Optimization Methods in Management Science

MIT 15.053

Recitation 5

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Problem 1

Suppose we are solving the linear program given in the tableau below:

Basic	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
$(-z)$	2	3	5	2	0	0	0	0
s_1	5	6	-1	2	1	0	0	6
s_2	-3	-1	2	-1	0	1	0	4
s_3	-2	0	2	-2	0	0	1	3

After three pivots, suppose we get the following tableau.

Basic	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
$(-z)$	a	b	c	d	-3	-4	0	e
x_4	1	$3.6\bar{6}$	0	1	$0.6\bar{6}$	$0.3\bar{3}$	0	$5.3\bar{3}$
x_3	-1	$1.3\bar{3}$	1	0	$0.3\bar{3}$	$0.6\bar{6}$	0	$4.6\bar{6}$
s_3	2	$4.6\bar{6}$	0	0	$0.6\bar{6}$	$-0.6\bar{6}$	1	$4.3\bar{3}$

(a) Determine the simplex multipliers, and then compute the optimal value and reduced costs of variables x_1, x_2, x_3, x_4, which are missing in the tableau (i.e., a, b, c, d, e).

Solution. The original tableau is in canonical form, so the simplex multipliers appear in the final tableau as negative of the reduced costs of the original basic variables. Therefore, simplex multipliers are 3,4, and 0 for the first, second, and third constraint, respectively. The reduced cost of each variable equals the original cost coefficient the sum-product of the simplex multipliers and the constraint coefficients of the variable. If c_i is the cost coefficient of x_i and A_i denotes the column vector of constraint coefficients of x_i , then the reduced cost of x_i is $c_i - \pi A_i$, where π is the (row vector of simplex multipliers. Here, we have $\pi = (3, 4, 0)$. Therefore,

$$a = c_1 - \pi A_1 = 2 - \begin{pmatrix} 5 & -3 & -2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} = -1;$$

$$b = c_2 - \pi A_2 = 3 - \begin{pmatrix} 6 & -1 & -0 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} = -11;$$

$$c = c_3 - \pi A_3 = 5 - \begin{pmatrix} -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} = 0;$$

$$d = c_4 - \pi A_4 = 2 - \begin{pmatrix} 2 & -1 & -2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} = 0;$$

Since we have the optimal solution and know the cost coefficients, then

$$e = -\text{optimal value} = -(4.6\bar{6} \times 5 + 2 \times 5.3\bar{3}) = -34.$$

(b) Assume that the cost coefficient of variable x_2 is increased by 5 (from 3 to 8). Does the current optimal solution remain optimal?

Solution. The reduced cost of x_2 is -11. So the optimal solution remains unchanged as long as the cost coefficient is increased by less than 11. Therefore, an increase by 5 will not affect the optimal solution.

Problem 2

You are given the following optimization problem:

$$\left.\begin{array}{rrcl} \max & x_1 - 2x_2 & & \\ \text{subject to:} & & & \\ \text{Constr1:} & 3x_1 + x_2 & \geq & 3 \\ \text{Constr2:} & x_1 + 2x_2 & = & 4 \\ & x_1, x_2 & \geq & 0. \end{array}\right\}$$

(a) Put the problem in standard form.

Solution.

$$\left.\begin{array}{rrcl} \max & x_1 - 2x_2 & & \\ \text{subject to:} & & & \\ \text{Constr1:} & 3x_1 + x_2 - s_1 & = & 3 \\ \text{Constr2:} & x_1 + 2x_2 & = & 4 \\ & x_1, x_2, s_1 & \geq & 0. \end{array}\right\}$$

(b) An initial basic feasible solution is not apparent. Therefore, we will apply Phase I to find an initial feasible solution. Write the Phase I problem.

Solution.

$$\left.\begin{array}{rrcl} \max & 4x_1 + 3x_2 - s_1 - 7 & & \\ \text{subject to:} & & & \\ \text{Constr1:} & 3x_1 + x_2 - s_1 + w_1 & = & 3 \\ \text{Constr2:} & x_1 + 2x_2 + w_2 & = & 4 \\ & x_1, x_2, s_1, w_1, w_2 & \geq & 0. \end{array}\right\}$$

(c) Write the initial tableau for the Phase I problem in canonical form below. The number of additional variables is at most 3. (Note: make sure it is in canonical form and the rhs value of the objective function row is correct!)

Solution.

Basic	x_1	x_2				Rhs
$(-z)$	4	3	-1	0	0	7
w_1	3	1	-1	1		3
w_2	1	2			1	4

Problem 3

Consider the following 2-person zero-sum game:

	C_1	C_2	C_3
R_1	2	3	-2
R_2	3	1	0
R_3	-3	-3	3

(a) Write a linear program to determine an optimal strategy for the row player. Do not solve the linear program.

Solution. For row i, we let the decision variables x_i be the probability the row player chooses row i for $i = 1, 2, 3$. The row player is going to maximize the minimum payoff. So he deals to the following optimization problem:

$$\begin{array}{rl} \max\ \min & \{2x_1 + 3x_2 - 3x_3, 3x_1 + x_2 - 3x_3, -2x_1 + x_3\} \\ \text{s.t.} & x_1 + x_2 + x_3 = 1, \\ & x_i \geq 1, \quad i = 1, 2, 3. \end{array}$$

We transform this into a linear program by introducing a new variable, z:

$$\begin{array}{lrcl} \max & z & & \\ & z & \leq & 2x_1 + 3x_2 - 3x_3 \\ & z & \leq & 3x_1 + x_2 - 3x_3 \\ & z & \leq & -2x_1 + x_3 \\ & x_1 + x_2 + x_3 & = & 1, \\ & x_1, x_2, x_3 & \geq & 0 \end{array}$$

(b) Write a linear program to determine an optimal strategy for the column player. Do not solve the linear program.

Solution. For column j, we let the decision variables y_j be the probability the column player chooses column j for $j = 1, 2, 3$. The column player wishes to minimize the maximum payoff. This leads to the following problem:

$$\begin{array}{rl} \min\ \max & \{2y_1 + 3y_2 - 2y_3, 3y_1 + y2, -3y_1 - 3y_2 + 3y_3\} \\ \text{s.t.} & y_1 + y_2 + y_3 = 1, \\ & y_j \geq 1, \quad j = 1, 2, 3. \end{array}$$

We transform this into a linear program by introducing a new variable, w:

$$\begin{array}{lrcl} \min & w & & \\ & w & \geq & 2y_1 + 3y_2 - 2y_3 \\ & w & \geq & 3y_1 + y2 \\ & w & \geq & -3y_1 - 3y_2 + 3y_3 \\ & y_1 + y_2 + y_3 & = & 1, \\ & y_1, y_2, y_3 & \geq & 0 \end{array}$$

Problem 4

Two players, say Player I and Player II, simultaneously call out one of the numbers one or two. Player Is name is Odd; he wins if the sum of the numbers is odd. Player IIs name is Even; she wins if the sum of the numbers is even. The amount paid to the winner by the loser is always the sum of the numbers in dollars. It turns out that one of the players has a distinct advantage in this game. Can you tell which one it is?

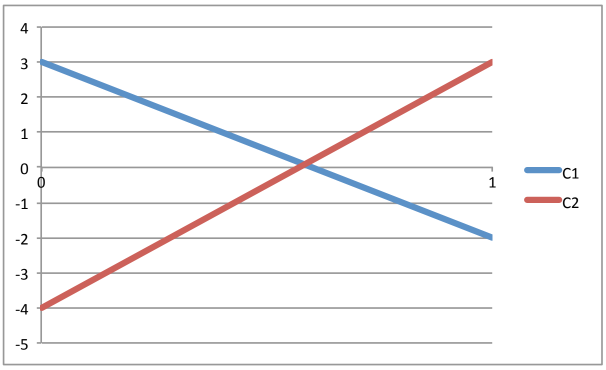
Solution. The payoff matrix is

		Even 1	Even 2
Odd	1	-2	3
	2	3	-4

We write an optimization problem to determine an optimal mixed strategy for the odd player. We let the decision variables p be the probability the odd player chooses the number one (row 1). The odd player wishes to maximize the minimum payoff. This leads to the following problem:

$$\begin{array}{lll} \min & \max & \{-2p + 3(1-p) = -5p + 3, 3p - 4(1-p) = 7p - 4\} \\ & \text{s.t.} & 0 \leq p \leq 1. \end{array}$$

We can solve this problem graphically as show in the figure.



The optimal strategy for the odd player is to choose the number one with probability 7/12 that would guarantee a payoff of 1/12.

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