Problem 1

Suppose branch and bound is being applied to a 0–1 integer program in which we are maximizing. By node 11 the first three variables have been fixed. An incumbent has already been found (in another part of the tree) and has value 15. Furthermore, x_4 has been chosen to be the next variable to branch on. This situation is illustrated in the following fragment of the branch and bound tree:
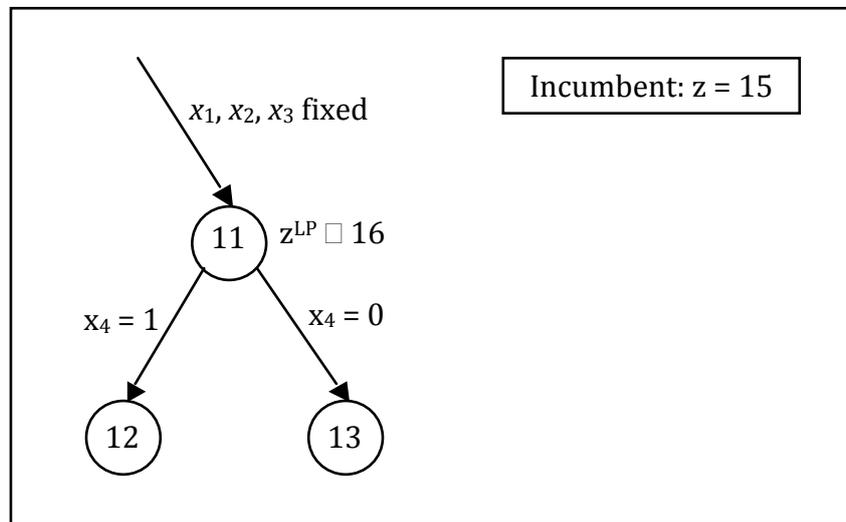
Figure 1: A fragment of the branch and bound tree for Problem 1.

The LP relaxation solved at node 11 (which has x_4 and x_5 as free variables) was:

$$\begin{aligned}
\max \quad & 18x_4 + ax_5 + 10 \\
\text{s.t.} \quad & 8x_4 + 10x_5 \leq 5, \\
& 0 \leq x_4 \leq 1, \\
& 0 \leq x_5 \leq 1,
\end{aligned}$$

(Hint: For the following questions, the LP relaxation essentially becomes a one variable problem that can be solved by inspection.)

Part A.1

Suppose x_4 is set to 1. Does node 12 get pruned (fathomed)? Justify your answer.

Solution. Yes, node 12 gets fathomed. If x4 is set to 1, then the LP relaxation becomes:

$$\begin{aligned} \max \quad & ax_5 + 28 \\ \text{s.t.} \quad & 10x_5 \le -3, \\ & 0 \le x_5 \le 1, \end{aligned}$$

The relaxation is infeasible because of the first constraint, so node 12 gets fathomed.

Part B.1

Suppose x_4 is set to 0 and parameter $a = 12$. Does node 13 get pruned (fathomed)? Justify your answer.

Solution. No, node 13 does not get fathomed. If $x_4 = 0$ and $a = 12$, then the LP relaxation becomes:

$$\begin{aligned} \max \quad & 12x_5 + 10 \\ \text{s.t.} \quad & 10x_5 \le 5, \\ & 0 \le x_5 \le 1, \end{aligned}$$

The optimal solution to this relaxation is $x_5 = 1/2$, with an objective value of 16. Because the solution is fractional and has a value better (higher) than the incumbent, the node wont be fathomed (since its possible that a better solution could be found in the subtree).

Part C.1

Suppose x_4 is set to 0 and parameter $a = 8$. Does node 13 get pruned (fathomed)? Justify your answer.

Solution. Yes, node 13 gets fathomed. If $x_4 = 0$ and $a = 8$, then the LP relaxation becomes:

$$\begin{aligned} \max \quad & 8x_5 + 10 \\ \text{s.t.} \quad & 10x_5 \le 5, \\ & 0 \le x_5 \le 1, \end{aligned}$$

The optimal solution to this relaxation is $x_5 = 1/2$ with an objective value of 14. Since this solution is not better than the incumbent, node 13 gets fathomed.

Problem 2

Consider the knapsack problem with the following decision variables for i = 1 to 4:

$$x_i = \begin{cases} 1 & \text{if item } i \text{ is selected;} \\ 0 & \text{otherwise.} \end{cases}$$

The knapsack problem is formulated as follows:

$$\begin{aligned} \max \quad & 19x_1 + 23x_2 + 30x_3 + 40x_4 \\ \text{s.t.} \quad & 6x_1 + 8x_2 + 10x_3 + 13x_4 \le 25, \qquad\qquad\qquad (1) \\ & x_i \in \{0,1\}, \quad \text{for } i = 1, \ldots, 4. \end{aligned}$$

Apply the branch-and-bound algorithm to solve the problem (notice that the the LP relaxation of a knapsack problem can be easily solved by selecting the items.

Solution. Let's first review the branch and bound algorithm: At each iteration, we select an **active** node j and make it **inactive**. Suppose that LP(j) represents the integer program corresponding to node j, in which the binary variables are relaxed to be fractional. Let $x(j)$ and $Z_{LP}(j)$ be the optimal solution and the optimal value of LP. There are three cases:

Case: 1 If $Z_{LP}(j) \le Z^*$, the node j gets **pruned**;

Case: 2 If $Z_{LP}(j) > Z^*$, and $x(j)$ is integral, then update the incumbent solution by setting $Z^* := Z_{LP}(j)$ and $x^* := x(j)$; In addition, node j gets **pruned**,

Case: 3 If $Z_{LP}(j) > Z^*$ and $x(j)$ is not integral, then make the childeren of node j as **active**;

We now apply this algorithm to solve Problem (1)). We first need an initial incumbent solution. We set $x^* = (0,0,0,0)$ with value $z^* = 0$ as the initial incumbent solution and then start with node 1, which represents problem IP(1) (notice that IP(1) is the original knapsack problem, that is, Problem (1)). Remember that this node is considered as an **active** node.

We select node 1 and then solve LP(1). Notice that LP(1) is IP(1) with relating the binary variable to be fractional. More precisely, LP(1) is as follows:

$$\begin{aligned} \max \quad & 19x_1 + 23x_2 + 30x_3 + 40x_4 \\ \text{s.t.} \quad & 6x_1 + 8x_2 + 10x_3 + 13x_4 \le 25, \qquad\qquad\qquad \text{(LP(1))} \\ & 0 \le x_i \le 1, \quad \text{for } i = 1, \ldots, 4. \end{aligned}$$

We next require to compute an optimal solution for LP(1). There is a simple way to solve LP(1) since there is a single constraint. In fcat, we want to select as much as possible of the items with the greatest profit per weight ratio. In Table 1, the profit per weight ratio of each item is given.

Table 1: Profit per weight ratio of each item for problem 2

item	1	2	3	4
Profit(P)	19	23	30	40
Weight(W)	6	8	10	13
P/W	3.16667	2.875	3	3.0769

Therefore, we first select item 4 since it has the most profit per weight ratio. The knapsack still have a capacity of 12, and we then select item 1. After selecting item 1 and 4, the knapsack still has a capacity of 6. So we can select 0.6 fraction of item 3. This gives an optimal solution for LP(1) as $x(1) = (1, 0, 0.6, 1)$ with value $Z_{LP}(1) = 75.75$. At this node, Case 3 occurs, so we make the children of node 1 (i.e., nodes 2 and 3) as active.

We next choose node 2 and make it inactive. We then solve LP(2). Notice that at node 2, the value of the firs binary variable is set to be zero, so LP(2) is as follows:

$$\begin{aligned} \max \quad & 23x_2 + 30x_3 + 40x_4 \\ \text{s.t.} \quad & 8x_2 + 10x_3 + 13x_4 \le 25, \\ & 0 \le x_i \le 1, \quad \text{for } i = 2, \dots, 4. \end{aligned} \qquad \text{(LP(2))}$$

The optimal solution of LP(2) is $x(2) = (0, 2/8, 1, 1)$ with value $Z_{LP}(2) = 75$. Notice that $Z^* < Z_{LP}(2)$ and $x(2)$ is not integral, so we make the children of node 2 (nodes 4 and 5) as active.

The above procedure is repeated until there is no active node, at which point the incumbent solution will be optimal. Figure 2 gives an overview of the branch and bound tree for solving Problem (1). As shown in the tree, the nodes 15,17,19,21 get pruned since the corresponding linear program at these nodes are all infeasible. In addition, we get $x(4) = (0, 0, 11)$ with value $Z_{LP}(4) = 70$ at node 4, so node gets pruned and the incumbent solution is updated. Later on, we get an improved incumbent solution at node 20 since we have $x(20) = (1, 1, 1, 0)$ with value $Z_{LP}(20) = 73$. This is the optimal solution.
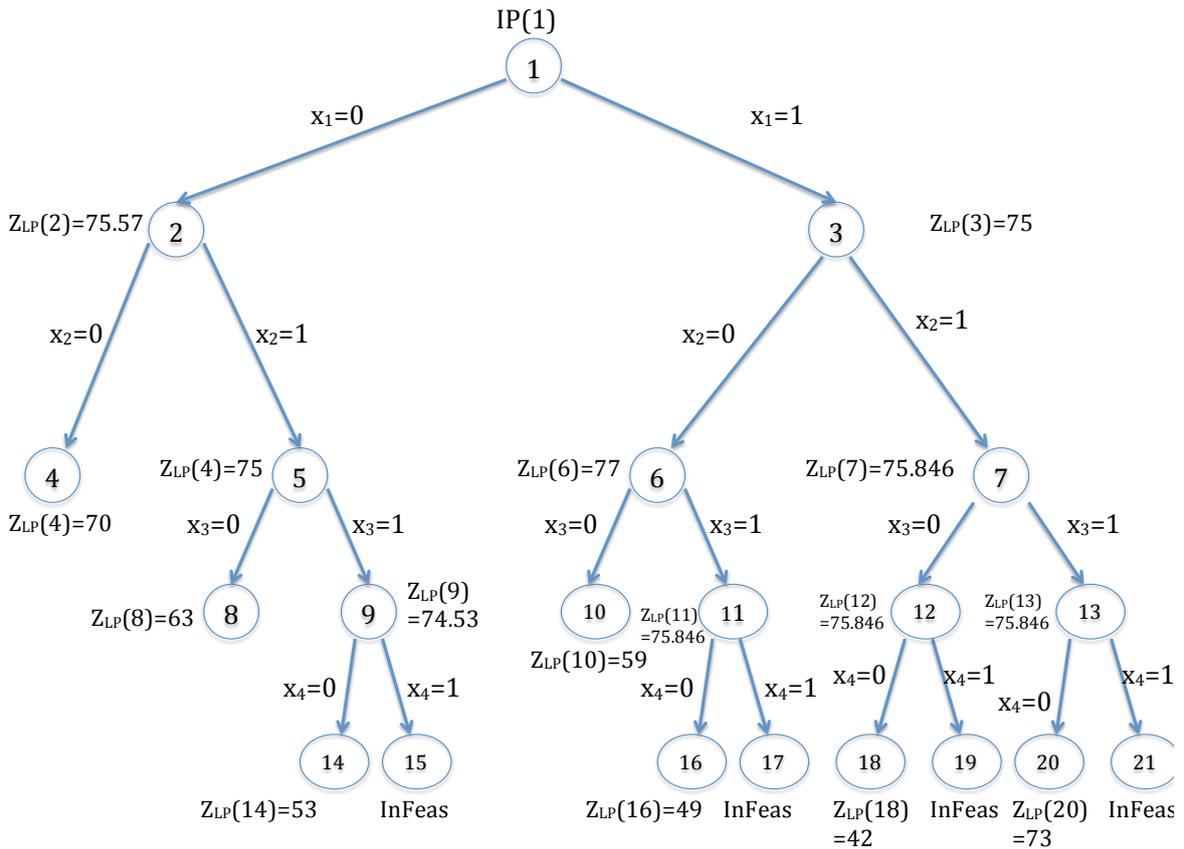
Figure 2: Branch and Bound Tree for Problem 1.

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