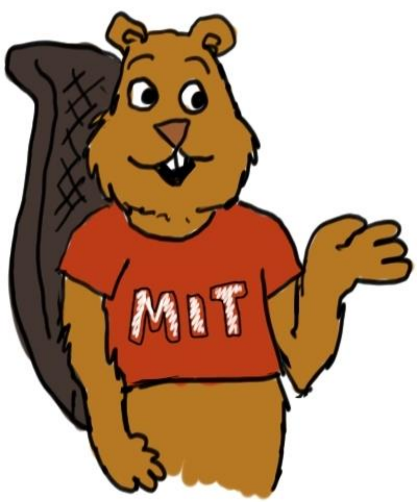
Optimization Methods in Management Science / Operations Research 15.053/058

LP Transformation Techniques

LP Transformation Techniques

Hello friends. Mita and I are here again to introduce a tutorial on LP transformation techniques [1].

The tutorial will show three different types of non-linear constraints that can be transformed into linear constraints. This is important since linear programs are so much easier to solve than non-linear programs.



Amit

[1] Donald Knuth claimed that a "technique" is a trick that is used more than once. Knuth was the developer of TeX, and one of the greatest computer scientists of the 20th century.

Photo of Donald Knuth removed due to copyright restrictions.

Donald E. Knuth

Mita

We'll begin with the MSR example which was described in the tutorial on algebraic formulations. The goal is to minimize the cost of reaching 1.5 million people using ads of different types.

Ella

	TV	Radio	Mail	Newspaper
Audience Size (in 1000s)	50	25	20	15
Cost/Impression	\$500	\$200	\$250	\$125
Max # of ads	20	15	25	15

Minimize $500\ x_1 + 200\ x_2 + 250\ x_3 + 125\ x_4$

subject to $50\ x_1 + 25\ x_2 + 20x_3 + 15\ x_4 \geq 1{,}500$

$0 \leq x_1 \leq 20 \quad 0 \leq x_2 \leq 15 \quad 0 \leq x_3 \leq 25 \quad 0 \leq x_4 \leq 15$

We are now going to introduce a non-linear constraint. Suppose that we require that the total of ads from the electronic media is within 5 of the number of ads of paper-based media. This can be modeled as follows: $|x_1 + x_2 - x_3 - x_4| \le 5$.

Ella

	TV	Radio	Mail	Newspaper
Audience Size	50,000	25,000	20,000	15,000
Cost/Impression	$500	$200	$250	$125
Max # of ads	20	15	25	15

Minimize $500 x_1 + 200 x_2 + 250 x_3 + 125 x_4$

subject to $50 x_1 + 25 x_2 + 20x_3 + 15 x_4 \ge 1{,}500$

$0 \le x_1 \le 20 \quad 0 \le x_2 \le 15 \quad 0 \le x_3 \le 25 \quad 0 \le x_4 \le 15$

$|x_1 + x_2 - x_3 - x_4| \le 5.$

The constraint "$|x_1 + x_2 - x_3 - x_4| \le 5$" is not a linear constraint. However, the constraint can be transformed into linear constraints using a simple trick. Er, I mean "technique".

The constraint "$|x_1 + x_2 - x_3 - x_4| \le 5$" is equivalent to the following two constraints:

1. $x_1 + x_2 - x_3 - x_4 \le 5$ and

2. $-x_1 - x_2 + x_3 + x_4 \le 5$

If you replace the original constraint by these two constraints, you obtain a linear program that is equivalent to the original non-linear program.

Minimize $500\,x_1 + 200\,x_2 + 250\,x_3 + 125\,x_4$

subject to $50\,x_1 + 25\,x_2 + 20x_3 + 15\,x_4 \geq 1{,}500$

$0 \leq x_1 \leq 20 \quad 0 \leq x_2 \leq 15 \quad 0 \leq x_3 \leq 25 \quad 0 \leq x_4 \leq 15$

$|x_1 + x_2 - x_3 - x_4| \leq 5.$

The non-linear program above is equivalent to the linear program below.

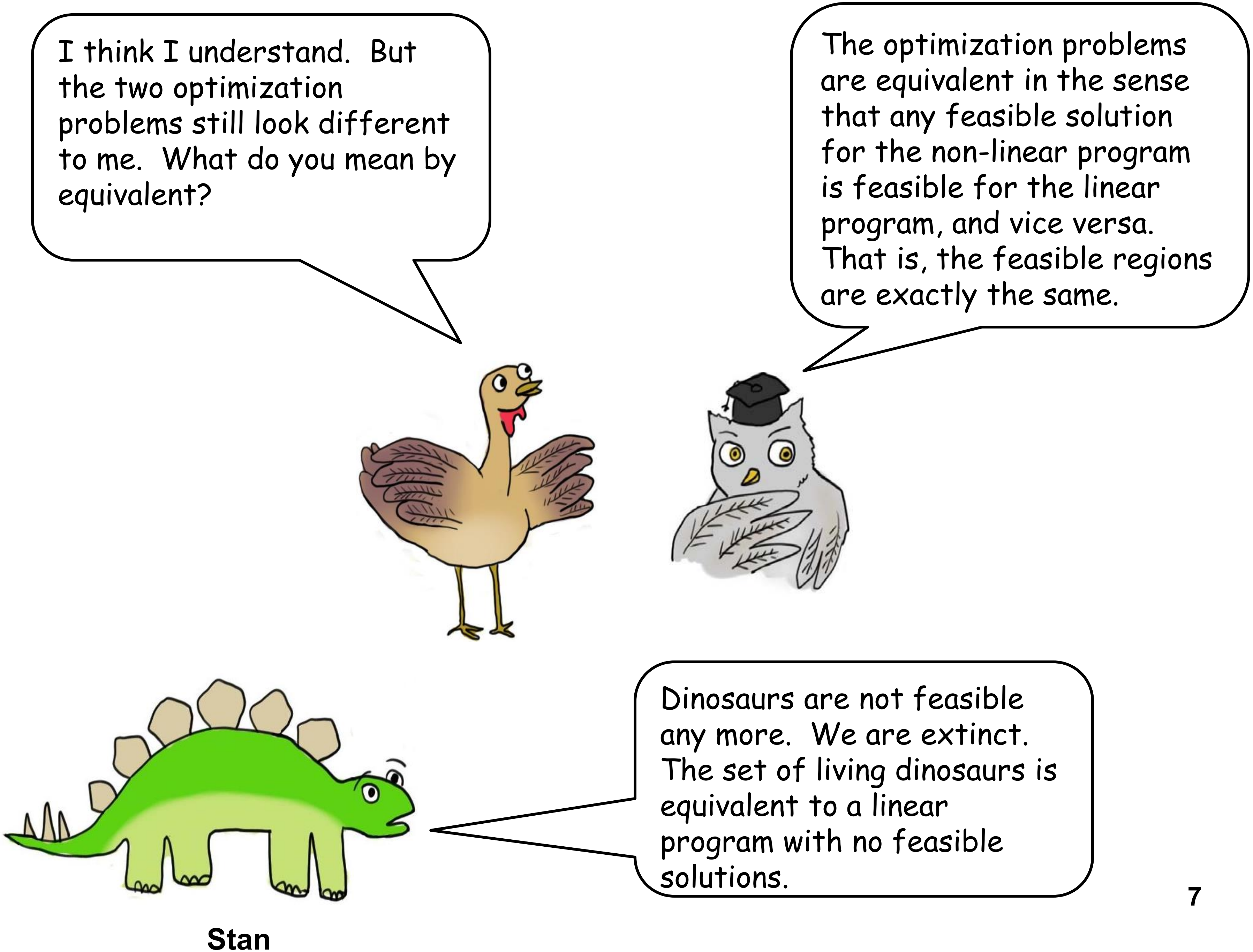
Minimize $500\,x_1 + 200\,x_2 + 250\,x_3 + 125\,x_4$

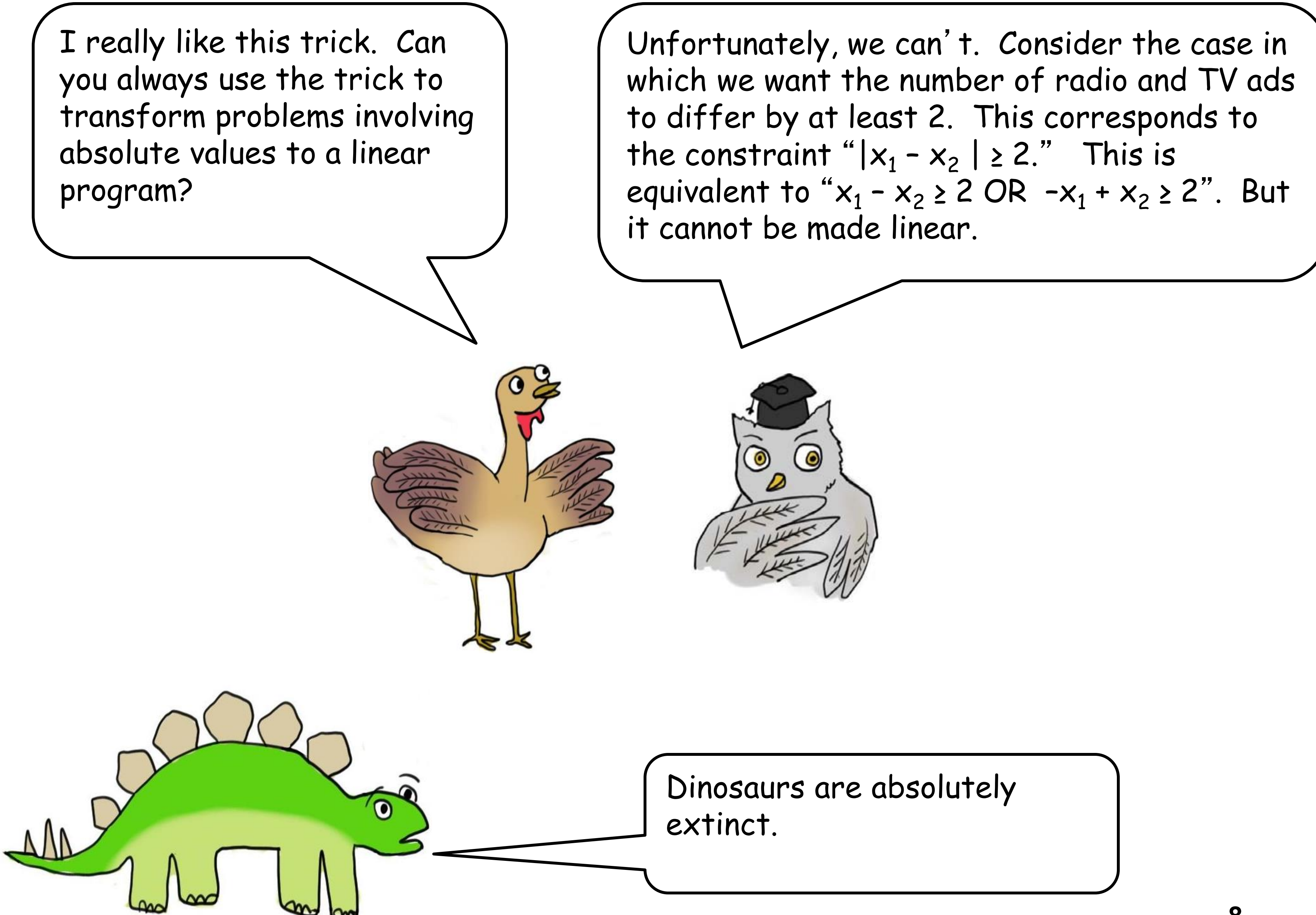
subject to $50\,x_1 + 25\,x_2 + 20x_3 + 15\,x_4 \geq 1{,}500$

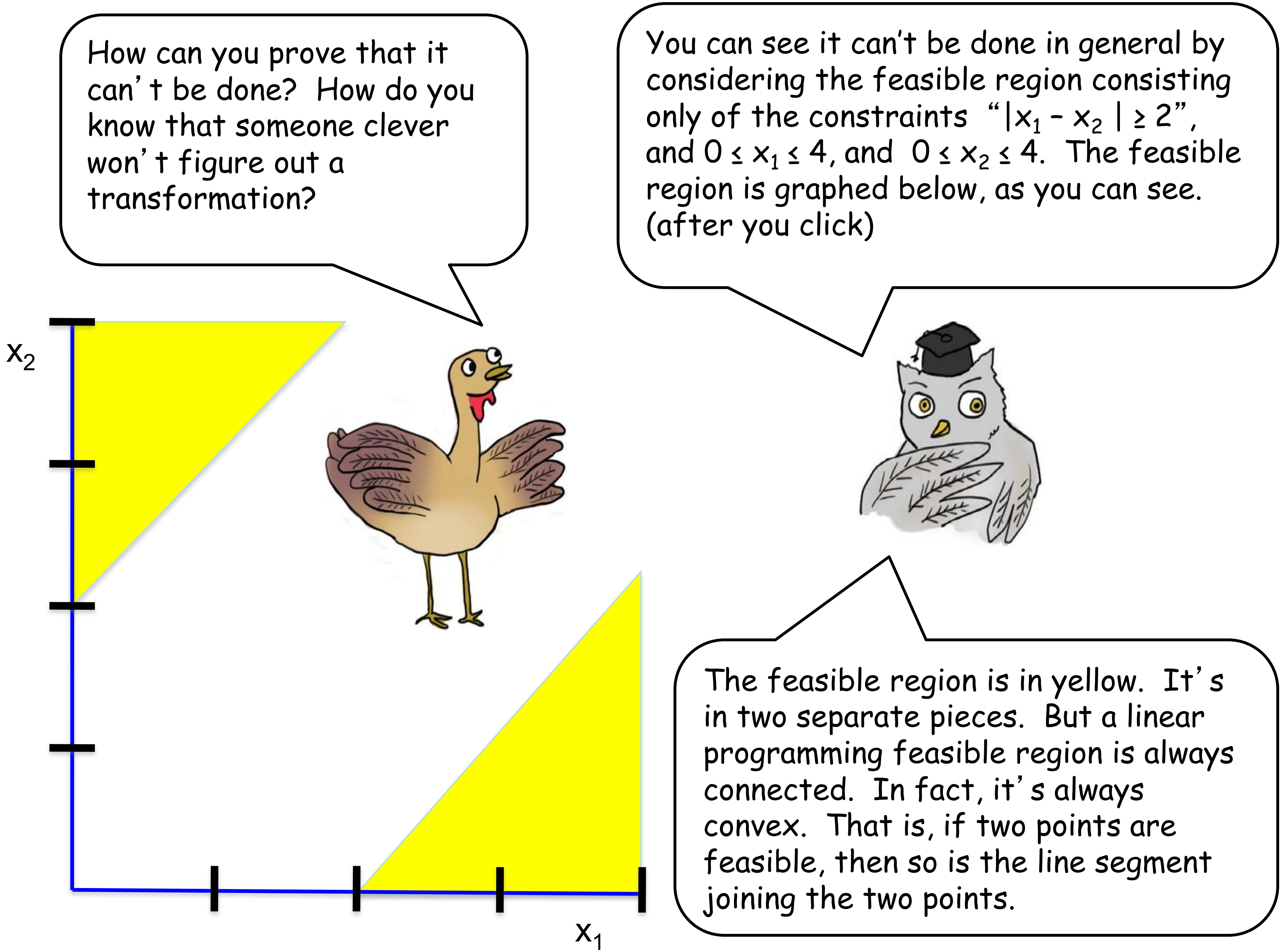
$0 \leq x_1 \leq 20 \quad 0 \leq x_2 \leq 15 \quad 0 \leq x_3 \leq 25 \quad 0 \leq x_4 \leq 15$

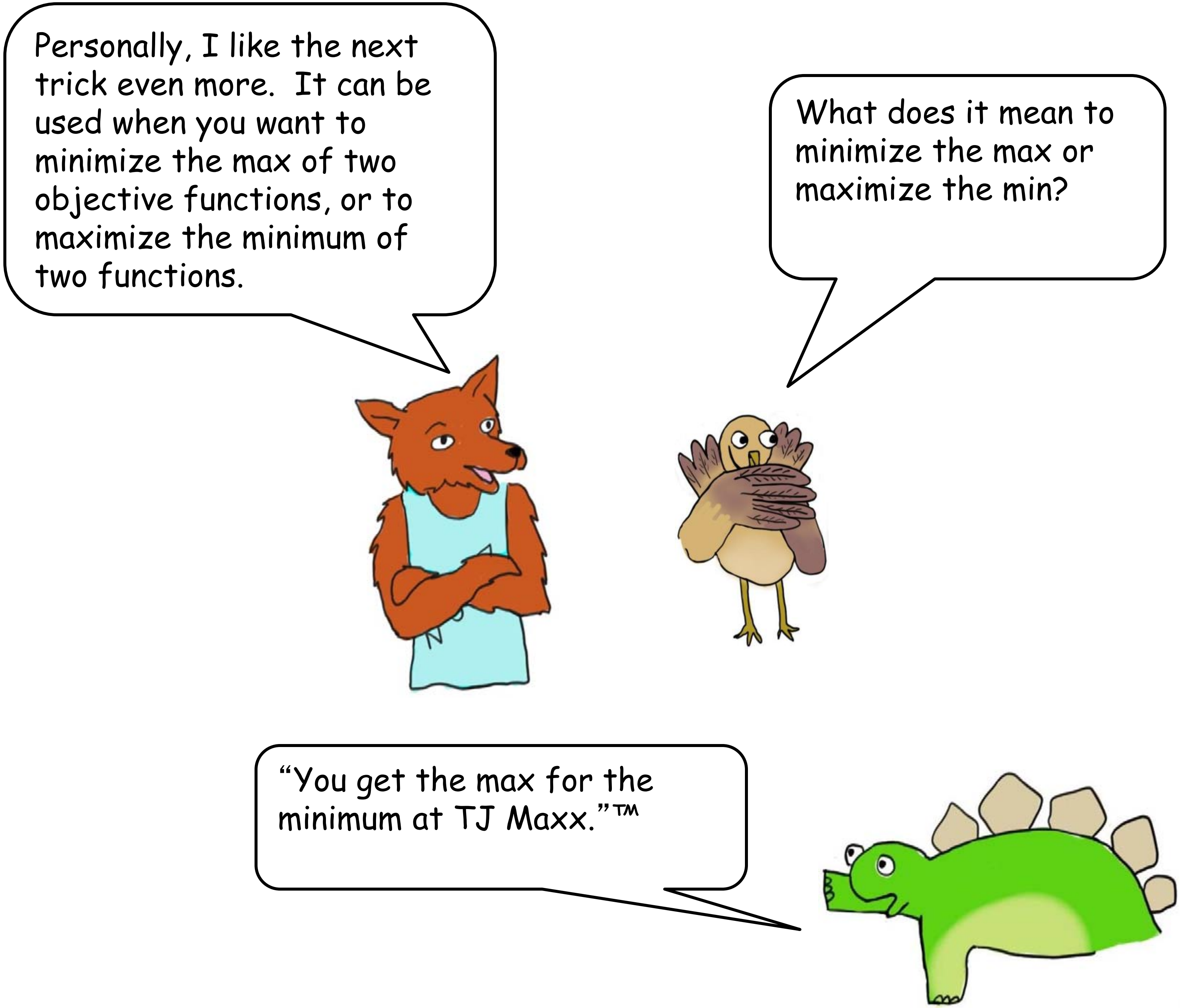
$x_1 + x_2 - x_3 - x_4 \leq 5$

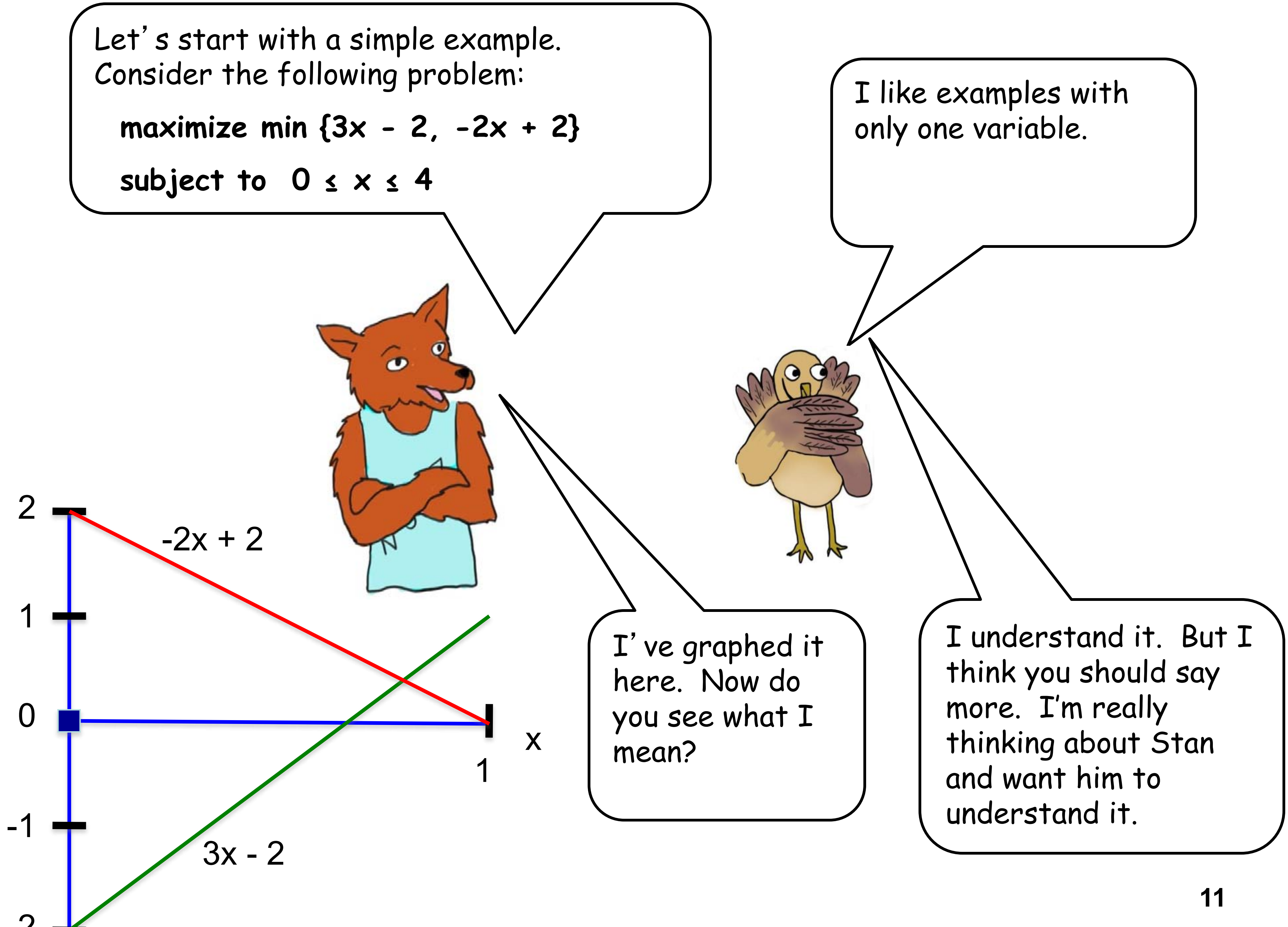
$-x_1 - x_2 + x_3 + x_4 \leq 5$

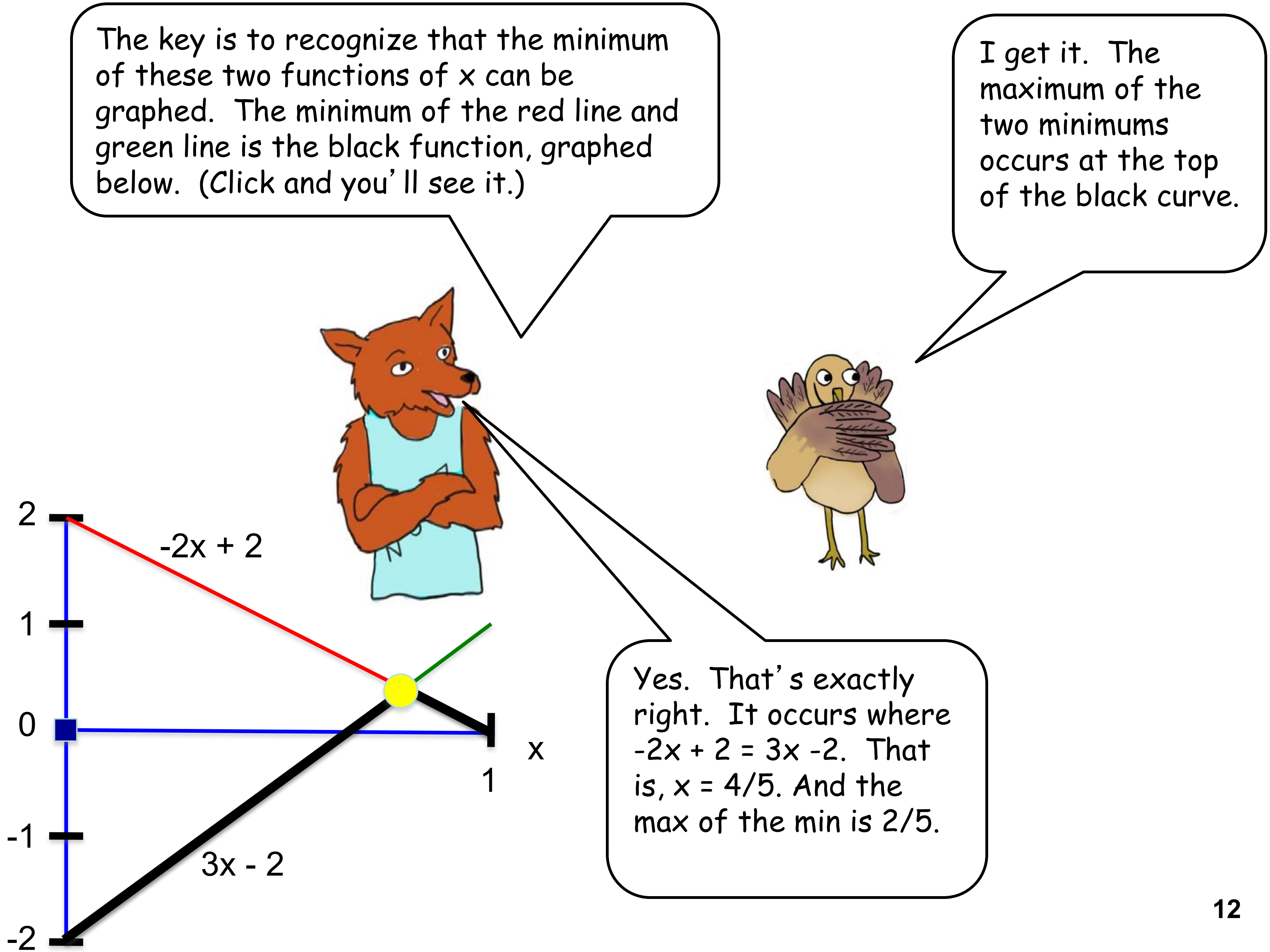

I think I understand. But the two optimization problems still look different to me. What do you mean by equivalent?
The optimization problems are equivalent in the sense that any feasible solution for the non-linear program is feasible for the linear program, and vice versa. That is, the feasible regions are exactly the same.
Dinosaurs are not feasible any more. We are extinct. The set of living dinosaurs is equivalent to a linear program with no feasible solutions.
Stan

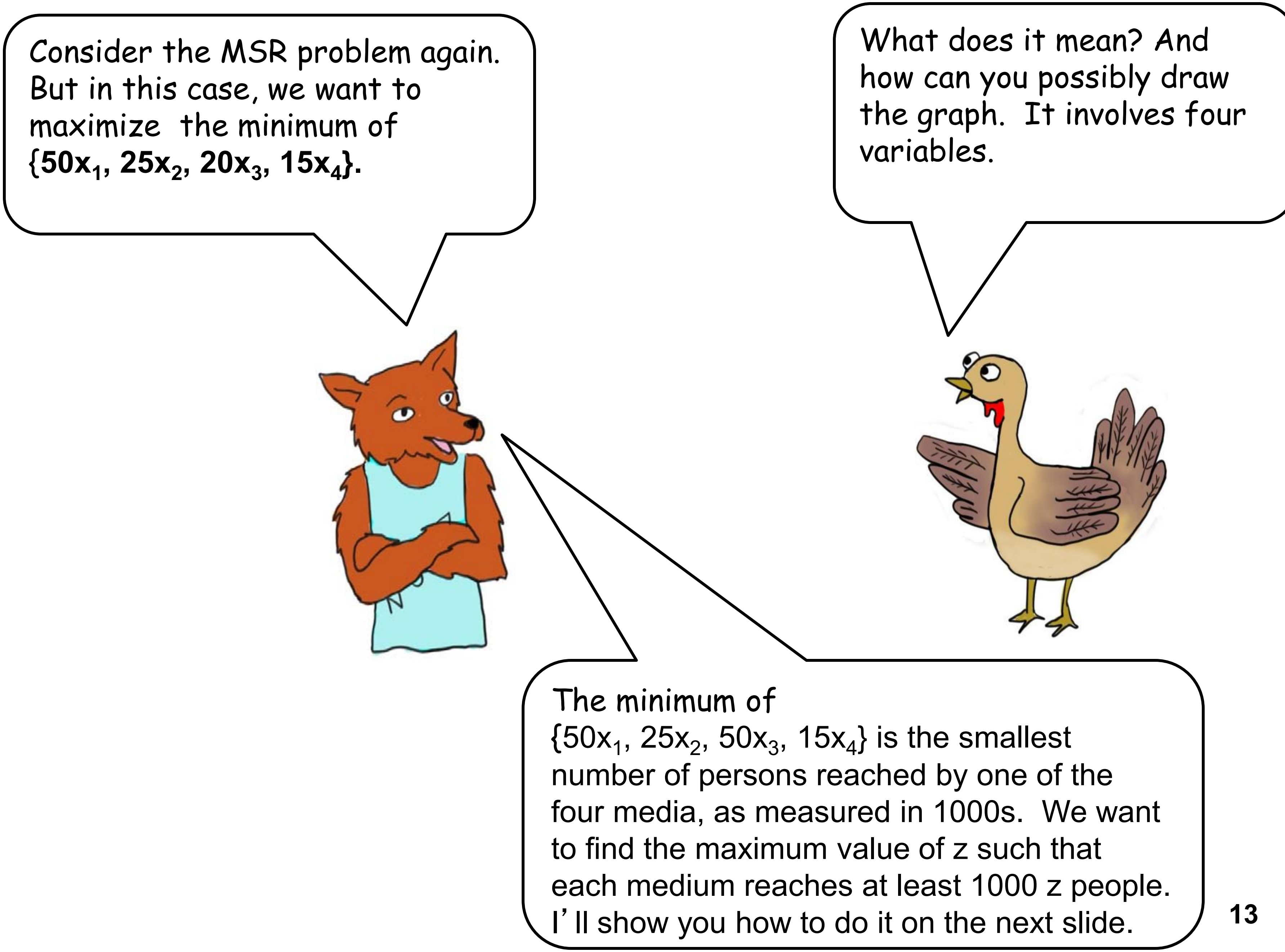

I really like this trick. Can you always use the trick to transform problems involving absolute values to a linear program?
Unfortunately, we can't. Consider the case in which we want the number of radio and TV ads to differ by at least 2. This corresponds to the constraint "$|x_1 - x_2| \geq 2$." This is equivalent to "$x_1 - x_2 \geq 2$ OR $-x_1 + x_2 \geq 2$". But it cannot be made linear.
Dinosaurs are absolutely extinct.

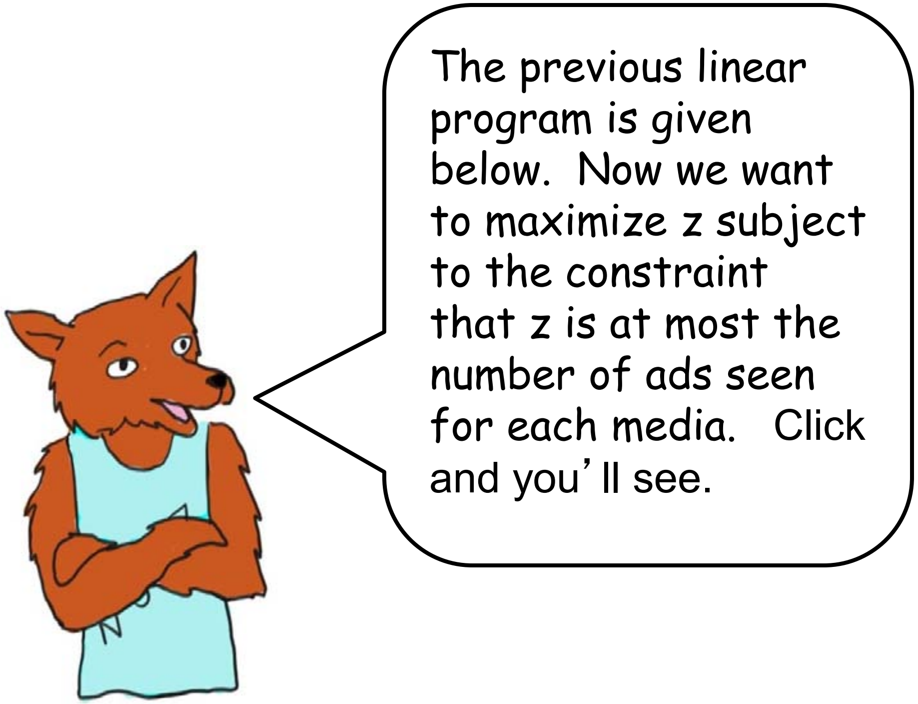

How can you prove that it can't be done? How do you know that someone clever won't figure out a transformation?
You can see it can't be done in general by considering the feasible region consisting only of the constraints "$|x_1 - x_2| \geq 2$", and $0 \leq x_1 \leq 4$, and $0 \leq x_2 \leq 4$. The feasible region is graphed below, as you can see. (after you click)
The feasible region is in yellow. It's in two separate pieces. But a linear programming feasible region is always connected. In fact, it's always convex. That is, if two points are feasible, then so is the line segment joining the two points.
x_2
x_1

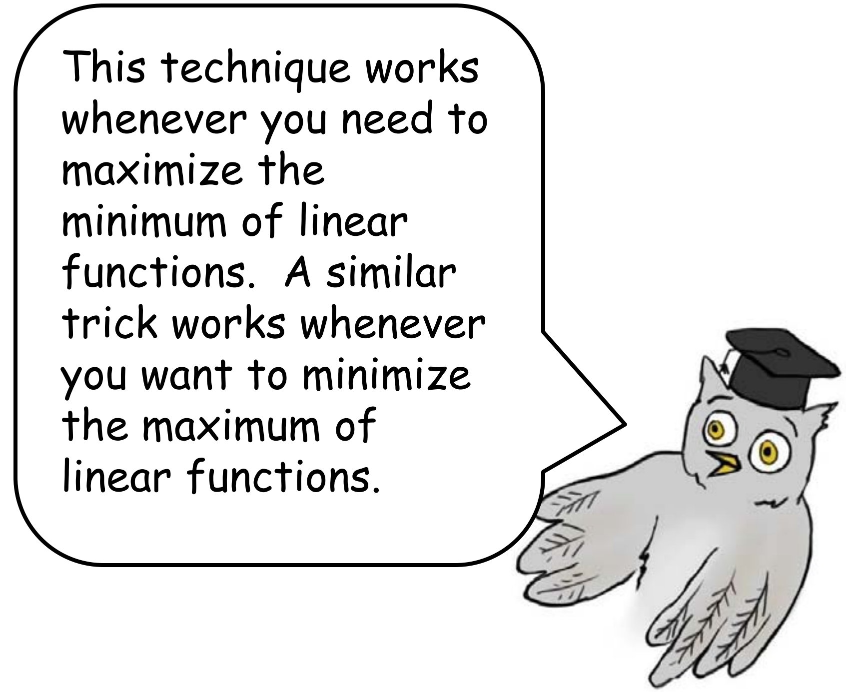

Personally, I like the next trick even more. It can be used when you want to minimize the max of two objective functions, or to maximize the minimum of two functions.
What does it mean to minimize the max or maximize the min?
"You get the max for the minimum at TJ Maxx."™


Let's start with a simple example. Consider the following problem:
maximize min {3x - 2, -2x + 2}
subject to 0 ≤ x ≤ 4
I like examples with only one variable.
I've graphed it here. Now do you see what I mean?
I understand it. But I think you should say more. I'm really thinking about Stan and want him to understand it.
2
1
0
-1
-2
-2x + 2
3x - 2
x
1


The key is to recognize that the minimum of these two functions of x can be graphed. The minimum of the red line and green line is the black function, graphed below. (Click and you' ll see it.)
I get it. The maximum of the two minimums occurs at the top of the black curve.
Yes. That's exactly right. It occurs where -2x + 2 = 3x -2. That is, x = 4/5. And the max of the min is 2/5.
2
1
0
-1
-2
-2x + 2
3x - 2
x
1


Consider the MSR problem again. But in this case, we want to maximize the minimum of **{$50x_1$, $25x_2$, $20x_3$, $15x_4$}.**
What does it mean? And how can you possibly draw the graph. It involves four variables.
The minimum of {$50x_1$, $25x_2$, $50x_3$, $15x_4$} is the smallest number of persons reached by one of the four media, as measured in 1000s. We want to find the maximum value of z such that each medium reaches at least 1000 z people. I'll show you how to do it on the next slide.





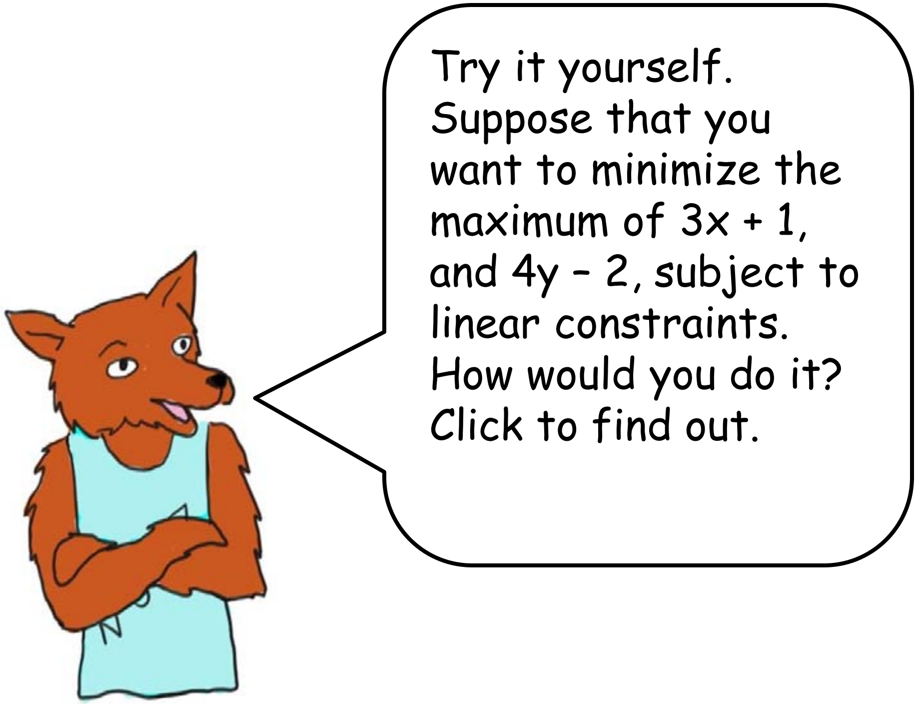
maximize z

subject to

$$50\,x_1 + 25\,x_2 + 20\,x_3 + 15\,x_4 \geq 1{,}500$$

$$0 \leq x_1 \leq 20 \qquad 0 \leq x_2 \leq 15 \qquad 0 \leq x_3 \leq 25 \qquad 0 \leq x_4 \leq 15$$

$$z \leq 50x_1, \qquad z \leq 25x_2, \qquad z \leq 20x_3, \qquad z \leq 15x_4$$



Minimize $\max\{3x + 1, 4y - 2\}$

subject to **<linear constraints>**

$x \geq 0, \; y \geq 0$

If you want to try another exercise, you can try exercise 28 parts b and c on page 36 of Applied Mathematical Programming.

It shows that linear programming can be used in data analysis as well.

Problem 28 is given below and continued on the next slide.

http://web.mit.edu/15.053/www/AMP-Chapter-01.pdf

28. The selling prices of a number of houses in a particular section of the city overlooking the bay are given in the following table, along with the size of the lot and its elevation:

Selling price P_i	*Lot size* (sq. ft.) L_i	*Elevation* (feet) E_i
$155,000	$12,000	350
120,000	10,000	300
100,000	9,000	100
70,000	8,000	200
60,000	6,000	100
100,000	9,000	200

A real-estate agent wishes to construct a model to forecast the selling prices of other houses in this section of the city from their lot sizes and elevations. The agent feels that a linear model of the form

$$P = b_0 + b_1 L + b_2 E$$

would be reasonably accurate and easy to use. Here b_1 and b_2 would indicate how the price varies with lot size and elevation, respectively, while b_0 would reflect a base price for this section of the city.

The agent would like to select the "best" linear model in some sense, but he is unsure how to proceed. It he knew the three parameters b_0, b_1 and b_2, the six observations in the table would each provide a forecast of the selling price as follows:

$$\hat{P}_i = b_0 + b_1 L_i + b_2 E_i \qquad i = 1, 2, \ldots, 6.$$

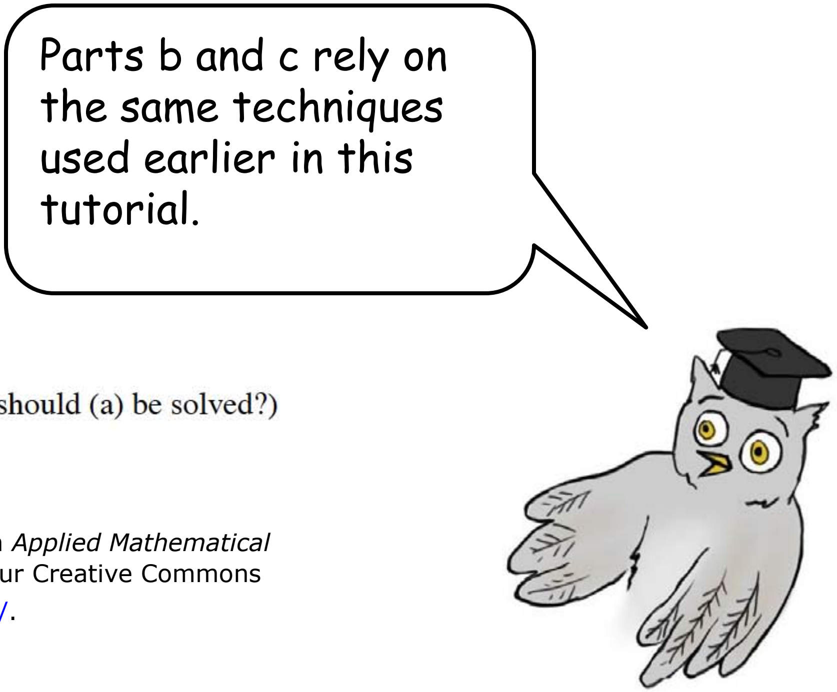
However, since b_0, b_1, and b_2 cannot, in general, be chosen so that the actual prices P_i are exactly equal to the forecast prices $\hat{P}_i$ for all observations, the agent would like to minimize the absolute value of the residuals $R_i = P_i - \hat{P}_i$. Formulate mathematical programs to find the "best" values of b_0, b_1, and b_2 by minimizing each of the following criteria:

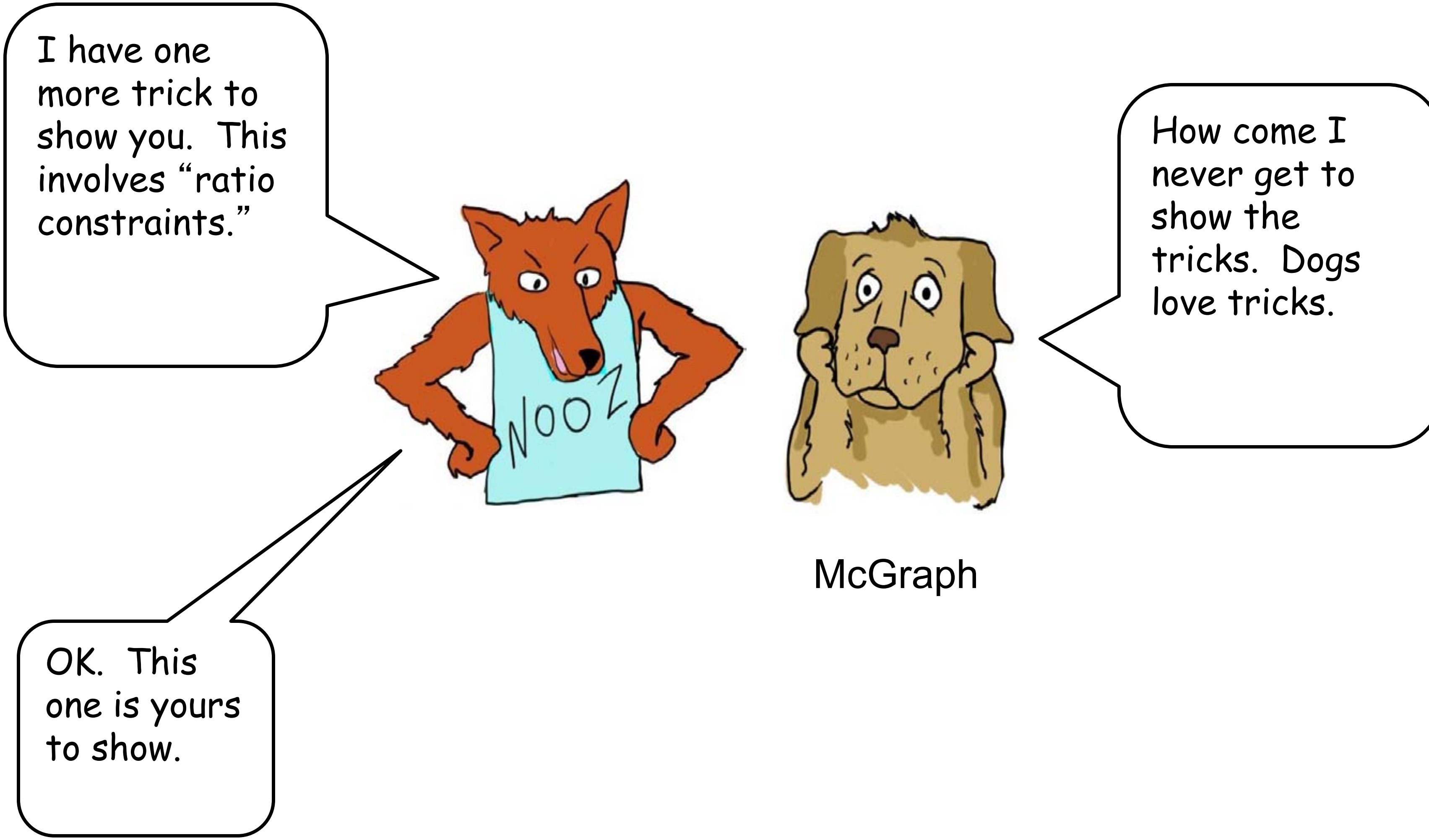
a) $\sum_{i=1}^{6} (P_i - \hat{P}_i)^2$, Least squares

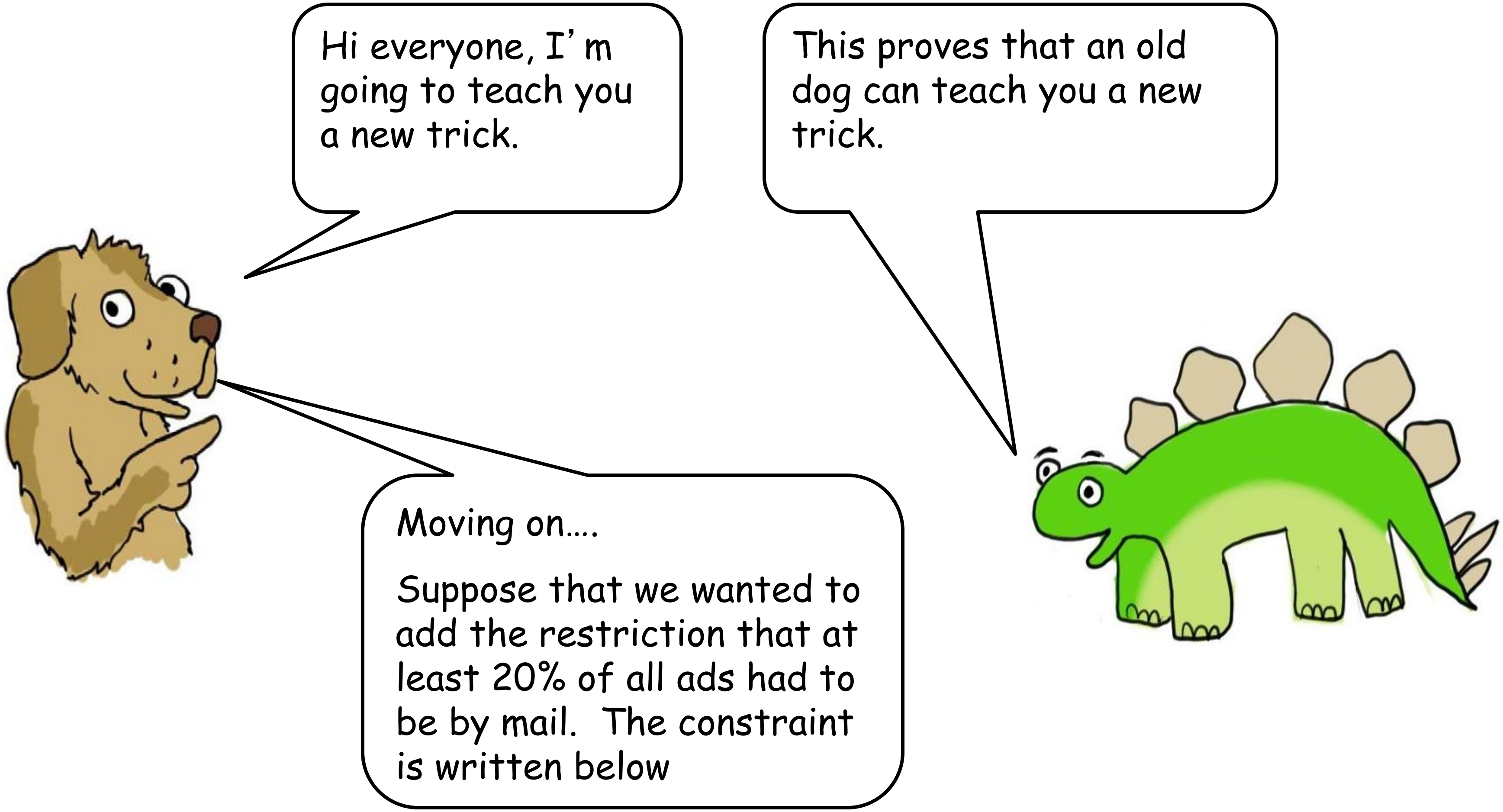
b) $\sum_{i=1}^{6} |P_i - \hat{P}_i|$, Linear absolute residual

c) $\max_{1 \le i \le 6} |P_i - \hat{P}_i|$, Maximum absolute residual

(*Hint*: (b) and (c) can be formulated as linear programs. How should (a) be solved?)




I have one more trick to show you. This involves "ratio constraints."
How come I never get to show the tricks. Dogs love tricks.
OK. This one is yours to show.
NOOZ
McGraph

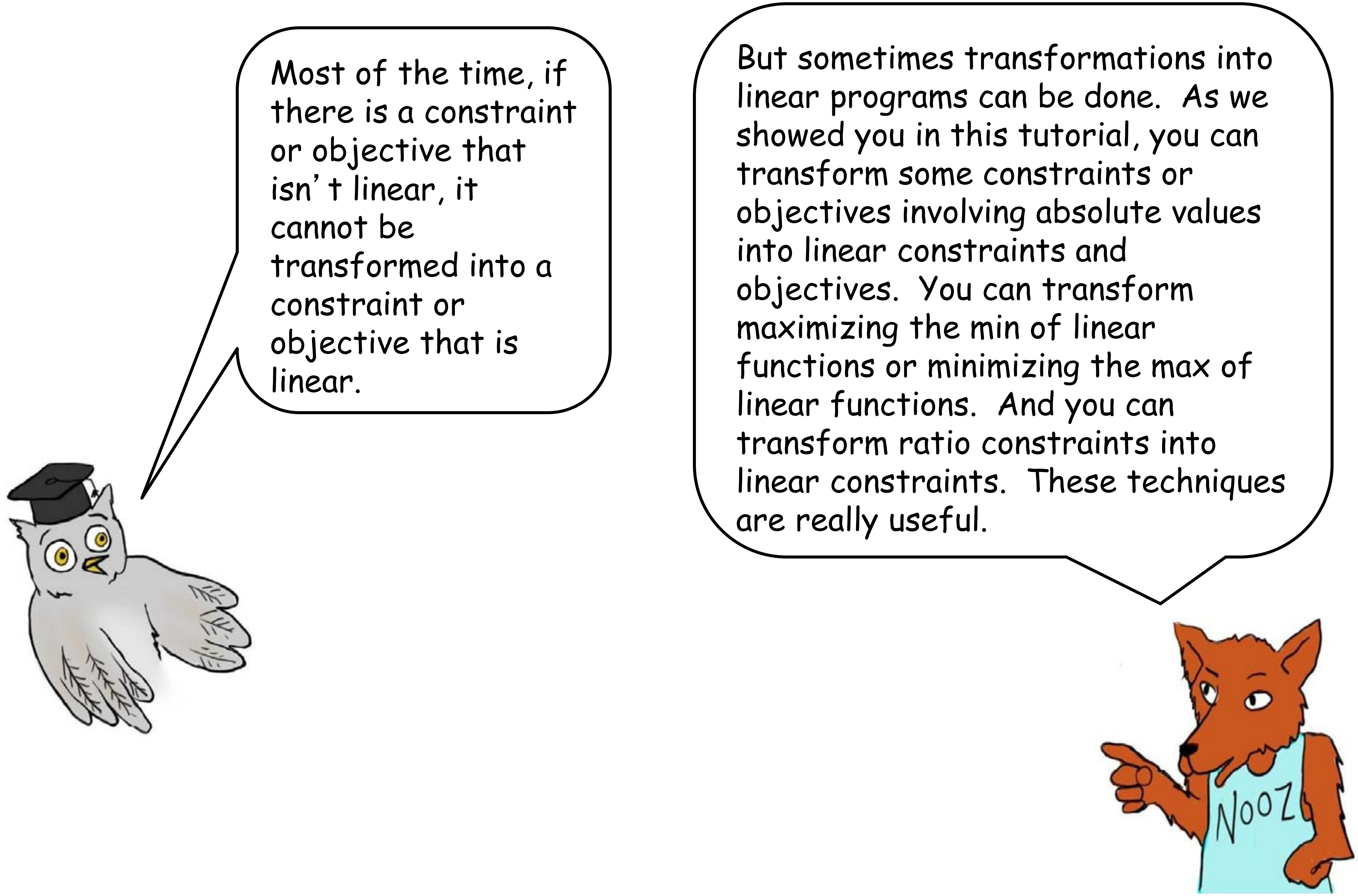


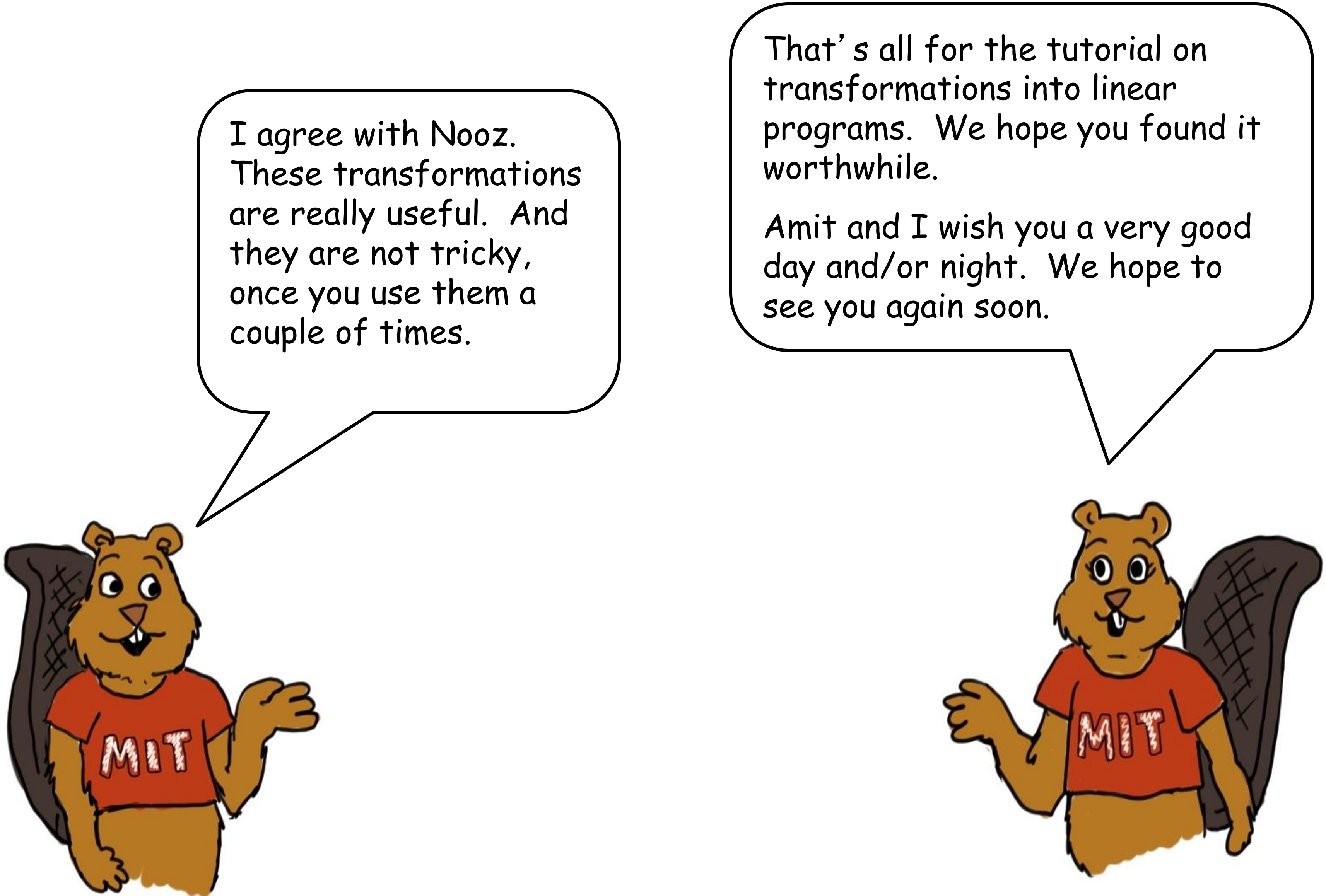
$$x_1/(x_1 + x_2 + x_3 + x_4) \geq .2$$

$x_1/(x_1 + x_2 + x_3 + x_4) \geq .2$

$x_1 \geq .2\ (x_1 + x_2 + x_3 + x_4)$

Equivalently,
$.8x_1 - .2\ x_2 - .2\ x_3 - .2\ x_4 \geq 0$


Most of the time, if there is a constraint or objective that isn't linear, it cannot be transformed into a constraint or objective that is linear.
But sometimes transformations into linear programs can be done. As we showed you in this tutorial, you can transform some constraints or objectives involving absolute values into linear constraints and objectives. You can transform maximizing the min of linear functions or minimizing the max of linear functions. And you can transform ratio constraints into linear constraints. These techniques are really useful.
NOOZ


I agree with Nooz. These transformations are really useful. And they are not tricky, once you use them a couple of times.
That's all for the tutorial on transformations into linear programs. We hope you found it worthwhile.
Amit and I wish you a very good day and/or night. We hope to see you again soon.
MIT
MIT