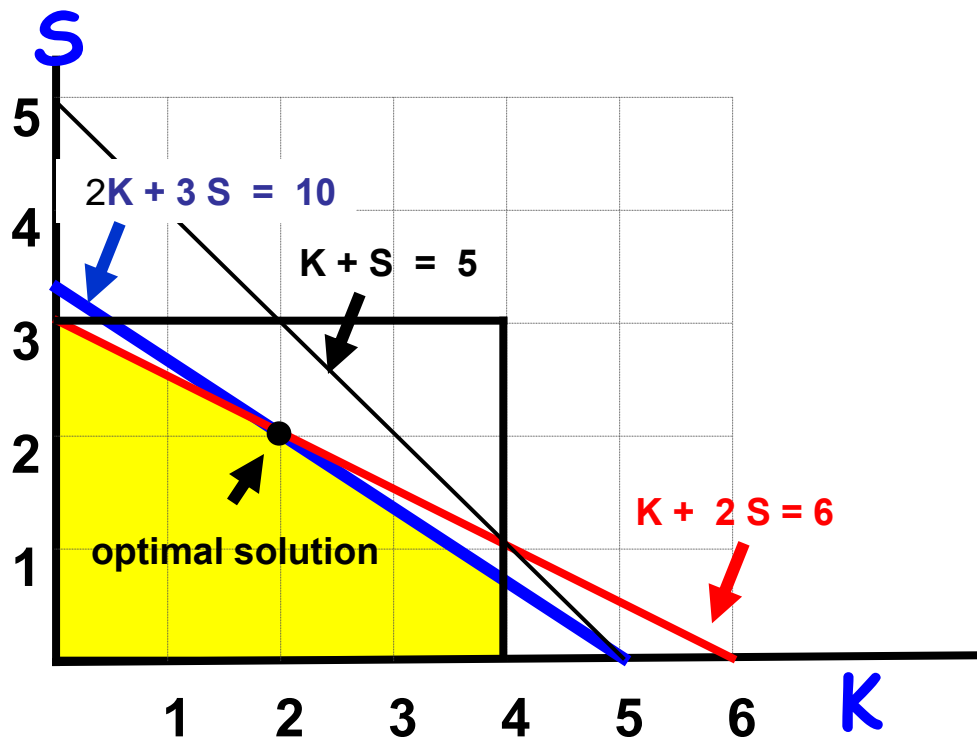
Mita, an MIT Beaver

For our running example, we will use David's Tool Company (DTC). You may recall that DTC makes slingshot kits and shields. They have time constraints for gathering stones, smoothing stones, and delivery. They also have upper bounds on the total demand for slingshot kits and shields. We use days instead of hours as the unit of time. We assume that each person works for 10 hours per day.

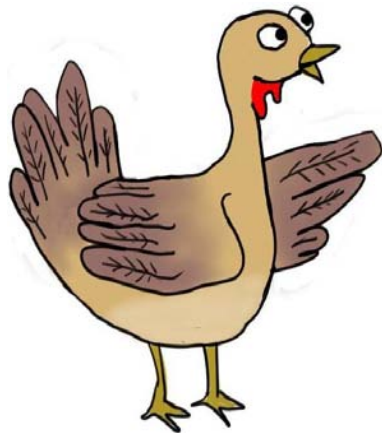


	Pack of 10 Slingshot Kits	Pack of 10 Stone Shields	Resources
Stone Gathering time	2 days	3 days	10 days
Stone Smoothing	1 day	2 days	6 days
Delivery time	1 day	1 days	5 days
Demand	4 packs	3 packs	
Profit	30 shekels	50 shekels	

The feasible region is in yellow. The optimal solution is at (2,2). The optimal objective value is $z = 160$. Sensitivity analysis deals with the following question: how does the optimal solution value change as the data for the problem changes. For example, suppose that we had 11 days of gathering time instead of 10 days. By how much would the optimum objective value improve?



Why should we want to change the data? If we are using data, then aren't the numbers correct?

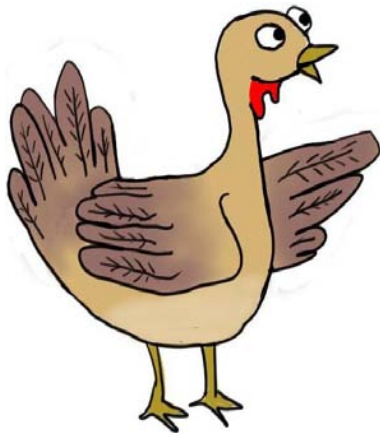


We all wish that numbers were correct, but they almost never are. They usually are estimates, and sometimes they are very bad estimates. For example, we write that it takes 2 days to smooth 10 shields. But this is at best an estimate based on previous experience. Each shield may take a different amount of time, and the time it takes may depend on how tired the stone smoother is.

The upper bound on demand is also an estimate, or forecast. And forecasts are often terrible. And don't even get me started on measures of costs. In large firms, everyone computes costs a little differently.



I don't understand. If the numbers are not correct, won't the answers be garbage? And what is the point of sensitivity analysis on bad data?



I said that the numbers were usually estimates, not that they were "garbage." By solving the LP, you often get a solution that is useful. It is usually far better than doing nothing or relying entirely on intuition.

As for sensitivity analysis, it has many uses. You can determine how sensitive the solution is to the data. For example, suppose that changing the demand of slingshot kits by 1 increased the profit by a lot. Then we may want to put some effort into advertising to increase demand. Or perhaps we would do surveys to better estimate the demand.

On the other hand, suppose that increased demand of slingshot kits does not improve the profit. Then there is no advantage in trying to increase the demand. And we probably would not be interested in surveys.

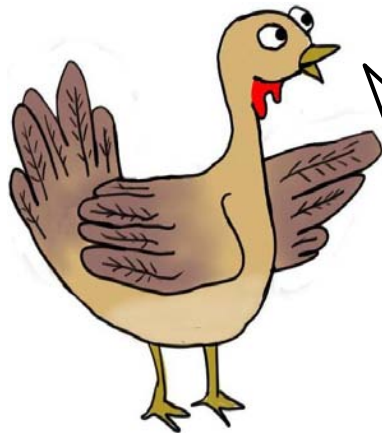


Often linear programs are solved in order to gain insights. Once the insights are gained, the problem becomes a little different. Management might consider different options.

And sensitivity analysis has other uses as well. For example, the DTC problem can be viewed as a “production mix” problem. In production mix problems, one may want to check to see if the optimal solution for the original problem is a good solution under a number of different scenarios.

We’ ll talk more about sensitivity analysis later in this course.

Now I’ m even more confused. Marketing and surveys are not part of the DTC problem.



OK. I think I get it. But why are we limiting ourselves to 2 variables and 2 dimensions?

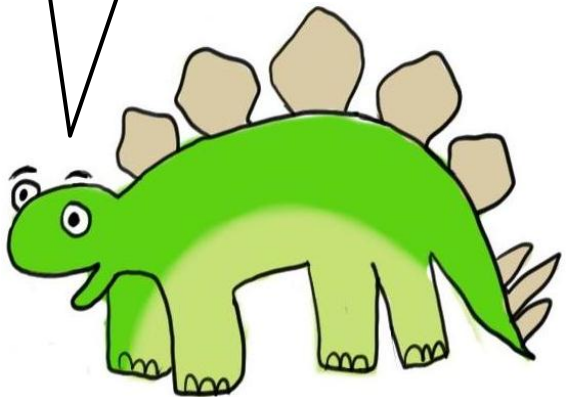
By restricting attention to 2 dimensions, we can use geometrical insights to help us understand sensitivity analysis.





I really like Tom, but I am also glad to be able to stop answering his questions and move on to the actual tutorial.

Our first topic is “shadow prices.” The shadow price of a constraint is the change in the optimal objective function if the RHS of the constraint is increased by 1. ¹



I wish I could ask questions as good as the ones that Tom asked. Turkeys are so smart.

¹ This definition is usually correct. But we will make it even more precise later on.

In order to motivate shadow prices, consider the following situation. Suppose that one of the children in town, say Beth, volunteered to do stone gathering for one day. What would be the extra profit for David, assuming that he does not pay her?



We could easily solve the linear program a second time with modified RHS and answer the question. But instead, we will carefully look at the graphical solution and see what insights we can obtain.

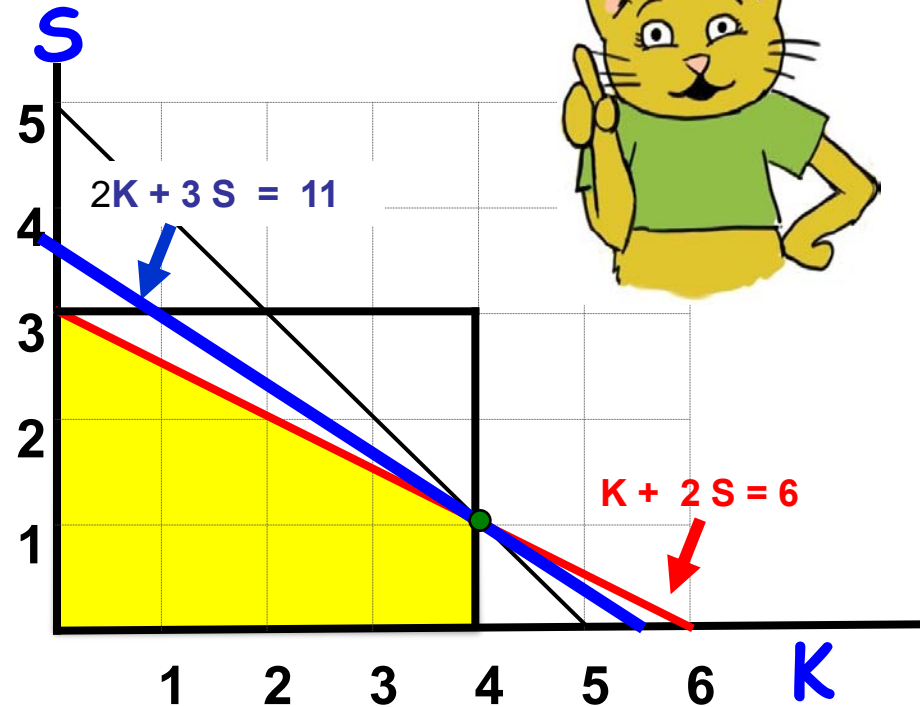
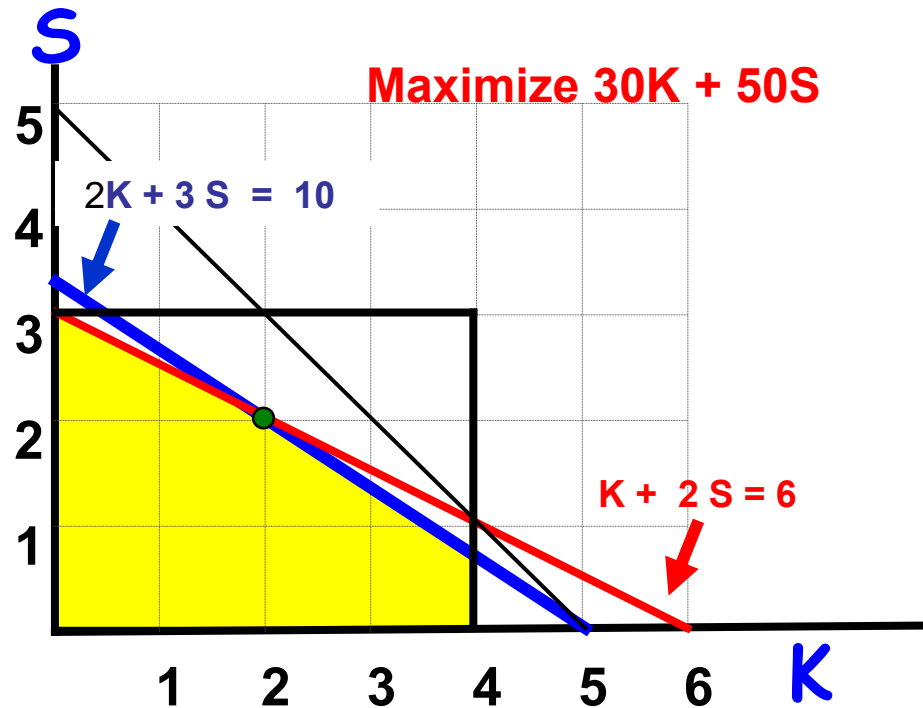
In order to do the analysis, we will assume that Beth's ability to do stone gathering is the same as given in the original data.





We've drawn the feasible region below. In the first case, there is a constraint $2K + 3S = 10$. The optimal solution is at $(2, 2)$. In the second case, we increased the RHS of this constraint to 11, and kept all other data constant. Notice that the feasible region got bigger. The new optimal solution is now at $(4, 1)$.

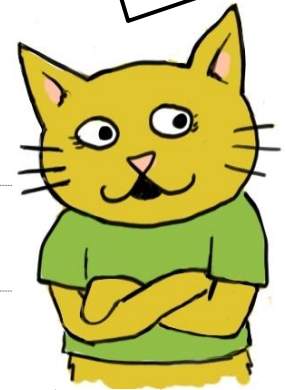
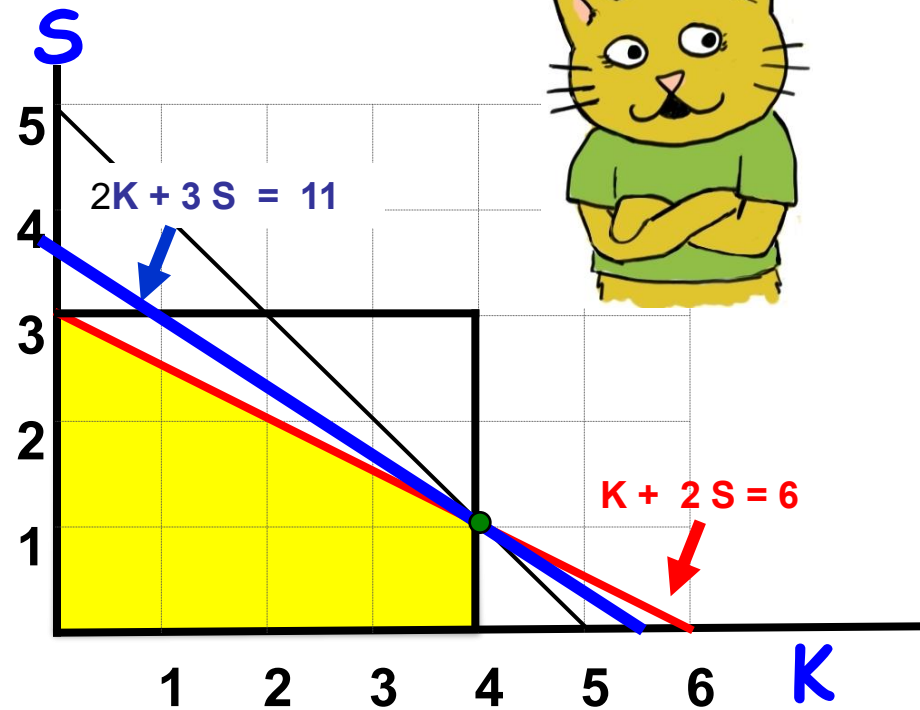
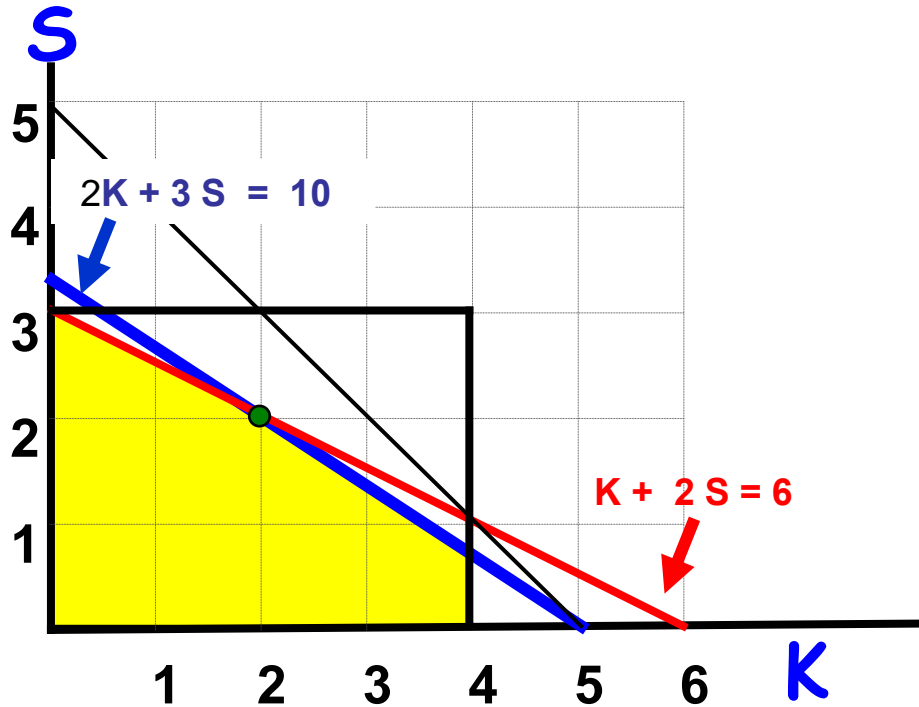
The original objective function was 160. The new optimum objective value is 170. Thus, the shadow price is $170 - 160 = 10$.



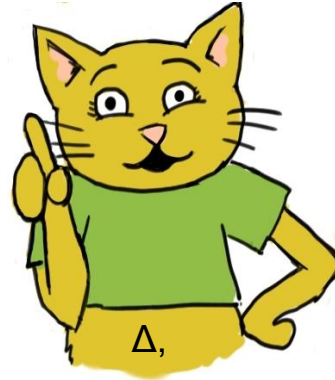


Before going any further about shadow prices or their interpretation, let's focus on the optimum solution. In the original problem, the optimum solution was at the intersection of the red and blue lines. When we increase the gathering time to 11, the optimum solution is still at the intersection of the red and blue lines.

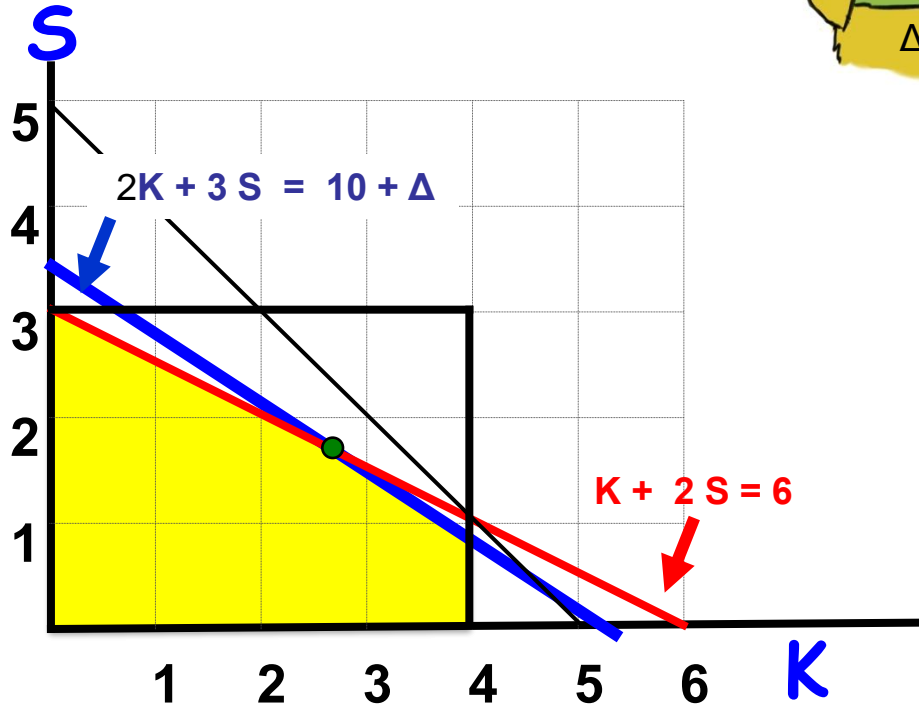
And if we had increased the amount of gathering time to a value between 10 and 11, the optimum solution will still be at the intersection of the red and blue lines. Next, we'll compute the optimum solution for any RHS.



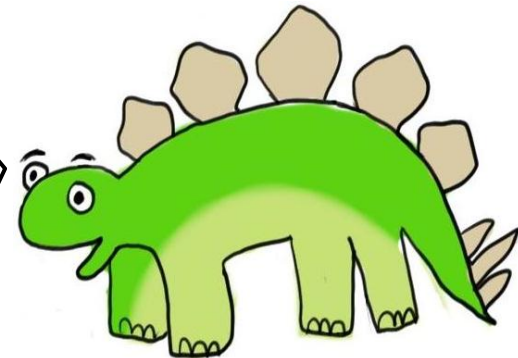
Suppose that the amount of gathering time is changed to $10 + \Delta$, where $0 \leq \Delta \leq 1$. The blue line becomes $2K + 3S = 10 + \Delta$. The red line is $K + 2S = 6$. The intersection of the two lines (expressed as a function of Δ) is $K = 2 + 2\Delta$; $S = 2 - \Delta$. The objective value is $30K + 50S = 160 + 10\Delta$.



Here is a key point. So long as $0 \leq \Delta \leq 1$, the optimal solution will be $K = 2 + 2\Delta$; $S = 2 - \Delta$, and the optimum objective value will be $160 + 10\Delta$. If we let $\Delta = .6$, then the objective value is 166. Thus, the increase in the optimum objective value is $6 = .6$ times the price. Thus the increase in optimal objective value is proportional to the increase in RHS.



Cathy does those calculations in her head. Wow!

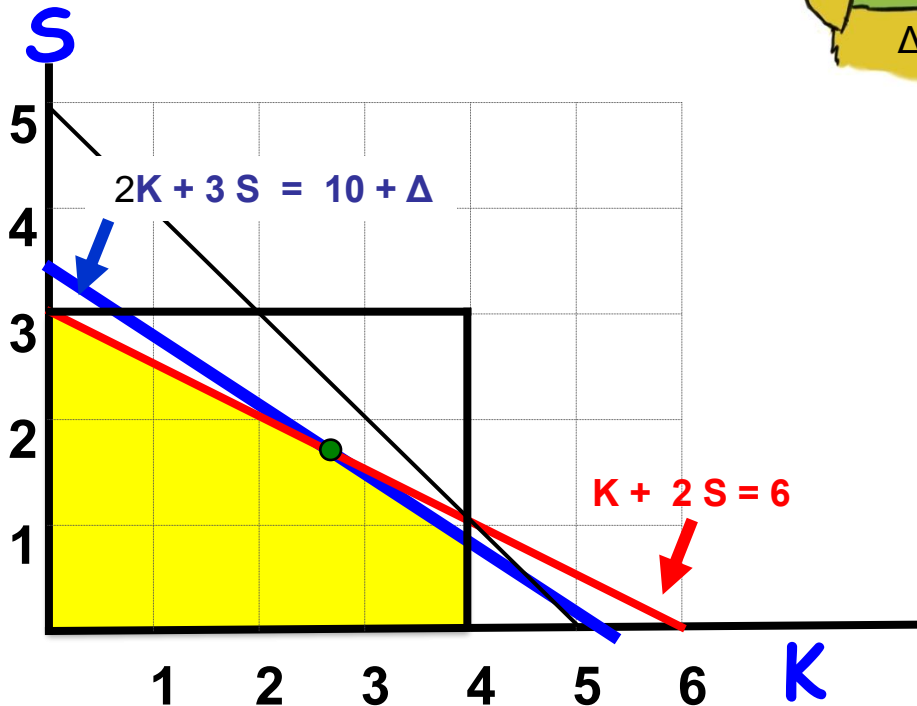


This is worth repeating. The optimal solution is $K = 2 + 2\Delta$; $S = 2 - \Delta$. The objective value is $30K + 50S = 160 + 10\Delta$.

I said before that this solution is optimal so long as $0 \leq \Delta \leq 1$. Actually, it is optimal so long as $-1 \leq \Delta \leq 1$, which is exactly the same range as where the solution is feasible.



If $\Delta < -1$, then $K = 2 + 2\Delta < 0$, which is infeasible. If $\Delta > 1$, then $K = 2 + 2\Delta > 4$, which violates the constraint " $K \leq 4$." For all values in between, the solution is optimal, and the objective value is $z(\Delta) = 160 + 10\Delta$.

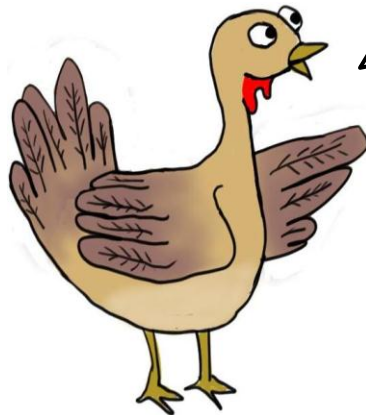


Before we said that the shadow price is the change in z if we increase the RHS by 1, which is 10 in this case. You can see that the shadow price is also the derivative of the optimal value function $z(\Delta)$.



We have illustrated shadow prices on one example with two variables. But what we have said is true for any linear program.

Suppose that the right hand side of some constraint (say constraint C) is increased by Δ , and all other data is unchanged. Then the optimal solution will be a linear function of Δ . And the optimal objective value $z(\Delta)$ will also be a linear function of Δ .

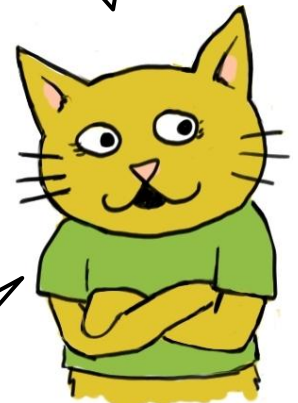


The shadow price of constraint C will be $z(1) - z(0)$, which is also the derivative of $z(\Delta)$ at $\Delta = 0$ (assuming that the derivative exists).

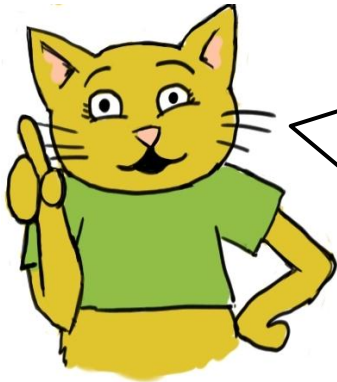
And the shadow price will be valid for all Δ such that the solution is feasible.

In your example, the solution was the intersection of the red and blue lines. But what are you talking about for other linear programs?

Great question, Tom.

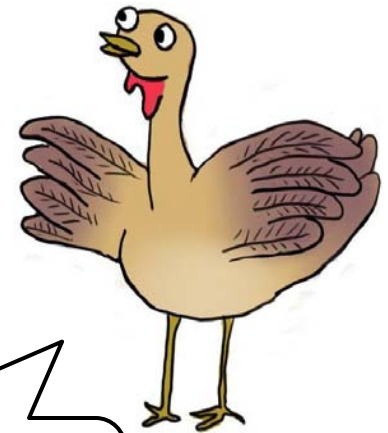


In two dimensional examples (that is, when there are two variables), the optimal solution can be described as the intersection of two lines. As Δ is increased or decreased from 0, the optimal solution is still the intersection of these two lines; however, now the solution will be a linear function of Δ .



Tom, I'll answer that question when we return to the subject of sensitivity analysis after showing how to solve linear programs with 3 or more variables. For now, I ask you to have some patience and trust.

But you still haven't answered my question. What happens if there are three or more variables?



We turkeys are very trusting and patient.

This is a good time to try it on your own. Can you determine the shadow price of the second constraint?



I'll get you started. Consider what happens when the second constraint becomes " $K + 2S = 6 + \Delta$ ". Then find the solution to the equations " $2K + 3S = 10$ " " $K + 2S = 6 + \Delta$ ", and express the solution as a function of Δ . Also compute the optimal objective value $z(\Delta)$.



Nooz, that isn't a hint. That's the answer.

That's pretty foxy.

I'll give you a hint.
The shadow price is 10.



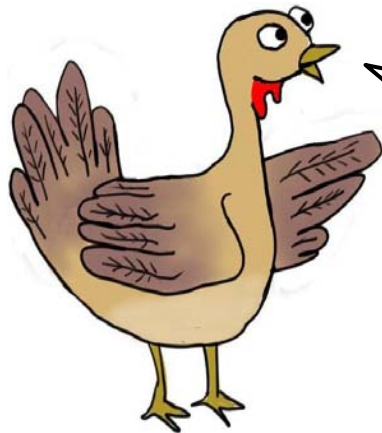
But I really want students to see how to do the work. By giving them the answer, they can verify that they did it correctly.

The interval in which the shadow price is valid is the interval in which the solution is feasible. In this case, the shadow price is not valid if $\Delta = 1$.

Recall that I first defined the shadow price of a constraint to be the change in the optimal objective function if the RHS of the constraint is increased by 1. Now you can see why that wasn't quite precise. This definition is valid so long as the interval in which the shadow price is valid contains "1". But the definition is close enough to the truth that it was worth stating.



I'm beginning to understand. But I still don't get why we are doing this. If I want to find out what happens if we change part of the data, it is really easy to solve a second linear program.



Tom, I'll answer that question when we return to the subject of sensitivity analysis after showing how to solve linear programs with 3 or more variables. For now, I ask you to have some patience and trust.



I just got a feeling of déjà vu.

OK. I'll try to be patient.

Summary for changes in RHS coefficients

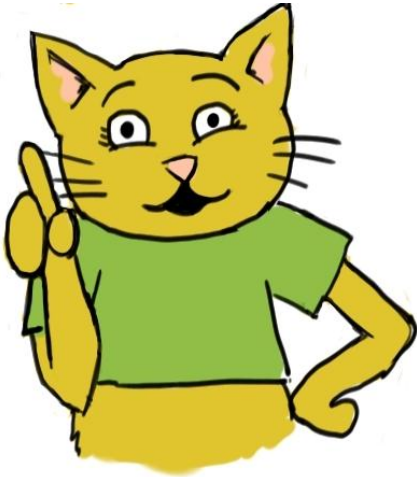
- Determine the binding constraints, that is, the constraints that hold with equality.
- Determine the change in the “corner point solution” as a function of Δ .
- Compute the largest and smallest values of Δ so that the solution stays feasible.
- The shadow price is valid so long as the “corner point solution” remains optimal, which is so long as it is feasible.
- If there are three binding constraints, then choose two of these to get the two equations to solve, and the technique still works. (But the shadow prices and ranges depends on which two constraints are chosen.)



And here is a shortcut. Determine the binding constraints, and determine the solution to the constraints if the RHS is increased by 1. (Don't worry about feasibility). The shadow price is the increase in the objective value.

We are going to cover one other type of sensitivity analysis in this tutorial. What is the effect on the optimum objective function of changing the cost coefficient of one of the variables?

Remember that the optimal solution occurs at a corner point. In this case, the optimum solution was $(2, 2)$, which was the intersection of the red and green line. What do you think will be the new optimal solution if the revenue from shields changes just a little bit?





If you answered that the optimal solution changes just a little bit, you are nearly right.

Ella is being tactful. Actually, that answer would be wrong.

If you answered that the optimal solution will not change, then you are completely right.

But this raises another question. How much can you change the cost coefficient so that the optimum solution stays the same?

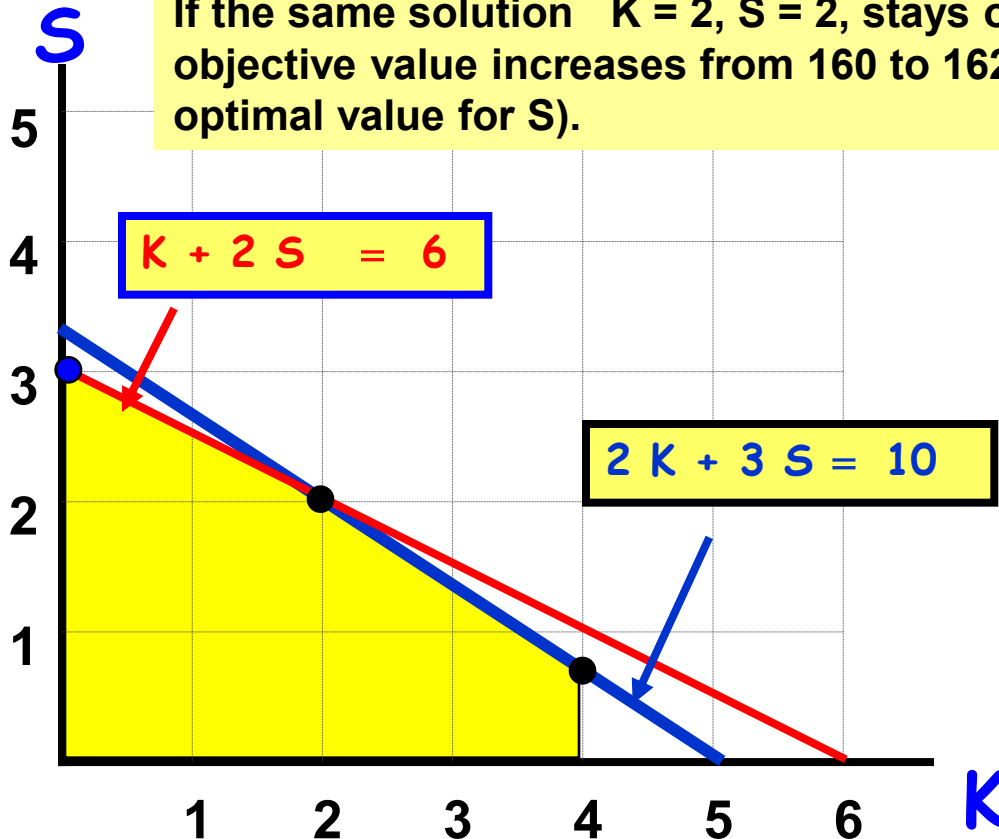


More on Cost Sensitivity Analysis

Suppose the objective function is changed to

$$z = 30 K + 51 S$$

If the same solution $K = 2, S = 2$, stays optimal, then the optimal objective value increases from 160 to 162. (Note that the increase is the optimal value for S).



If the cost coefficient of S is increased by Δ , then the revised objective function is

$$z = 30 K + (50 + \Delta) S$$

Assuming that the current optimal solution stays optimal, then the objective increases to

$$z = 160 + 2\Delta$$

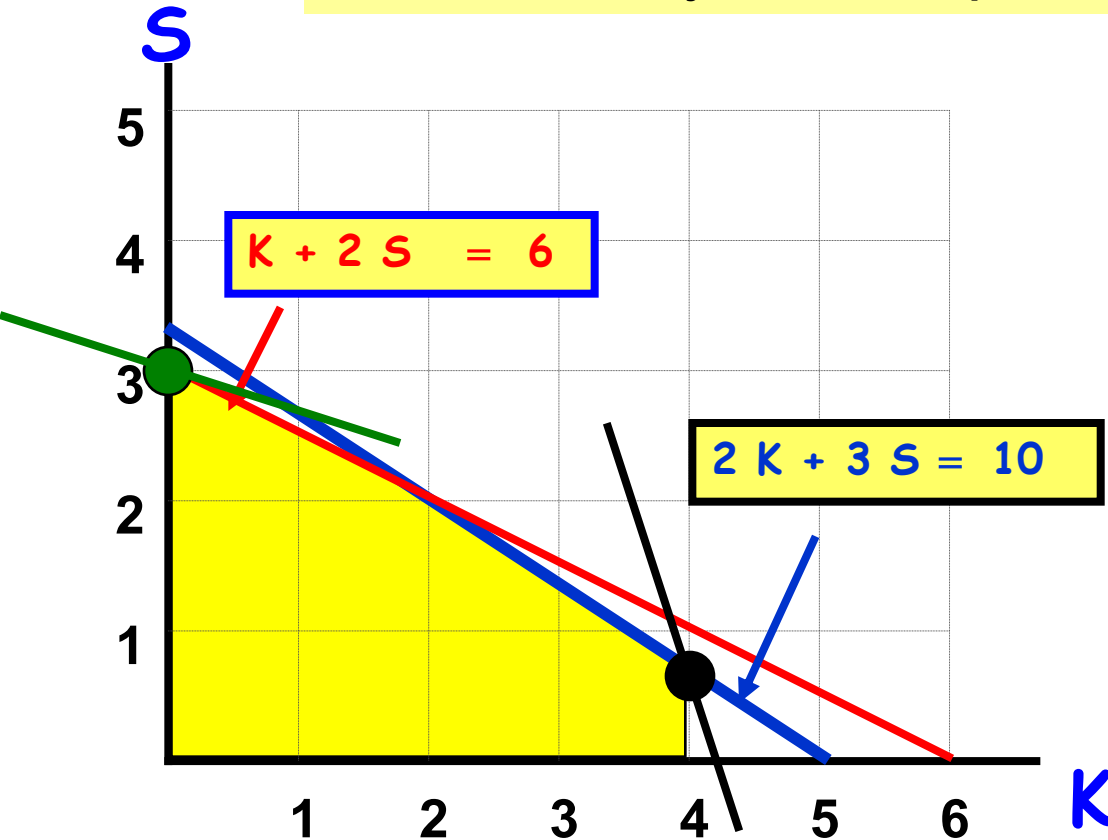
The key issue is the following: how large and small can Δ be so that the optimal solution remains optimal?

Determining Bounds on Cost Coefficients

Suppose the objective function is changed to

$$z = 30K + (50 + \Delta)S$$

As Δ increases or decreases, the slope of the objective line changes. Currently the optimum solution is at the corner point (2, 2). If the slope of the objective function changes enough, then the optimum solution will become an adjacent corner point, either (0, 3) or (4, 1).



To figure out the answer, remember the geometric method for solving a 2-variable LP. One shifts the isoprofit line until it reaches the maximum level that touches the feasible region.



Determining Bounds on Cost Coefficients

If $\Delta = -5$, the objective is parallel to $2K + 3S = 10$

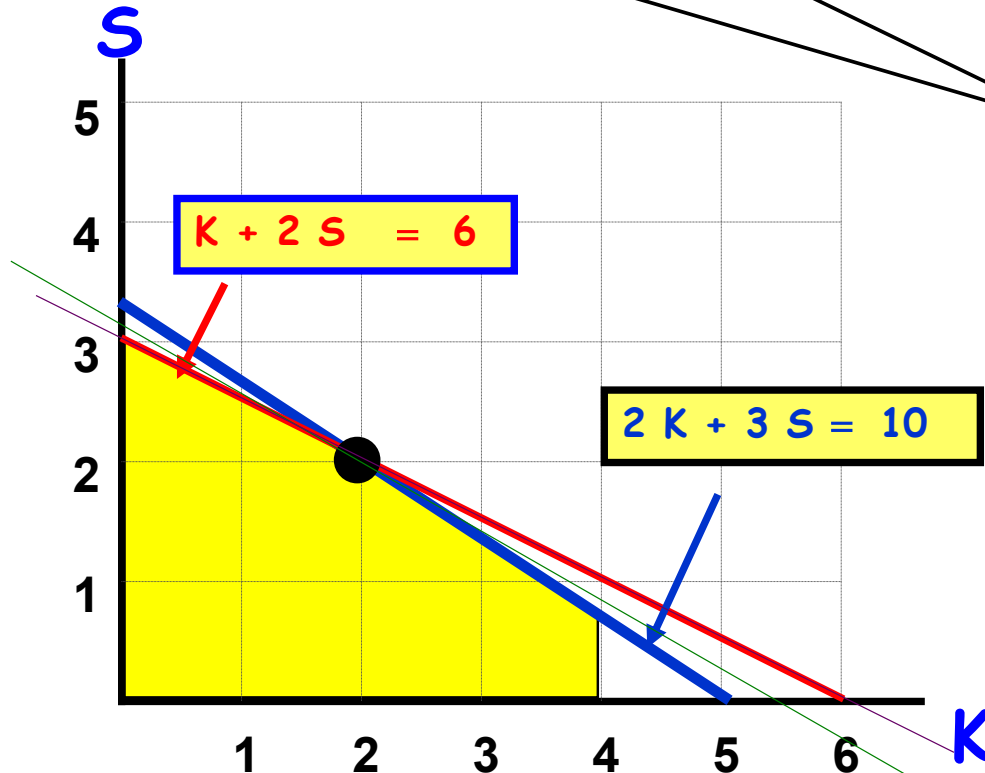
So $(2, 2)$ remains optimal if $-5 \leq \Delta \leq 10$.

Suppose the objective function is changed to

$$z = 30K + (50 + \Delta)S$$

If $\Delta = 0$, the optimal solution is $(2, 2)$. This solution stays optimal for small positive Δ so long as $\Delta < 10$.

If $\Delta = 10$, the objective is parallel to " $K + 2S = 6$ ". When $\Delta = 10$, then, every point on the line segment from $(0,3)$ to $(2, 2)$ is optimal.



When $\Delta = 10$, the isoprofit line is $30K + 60S = 180$. This is the same line as $K + 2S = 6$.

Determining Bounds on Cost Coefficients

If $\Delta = -5$, the objective is parallel to $2K + 3S = 10$

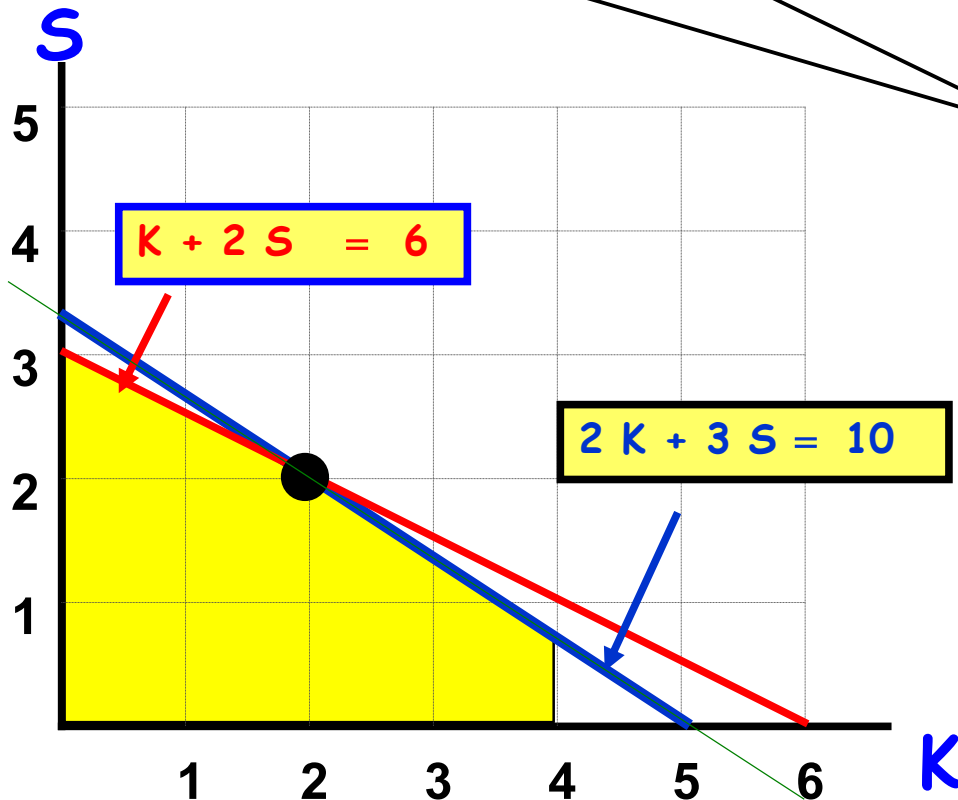
So $(2, 2)$ remains optimal if $-5 \leq \Delta \leq 10$.

We still assume that the objective function is
 $z = 30K + (50 + \Delta)S$.

If Δ is between 0 and -5, the optimal solution is $(2, 2)$. If $\Delta = -5$, then, every point on the line segment from $(2, 2)$ to $(4, 1)$ is optimal.

When $\Delta = -5$, the isoprofit line is
 $30K + 45S = 150$.

This is the same line as
 $2K + 3S = 10$.



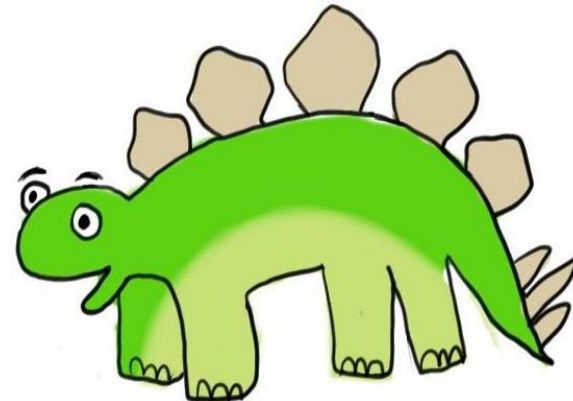
As Δ changes from $-\infty$ to ∞ , each of the five corner points will be optimal within a segment.

Summary for Changes in the Cost Coefficients

Suppose that we want to modify the cost coefficient for a variable x_j . We want to increase it from c_j to $c_j + \Delta$.

- Determine the binding constraints and the current corner point solution, say x^* .
- Compute the largest and smallest values of Δ so that the x^* remains optimal. In two dimensions, this will occur when the revised objective function is parallel to one of the constraints.

I really like 2 variable LPs! They are my favorite LPs, except possibly for LPs with only one variable.



Last Slide

By the way, 2-D LPs are not so important in and of themselves. But by knowing how to do sensitivity analysis in 2-D, it's easier to understand what happens with problems with more variables.



That concludes this tutorial of sensitivity analysis for Linear Programming in two dimensions.

I hope it was helpful. So long!



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