Optimization Methods in Management Science

MIT 15.053, Spring 2013

Problem Set 1 (Second Group of Students)

Students with first letter of surnames G–Z

Due: February 12 , 2013

Problem Set Rules:

1. Each student should hand in an individual problem set.
2. Discussing problem sets with other students is permitted. Copying from another person or solution set is *not* permitted.
3. Late assignments will *not* be accepted. No exceptions.
4. The non-Excel solution should be handed in at the beginning of class on the day the problem set is due. The Excel solutions, if required, should be posted on the website by the beginning of class on the day the problem set is due. Questions that require an Excel submission are marked with Excel submission . For Excel submission questions, only the Excel spreadsheet will be graded.

Problem 1

(35 points total) The first problem asks you to formulate a 3-variable linear program in three different ways (four ways if you also count the algebraic formulation). Both of the first two ways are fairly natural. The third way is a bit obscure. And the algebraic formulation may seem overly complex. In practice, there are advantages to formulating linear programs in different ways. And there are huge advantages in the algebraic formulation. (One can express huge problems efficiently on a computer using a modeling language, which is based on the algebraic formulation.) In addition, formulating an LP in multiple ways provides insight into the LP models.

Accessories & co. is producing three kinds of covers for Apple products: one for iPod, one for iPad, and one for iPhone. The company's production facilities are such that if we devote the entire production to iPod covers, we can produce 6000 of them in one day. If we devote the entire production to iPhone covers or iPad covers, we can produce 5000 or 3000 of them in one day. The production schedule is one week (5 working days), and the week's production must be stored before distribution. Storing 1000 iPod covers (packaging included) takes up 40 cubic feet of space. Storing 1000 iPhone covers (packaging included) takes up 45 cubic feet of space, and storing 1000 iPad covers (packaging included) takes up 210 cubic feet of space. The total storage space available is 6000 cubic feet. Due to commercial agreements with Apple, Accessories & co. has to deliver at least 5000 iPod covers and 4000 iPad covers per week in order to strengthen the product's diffusion. The marketing department estimates that the weekly demand for iPod covers, iPhone, and iPad covers does not exceed 10000 and 15000, and 8000 units, therefore the company does not want to produce more than these amounts for iPod, iPhone, and iPad covers. Finally, the net profit per each iPod cover, iPhone cover, and iPad cover is \$4, \$6, and \$10, respectively.

The aim is to determine a weekly production schedule that maximizes the total net profit.

(a) (5 points) Write a Linear Programming formulation for the problem. Start by stating any assumptions that you make. Label each constraint (except nonnegativity). For this first formulation, the decision variables should represent the proportion of time spent each day on producing each of the two items:

$$x_1 = \text{ proportion of time devoted each day to iPod cover production,}$$
$$x_2 = \text{ proportion of time devoted each day to iPhone cover production,}$$
$$x_3 = \text{ proportion of time devoted each day to iPad cover production.}$$

(Different formulations will be required for parts (b) and (c).)

Solution. As required, we let: x_1 = proportion of time devoted each day to iPod smart cover production, x_2 = proportion of time devoted each day to iPhone smart cover production, and x_3 = proportion of time devoted each day to iPad smart cover production. We assume:

(a) that the production can be split between the two products in any desired way (that is, fractional values for x_1, x_2, and x_3 are acceptable), and
(b) that the number of produced item of each type is directly proportional to the time devoted to producing the item.

Given these assumptions, we can formulate the problem as an LP as follows:

$$\left.\begin{array}{rrcl}
\text{max} & 120000x_1 + 150000x_2 + 150000x_3 & & \\
\text{s.t.:} & & & \\
\text{Max daily production:} & x_1 + x_2 + x_3 & \leq & 1 \\
\text{Storage:} & 1200x_1 + 1125x_2 + 3150x_3 & \leq & 6000 \\
\text{Min iPod production:} & 30000x_1 & \geq & 5000 \\
\text{Min iPad production:} & 15000x_3 & \geq & 4000 \\
\text{Max iPod demand:} & 30000x_1 & \leq & 10000 \\
\text{Max iPhone demand:} & 25000x_2 & \leq & 15000 \\
\text{Max iPad demand:} & 15000x_3 & \leq & 8000 \\
 & x_1, x_2, x_3 & \geq & 0.
\end{array}\right\} \quad (1)$$

Because x_1 already has a lower bound, omitting the nonnegativity constraint for x_1 is not considered an error (no penalty).

(b) (5 points) Write a second Linear Programming formulation for the problem. Label each constraint (except nonnegativity). For this second formulation, the decision variables should represent the number of items of each type produced over the week:

$$y_1 = \text{number of iPod covers produced over the week,}$$
$$y_2 = \text{number of iPhone covers produced over the week,}$$
$$y_3 = \text{number of iPad covers produced over the week.}$$

The problem data is the same but you must make sure that everything matches the new decision variables.

Solution. As required, we let: y_1 = number of iPod covers produced over the week, y_2 = number of iPhone covers produced over the week, and y_3 = number of iPad covers produced over the week. We make the same assumptions as for part (a). We can formulate the problem as an LP as follows:

$$\left.\begin{array}{rrcl}
\text{max} & 4y_1 + 6y_2 + 10y_3 & & \\
\text{s.t.:} & & & \\
\text{Max weekly production:} & 1/6000y_1 + 1/5000y_2 + 1/3000y_1 & \leq & 5 \\
\text{Storage:} & 0.04y_1 + 0.045y_2 + 0.21y_3 & \leq & 6000 \\
\text{Min iPod production:} & y_1 & \geq & 5000 \\
\text{Min iPad production:} & y_3 & \geq & 4000 \\
\text{Max iPod demand:} & y_1 & \leq & 10000 \\
\text{Max iPhone demand:} & y_2 & \leq & 15000 \\
\text{Max iPad demand:} & y_3 & \leq & 8000 \\
 & y_1, y_2, y_3 & \geq & 0.
\end{array}\right\} \quad (2)$$

Because y_1 already has a lower bound, omitting the nonnegativity constraint for y_1 is not considered an error (no penalty).

(c) (5 points) Write a third Linear Programming formulation for the problem. Label each constraint (except nonnegativity). Assume that each working day has 8 working hours. For this third formulation, the decision variables should be:

$$\begin{aligned}
z_1 &= \text{number of hours devoted to the production of iPod smart covers in one week ,} \\
z_2 &= \text{number of hours devoted to the production of iPhone smart covers in one week,} \\
z_3 &= \text{total number of production hours employed during the week.}
\end{aligned}$$

Express the objective function in thousands of dollars. The problem data is the same but you must make sure that everything matches the new decision variables.

Solution. As requested, we use two decision variables: z_1 is the number of hours devoted to iPod cover production, z_2 is the number of hours devoted to iPhone cover production, and z_3 is the total number of production hours employed for the week. It follows that the number of hours devoted to iPad cover production is $z_3 - z_1 - z_2$, which should be a nonnegative number (i.e. we must impose $z_1 + z_2 \leq z_3$). We can therefore formulate the problem as follows:

$$\left.\begin{array}{rrcl}
\text{max} & 3000z_1 + 3750z_2 + 3750(z_3 - z_1 - z_2) & & \\
\text{s.t.:} & & & \\
\text{iPad Production:} & z_1 & \leq & 40 \\
\text{iPhone Production:} & z_2 & \leq & 40 \\
\text{Storage:} & 30z_1 + 28.125z_2 + 78.75(z_3 - z_1 - z_2) & \leq & 6000 \\
\text{Min iPod production:} & 750z_1 & \geq & 5000 \\
\text{Min iPad production:} & 375(z_3 - z_1 - z_2) & \geq & 4000 \\
\text{Max iPod demand:} & 750z_1 & \leq & 10000 \\
\text{Max iPhone demand:} & 625z_2 & \leq & 15000 \\
\text{Max iPad demand:} & 375(z_3 - z_1 - z_2) & \leq & 8000 \\
 & z_1, z_2 & \geq & 0.
\end{array}\right\}$$

(d) (5 points) What is the relationship between the variables z_1, z_2, z_3 of part (c) and the variables x_1, x_2, x_3 of part (a) of this problem? Give a formula to compute z_1, z_2, z_3 from

x_1, x_2, x_3.

Solution. The relationship between z_1, z_2, z_3 of part (d) and x_1, x_2, x_3 of part (a) is the following. $z_1 = 40x_1$ because x_1 represents the percentage of time of each day dedicated to iPad cover production; multiplying x_1 by the total number of hours in a production period (= 40) yields the number hours spent on iPods. Similarly, we have $z_2 = 40x_2$. Then, $z_3 = 40(x_1 + x_2 + x_3)$ because $x_1 + x_2 + x_3$ represents the fraction of time used during the day; multiplying this by the total number of hours in a production period yields the total number of hours employed during the week.

(e) (5 points) <u>Excel submission</u> Solve the problem using Excel Solver, following the guidelines given in the Excel Workbook that comes with this problem set. Pay attention to the formulation in the Excel Workbook: it is similar to the one required for part (b), but it is not exactly the same.

Solution. Nonzero variables in the optimal solution: $x_1 = 5, x_2 = 7.5, x_3 = 8$. Objective value: 145000.

(f) (10 points) Write an algebraic formulation of the weekly production schedule problem described above using the following notation:

- n is the number of product types,
- x_j is the number of days devoted to the production of products of type j,
- p_j is the number of items of type j that can be manufactured in one day, assuming that the process is devoted to products of type j.
- P is the number of production days in one week,
- s_j is the storage space required by *one* item of type j,
- S is the total storage space available for the week's production,
- r_j is the unit profit for each product of type j,
- d_j is the weekly maximum demand for an item of type j.
- b_j is the weekly minimum demand for an item of type j.

Solution. We denote by x_j the number of days devoted to the production of the j-th item. This is one possible formulation. Other correct formulations are possible.

$$\left.\begin{array}{rrcl}
\max & \sum_{j=1}^{n} r_j p_j x_j & & \\
\text{s.t.:} & & & \\
\text{Production:} & \sum_{j=1}^{n} x_j & \leq & P \\
\text{Storage:} & \sum_{j=1}^{n} s_j p_j x_j & \leq & S \\
\text{Max demand: } \forall j = 1, \ldots, T & p_j x_j & \leq & d_j \\
\text{Min demand: } \forall j = 1, \ldots, T & p_j x_j & \geq & b_j \\
\forall j = 1, \ldots, T & x_j & \geq & 0.
\end{array}\right\}$$

Problem 2

(10 points total). Problem 2 reviews the transformations from nonlinear constraints or objectives into linear constraints and objectives, as mentioned in the second lecture and discussed in

the tutorial “LP Transformation Tricks”.

In each part, transform the corresponding mathematical program to an equivalent linear program. Do not solve the linear program.

(a) (5 points) Problem formulation:

$$\left.\begin{array}{rrcl} \text{min} & \max\{2.3x_1 + x_2, 4.3x_1 - 0.5x_2, 2.5x_1 + 3.5x_2\} & & \\ \text{s.t.:} & & & \\ \text{Constr1}: & x_1/(x_1 + x_2) & \leq & 0.5 \\ \text{Constr2}: & 10x_1 + 28x_2 & = & 3.4 \\ \text{Constr3}: & x_1 + x_2 & \geq & 0 \\ & x_1 & \text{free} & \\ & x_2 & \text{free} & \end{array}\right\} \quad (3)$$

Solution. The objective function can be reformulated as a linear objective function by introducing an extra variable w and adding three constraints. In addition, we require to multiply through by $x_1 + x_2$ (which is nonnegative) the constraint that has x_3 the denominator. The resulting problem is as follows

$$\left.\begin{array}{rrcl} \text{min} & w & & \\ \text{s.t.:} & & & \\ \text{Obj Ref1}: & 2.3x_1 + x_2 - w & \leq & 0 \\ \text{Obj Ref2}: & 4.3x_1 - 0.5x_2 - w & \leq & 0 \\ \text{Obj Ref2}: & 2.5x_1 + 3.5x_2 - w & \geq & 0 \\ \text{Constr1}: & 0.5x_1 - 0.5x_2 & \leq & 0 \\ \text{Constr2}: & 10x_1 + 28x_2 & = & 3.4 \\ \text{Constr3}: & x_1 + x_2 & \geq & 0 \\ & x_1 & \text{free} & \\ & x_2 & \text{free} & \end{array}\right\}$$

(b) (5 points) Problem formulation:

$$\left.\begin{array}{rrcl} \text{min} & |0.8x_1 + 0.9x_2| & & \\ \text{s.t.:} & & & \\ \text{Constr1}: & |0.9x_1 + 1.2x_2| & \leq & 10 \\ & x_1 & \geq & 0 \\ & x_2 & & \text{free} \end{array}\right\} \quad (4)$$

Solution. Can be reformulated by introducing an extra variable w to reformulate the objective function, and splitting Constr1 into two constraints:

$$\left.\begin{array}{rrcl} \text{min} & w & & \\ \text{s.t.:} & & & \\ \text{Obj Ref1}: & 0.8x_1 + 0.9x_2 - w & \leq & 0 \\ \text{Obj Ref2}: & -0.8x_1 - 0.9x_2 - w & \leq & 0 \\ \text{Constr1 Ref1}: & 0.9x_1 + 1.2x_2 & \leq & 10 \\ \text{Constr1 Ref2}: & -0.9x_1 - 1.2x_2 & \leq & 10 \\ & x_1 & \geq & 0 \\ & x_2 & & \text{free} \end{array}\right\}$$

Problem 3 (Second group of students)[1]

(55 points total) Charles Watts Electronics manufactures the following six peripheral devices used in computers especially designed for jet fighter planes: internal modems, external modems, graphics circuit boards, USB memory stick, hard disk drives, and memory expansion boards. Each of these technical products requires time, in minutes, on three types of electronic testing equipment as shown in the following table:

	Internal Modem	**External Modem**	**Circuit Board**	**USB Stick**	**Hard Drives**	**Memory Boards**
Test device 1	7	3	12	6	18	17
Test device 2	2	5	3	2	15	17
Test device 3	5	1	3	2	9	2

The first two test devices are available 130 hours per week. The third (device 3) requires more preventive maintenance and may be used only 100 hours each week. Watts Electronics believes that it cannot sell more than 2000, 1500, 1800, 1200, 1000, 1000 units of each device, respectively. Thus, it does not want to produce more than these units. The table that follows summarizes the revenues and material costs for each product:

Device	**Revenue per unit sold ($)**	**Material Cost per unit ($)**
Internal modem	200	35
External modem	120	25
Circuit board	180	40
USB memory stick	130	45
Hard disk drive	430	170
Memory expansion board	260	60

In addition, variable labor costs are $16 per hour for test device 1, $12 per hour for test device 2, and $18 per hour for test device 3. Watts Electronics wants to maximize its profits.

a) (10 points) Write a linear program for this problem.

Solution.
We define six decision variables that indicate the number of devices as:

$x_I =$ the number of internal modems $x_E =$ the number of external modems
$x_C =$ the number of circuit boards $x_U =$ the number of USB memory sticks
$x_H =$ the number of hard disk derives $x_M =$ the number of memory expansion boards

The challenging part of this problem is to determine the objective function. The objective function is the sum of profits from selling each device. The profit from each device is the selling price minus the material costs minus the costs of testing. For example, the profit for selling x_I units of internal modem device is given as

$$x_I\Big(200 - 35 - (7/60)16 - (2/60)12 - (5/60)18\Big).$$

[1]This problem is based on Problem B.29 of Operations Management by Heizer and Render (2010).

Similarly, we can determine the profit for other devices. Thus, we can formulate the problem as follows:

$$
\left.\begin{array}{rrcl}
\max & x_I\Big(200-35-(7/60)16-(2/60)12-(5/60)18\Big)+ & & \\
& x_E\Big(120-25-(3/60)16-(5/60)12-(1/60)18\Big)+ & & \\
& x_C\Big(180-40-(12/60)16-(3/60)12-(3/60)18\Big)+ & & \\
& x_U\Big(130-45-(6/60)16-(2/60)12-(2/60)18\Big)+ & & \\
& x_H\Big(430-170-(18/60)16-(15/60)12-(9/60)18\Big)+ & & \\
& x_M\Big(260-60-(17/60)16-(17/60)12-(2/60)18\Big) & & \\
\text{s.t.:} & & & \\
\text{Availability Test Dev. 1}: & 7x_I+3x_E+12x_C+6x_U+18x_H+17x_M & \leq & 130*60 \\
\text{Availability Test Dev. 2}: & 2x_I+5x_E+3x_C+2x_U+15x_H+17x_M & \leq & 130*60 \\
\text{Availability Test Dev. 3}: & 5x_I+1x_E+3x_C+2x_U+9x_H+2x_M & \leq & 100*60 \\
\text{Max int modem demand}: & x_I & \leq & 2000 \\
\text{Max ext modem demand}: & x_E & \leq & 1500 \\
\text{Max circuit demand}: & x_C & \leq & 1800 \\
\text{Max USB memory demand}: & x_U & \leq & 1200 \\
\text{Max hard disk demand}: & x_H & \leq & 1000 \\
\text{Max memory demand}: & x_M & \leq & 1000 \\
& x_I,x_E,x_C,x_U,x_H,x_M & \geq & 0.
\end{array}\right\}
$$

b) (10 points) <u>Excel submission</u> Write a spreadsheet for the problem and solve the problem using Excel Solver, following the guidelines given in the Excel Workbook that comes with this problem set. (Hint: the optimal value is $102986.7 .)

c) Use the Excel spreadsheet to answer the following questions:

(i) (4 points) What is the value of an additional minute of time per week on test device 1? Test device 2? Test device 3? Should Watts Electronics add more test device time? If so, on which equipment?

Solution. In the optimal solution, there constraint on time for device is non-binding as not all of the available time is used; therefore, there is no value in additional hours for device 3. However, for device 1, the solution changes by $211688-$211667 = $21 when one additional minute is allowed, indicating an additional minute on device 1 is worth $21. Similarily, when one additional minute of availability is added for device 2, the solution changes by $211673-$211667 = $6; one additional minute is worth $6 (or $5.70 if not rounding).

(ii) (4 points) Suppose that Watts Electronics is considering to increase the available time of test device 2 for the next week. What would be the increase in the profit if the availabile time increases to t for $t = 131$, 132, and 133. (Assume that there are still 130 hours of test device 1 and 100 hours of test device 3.) The increase is the difference between the new profit and the profit from Part (a).

Solution. The increase in profit for $t = 131$, 132, 133 will be $344.69, $689.38, and $1034.07, respectively

(iii) (4 points) Based on your answer to part (ii), what do you think will be the contribution if the availability time of test device 2 increases to 135? (Verify that you are correct.)

What is the formula for the optimum profit if the availability time increased by $130+t$? (You may assume that t is between 1 and 10).

> **Solution.** The increase in profit will be \$1723.45; the formula for profit increase in this region of t will be \$344.69*t.

(iv) (4 points) Based on your formula in part (iii), what is the contribution if the availability time of test device 2 increases to 150. Use Excel solver to see if the formula is correct. Use Excel solver to determine the maximum value of t for which your formula is correct (Be accurate to within an hour).

> **Solution.** The formula predicts an increase of \$6893.80; however, there is only an increase of \$3693.10 in reality. The formula works for an increase of up to t=10 hours.

(v) (4 points) How would the optimal solution in Part (b) change if the labor costs increased to \$18 per hour for test device 1, \$13 per hour for test device 2, and \$20 per hour for test device 3?

> **Solution.** The optimal profit then reduces to \$211142.

(vi) (5 points) Over what range of the labor cost for test device 3 will the optimal production-mix in Part (b) remain optimal? (Be accurate to within one dollar.)

> **Solution.** The solution in part (b) is stable over all labor rates.

d) (10 points) Write an algebraic formulation for the problem using the following notation:

- m is the number of peripheral devices,
- n is the number of test devices,
- a_i is the maximum availability time of test device i (in hour),
- c_i^L is the labor cost per hour of test device i (in \$),
- c_j^M is the material cost per unit for device j (in \$),
- r_j is the revenue cost per unit for device j (in \$),
- a_i is the maximum availability of material i,
- b_j is the maximum demand of device j,
- q_{ij} is the amount of time in minute on test service i that is required in device j,

Solution. Let x_j indicate the number of units of device j for $j = 1, \ldots, m$. Then, the problem can be formulated as follows:

$$\begin{aligned}
\max \quad & \sum_{j=1}^{m} \Big(r_j - c_j^M - \frac{1}{60} \big(\sum_{i=1}^{n} c_i^L q_{ij} \big) \Big) x_j \\
\text{s.t.} \quad & \sum_{j=1}^{m} q_{ij} x_j \le a_i * 60 \quad \forall i = 1, \ldots, n, \quad \text{(Availability time for test device } i\text{)} \\
& x_j \le b_j, \quad \forall j = 1, \ldots, m, \quad \text{(Maximum demand for device } j\text{)} \\
& x_j \ge 0, \quad \forall j = 1, \ldots, m, \quad \text{(Nonnegative constraints).}
\end{aligned}$$