for $n_1, n_2, n_3, \dots$ collinear modes in the sum on n. The collinear modes are distinct only if

$$n_i \cdot n_j \gg \lambda^2 \quad \text{for } i \neq j\,. \tag{5.36}$$

We may understand this result by a counter argument: If a momentum $p_2 = Qn_2$, then $n_1 \cdot p_2 = Qn_1 \cdot n_2 \sim \lambda^2$ iff $n_1 \cdot n_2 \sim \lambda^2$. Hence p_2 is n_1-collinear, and n_2 is not a distinct collinear direction from n_1. If $n_i \cdot n_1 \sim \lambda^2$ then we say that n_i is within the RPI equivalence class $[n_1]$ defined by the member n_1. Distinct collinear directions correspond to the different equivalence classes, and we only sum over distinct directions in Eq. (5.35).

Essentially all of the things we derived with one collinear direction get repeated when we have more than one collinear direction.

- For each light-like n_i we define an auxillary light-like $\bar{n}_i$ where $n_i \cdot \bar{n}_i = 2$. Collinear momenta in the n_i direction are decomposed with the $\{n_i, \bar{n}_i\}$ basis vectors since the components have a definite power counting: $(n_i \cdot p, \bar{n}_i \cdot p, p_{n_i\perp}) \sim (\lambda^2, 1, \lambda)$. Note that the meaning of $\perp$ depends on which n_i-collinear sector we are discussing.

- There is a separate RPI for each n_i-collinear sector that only acts on the n_i-collinear fields, and on objects decomposed with the $\{n_i, \bar{n}_i\}$ basis vectors. Here there is no simple connection to an overall Lorentz transformation because the fields in other sectors do not transform.

- There is a collinear gauge transformation U_{n_i} for each type of collinear field. Only the fields in the n_i-collinear direction transform (fields in other collinear sectors do not transform with U_{n_i} since such transformations would yield offshell momenta that are outside the effective theory).

- Matching calculations generate multiple collinear Wilson lines $W_{n_i} = W_{n_i}[\bar{n}_i \cdot A_{n_i}]$. The definitions are identical to Eq. (4.51) with $n \to n_i$, $\bar{n} \to \bar{n}_i$, including $\overline{P} \to \bar{n}_i \cdot \mathcal{P}$. They are again always built only out of the $\mathcal{O}(\lambda^0)$ gluon fields, and correspond to straight Wilson lines. These matching calculations lead to operators in SCET that are gauge invariant under U_{n_i} transformations.

As an example of the last point consider the process $e^+e^- \to \gamma^* \to$ two-jets. The QCD current is $J^\mu = \bar{\psi}\gamma^\mu\psi$. By integrating out offshell fields to match onto SCET_I we obtain the leading order current

$$J^\mu_{\text{SCET}} = (\bar{\xi}_{n_1} W_{n_1})\gamma^\mu (W^\dagger_{n_2}\xi_{n_2})\,. \tag{5.37}$$

Here n_1 and n_2 are the directions of the two jets. The Wilson line $W_{n_1} = W_{n_1}[\bar{n}_1 \cdot A_{n_1}]$ is generated by integrating out the attachment of $\bar{n}_1 \cdot A_{n_1}$ gluons to n_2-collinear quarks and gluons, and analogously for W_{n_2}. The resulting operator in Eq. (7.29) is invariant under n_1-collinear, n_2-collinear, and ultrasoft gauge transformations. In general one can carry out all orders tree level matching computations to derive the presence of these Wilson lines. For situations with multiple lines in different directions these calculations are greatly facilitated by using the auxillary field method (see the appendices of [6, 8]).

6 Factorization from Mode Separation

One of the benefits of the SCET formalism is the clear separation of scales at the level of the Lagrangian and of operators that mediate hard interatctions. We will explore the factorization between various types of modes in this section.

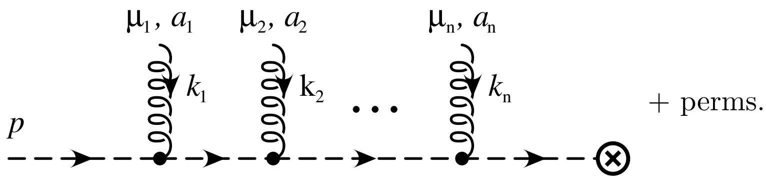


Figure 9: The attachments of ultrasoft gluons to a collinear quark line which are summed up into a path-ordered exponential.

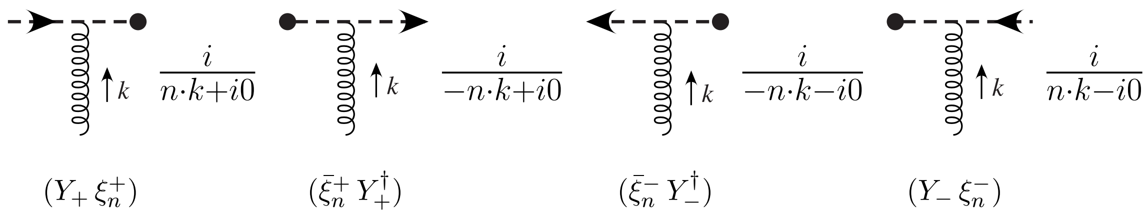


Figure 10: Eikonal $i0$ prescriptions for incoming/outgoing quarks and antiquarks and the result that reproduces this with an ultrasoft Wilson line and sterile quark field.

6.1 Ultrasoft-Collinear Factorization

Recall that only the $n \cdot A_{us}$ component couples to n-collinear quarks and gluons at leading order in λ. This is explicit in the Feynman rules in Figs. 6 and 7 where only n_μ appears for the ultrasoft gluon with index μ. Furthermore due to the multipole expansion the collinear particles only see the $n \cdot k$ ultrasoft momentum of the $n \cdot A_{us}$ gluons. For example, if we consider Fig. 9 with only one ultrasoft gluon then the collinear quark propagator is

$$\frac{\bar n \cdot p}{\bar n \cdot p\, n \cdot (p_r + k) + p_\perp^2 + i0} = \frac{\bar n \cdot p}{\bar n \cdot p\, n \cdot k + p^2 + i0} = \frac{\bar n \cdot p}{\bar n \cdot p\, n \cdot k + i0}\,, \tag{6.1}$$

where in the last equality we used the onshell condition $p^2 = 0$ for the external collinear quark. Together with the n_μ from the vertex this result corresponds to the eikonal propagator for the coupling of soft gluons to an energetic particle. The appropriate sign for the $i0$ is determined by dividing through by $\bar n \cdot p$ and noting the sign of this momentum, which differs for quark and antiquarks. Accounting for attachments to incoming or outgoing particles this leads to the four eikonal propagator results summarized in Fig. 10.

Now, we consider the case of multiple usoft gluon emission. Calculating within SCET the graphs in Fig. 9 gives $\Gamma\, \tilde Y_n u_n$ where Γ is the structure at the $\otimes$ vertex, and u_n is a collinear quark spinor. Here

$$\tilde Y_n = \sum_{m=0}^{\infty} \sum_{\text{perms}} \frac{(-g)^m n \cdot A^{a_1}(k_1) \cdots n \cdot A^{a_m}(k_m) T^{a_m} \cdots T^{a_1}}{n \cdot k_1 n \cdot (k_1 + k_2) \cdots n \cdot (\sum_i k_i)} \tag{6.2}$$

where all propagators are $+i0$. These eikonal propagators come from collinear quarks with offshellness $\sim \lambda^2$, which is near their mass shell, and hence are a property of fields in the EFT itself (as opposed to the Wilson lines W_n which were generated by matching onto the EFT). This corresponds to the momentum space formula for an ultrasoft Wilson line Y_n. In position space this formula becomes

$$Y_n(x) = \text{Pexp}\left[ig \int_{-\infty}^{0} ds\, n \cdot A^a_{us}(x + ns) T^a\right]. \tag{6.3}$$

It satisfies a defining equation and unitarity condition:

$$in \cdot D\, Y_n = 0, \qquad Y_n^\dagger Y_n = 1. \tag{6.4}$$

When we wish to be specific in the notation for our Wilson lines to show whether they extend from $-\infty$ or out to $+\infty$, and whether they are path-ordered or antipath-ordered, we will use the following notations

$$Y_{n+} = \mathrm{P}\exp\Big(ig\int_{-\infty}^{0} ds\; n\cdot A_{us}(x+sn)\Big)\,, \qquad Y_{n-} = \overline{\mathrm{P}}\exp\Big(-ig\int_{0}^{\infty} ds\; n\cdot A_{us}(x+sn)\Big)\,, \tag{6.5}$$
$$Y_{n-}^{\dagger} = \overline{\mathrm{P}}\exp\Big(-ig\int_{-\infty}^{0} ds\; n\cdot A_{us}(x+sn)\Big)\,, \qquad Y_{n+}^{\dagger} = \mathrm{P}\exp\Big(ig\int_{0}^{\infty} ds\; n\cdot A_{us}(x+sn)\Big)\,.$$

Here $(Y_{n\pm})^\dagger = Y_{n\mp}^\dagger$, and the subscript on $Y_{n\pm}^\dagger$ should be read as $(Y_n^\dagger)_\pm$ rather than $(Y_\pm)^\dagger$. The $+$ denotes Wilson lines obtained from attachments to quarks, and the $-$ denotes Wilson lines from attachments to antiquarks. The Wilson lines obtained for various situations are shown in Fig. 10.

The generation of the Wilson line Y_n from the example above motivates us to consider whether all the leading order usoft-collinear interactions within $\mathrm{SCET_I}$ (to all orders in α_s and with loop corrections) can be encoded through the non-local interactions contained in the Wilson line $Y_n(x)$. To show that this is indeed the case we consider the BPS field redefinitions [6]

$$\xi_{n,p}(x) = Y_n(x)\xi_{n,p}^{(0)}(x), \qquad A_{n,p}^{\mu}(x) = Y_n(x)\,A_{n,p}^{(0)\mu}(x)Y_n^\dagger(x)\,. \tag{6.6}$$

They include in addition $c_{n,p}(x) = Y_n(x)\,c_{n,p}^{(0)}Y_n^\dagger(x)$ for the ghost field in any general covariant gauge.

The defining equation for Y_n implies the operator equation

$$Y_n^\dagger in\cdot D_{us} Y_n = in\cdot\partial. \tag{6.7}$$

Also because the label operator $\overline{\mathcal{P}}$ commutes with Y_n the redefinition on $\bar n\cdot A_n$ in (6.6) implies that

$$W_n \to Y_n W_n^{(0)} Y_n^\dagger\,, \tag{6.8}$$

where $W_n^{(0)}$ is built from $\bar n\cdot A_n^{(0)}$ fields. Implementing these transformations into our leading collinear quark Lagrangian we find

$$\begin{aligned}
\mathcal{L}_{n\xi}^{(0)} &= \bar\xi_{n,p'}\left(in\cdot D + i\not{D}_{n\perp}\frac{1}{i\bar n\cdot D_n}i\not{D}_{n\perp}\right)\frac{\bar{\not n}}{2}\xi_{n,p}\\
&= \bar\xi_{n,p'}\left(in\cdot D_{us} + gn\cdot A_{n,q} + (\not{\mathcal{P}}_\perp + g\not{A}_{n,q\perp})W\frac{1}{\overline{\mathcal{P}}}W^\dagger(\not{\mathcal{P}}_\perp + g\not{A}_{n,q\perp})\right)\frac{\bar{\not n}}{2}\xi_{n,p}\\
&= \bar\xi_{n,p'}^{(0)}Y^\dagger\Big(in\cdot D_{us} + gYn\cdot A_{n,q}^{(0)}Y^\dagger\\
&\quad +(\not{\mathcal{P}}_\perp + gY\not{A}_{n,q\perp}^{(0)}Y^\dagger)YW^{(0)}Y^\dagger\frac{1}{\overline{\mathcal{P}}}YW^{(0)\dagger}Y^\dagger(\not{\mathcal{P}}_\perp + g\not{A}_{n,q\perp}^{(0)})\Big)\frac{\bar{\not n}}{2}Y\xi_{n,p}^{(0)}\\
&= \bar\xi_{n,p'}^{(0)}\left(in\cdot\partial + gn\cdot A_{n,q}^{(0)} + (\not{\mathcal{P}}_\perp + g\not{A}_{n,q\perp}^{(0)})W^{(0)}\frac{1}{\overline{\mathcal{P}}}W^{(0)\dagger}(\not{\mathcal{P}}_\perp + g\not{A}_{n,q\perp}^{(0)})\right)\frac{\bar{\not n}}{2}\xi_{n,p}^{(0)}\,,
\end{aligned} \tag{6.9}$$

where the last line is completely independent of the usoft gluon field. With similar steps we can easily show that the collinear gluon Lagrangian $\mathcal{L}_{ng}^{(0)}$ in (4.55) also completely decouples from the $n\cdot A_{us}$ usoft gluon field. In summary, we see that the usoft gluons have completely decoupled from collinear particles in the leading order collinear Lagrangian $\mathcal{L}_n^{(0)} = \mathcal{L}_{n\xi}^{(0)} + \mathcal{L}_{ng}^{(0)}$ via

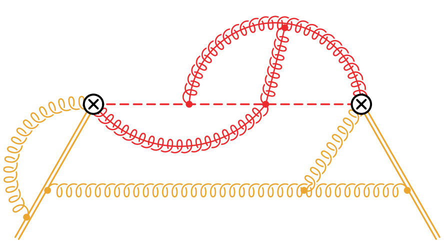
$$\mathcal{L}_n^{(0)}\left[\xi_{n,p}, A_{n,q}^{\mu}, n\cdot A_{us}\right] = \mathcal{L}_n^{(0)}\left[\xi_{n,p}^{(0)}, A_{n,q}^{(0)\mu}, 0\right]. \tag{6.10}$$

However, it is important to note that the usoft interactions for our collinear field have not disappeared, but have simply moved out of the Lagrangian and into the currents. We must make the field redefinition everywhere, including external operators and currents, as well as on interpolating fields for partons and hadrons. The field redefinition on the interpolating fields that describe incoming and outgoing states will determine whether the final usoft Wilson lines are Y_+, $Y_+^\dagger$, Y_-, or $Y_-^\dagger$ since these interplolating field operators are localized either at $-\infty$ or $+\infty$.

Eg.1: Consider our standard heavy-to-light current. Performing the field redefinitions we have

$$\begin{aligned} J^\mu = \bar{\xi}_{n,} W \Gamma^\mu h_v &= \bar{\xi}_n^{(0)} Y_n^\dagger Y_n W^{(0)} Y_n^\dagger \Gamma^\mu h_v \\ &= \bar{\xi}_n^{(0)} W^{(0)} \Gamma^\mu Y_n^\dagger h_v \,. \end{aligned} \tag{6.11}$$

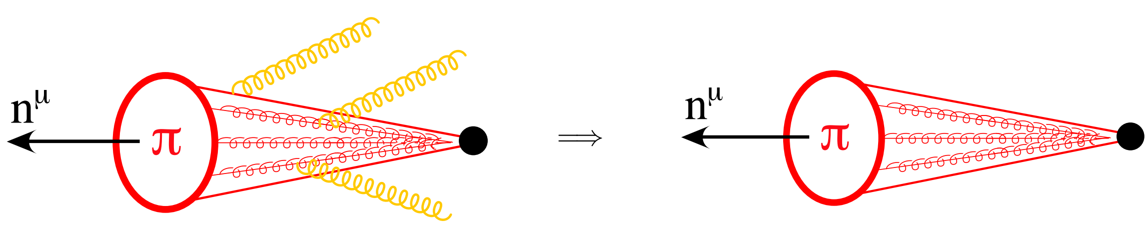
The last line gives us our first factorization result. Since $\bar{\xi}_n$ is an outgoing quark, here $Y_n^\dagger = Y_+^\dagger$. As is necessary for effective theories, we will need to include a Wilson coefficient encoding higher energy dynamics, but we can already clearly see how different scales have separated into distinct gauge invariant quantities $(\bar{\xi}_{n,p}^{(0)} W^{(0)})$ and $(Y_n^\dagger h_v)$ at the level of operators. We can demonstrate this ultrasoft-collinear factorization diagrammatically by considering the time ordered product of two currents $T J^\mu(x) J^{\dagger\nu}(0)$ (whose imaginary part is related to the inclusive decay rate). Rather than having diagrams with ultrasoft gluons coupling to collinear lines they decouple into distinct parts:



Eg.2: Consider a current that is a global color singlet within the n-collinear sector

$$J^\mu = (\bar{\xi}_n W) \Gamma^\mu W^\dagger \xi_n = (\bar{\xi}_n^{(0)} W^{(0)}) \Gamma^\mu (W^{(0)\dagger} \xi_n^{(0)}) \,. \tag{6.12}$$

Here all the usoft gluons have cancelled using $Y_n^\dagger Y_n = 1$, so all the usoft gluons decouple at leading order. Diagramatically we can imagine this current producing an energetic color singlet state like a collinear pion (ignoring the fact that we're in SCET_I for a moment):



This decoupling is called color transparency, the long wavelength usoft gluons only see the overall color charge of the energetic fields in the pion, and hence cancel out for this color singlet object.

Eg.3: As a third example, consider our operator for $e^+e^- \to$ dijets. Here we have two types of collinear fields, n_1 and n_2, and the BPS field redefinitions give Y_{n_1} and Y_{n_2} ultrasoft Wilson lines:

$$J = (\bar{\xi}_{n_1} W_{n_1}) \Gamma (W_{n_2}^\dagger \xi_{n_2}) = \big(\bar{\xi}_{n_1}^{(0)} W_{n_1}^{(0)}\big) \big(Y_{n_1}^\dagger Y_{n_2}\big) \Gamma \big(W_{n_2}^{(0)\dagger} \xi_{n_2}^{(0)}\big) \,. \tag{6.13}$$

This result involves the product of three factored sectors (n_1-collinear)(ultrasoft)(n_2-collinear). Here the lines are both outgoing, $Y_{n_1}^\dagger = Y_{n_1+}^\dagger$ and $Y_{n_2} = Y_{n_2-}$.

Remark: It is possible to formulate a gauge symmetry for the decoupled collinear fields via $U_n^{(0)} = Y_n^\dagger(x)U_n(x)Y_n(x)$, that then acts on the collinear (0) fields without ultrasoft components. However, there is not new content to this gauge symmetry beyond the ones we considered earlier.

6.2 Wilson Coefficients and Hard Factorization

As is standard in effective field theories, the high energy behavior of the theory is encoded in Wilson coefficients C. In SCET the Wilson coefficients can depend on the large momenta of collinear fields that are $\mathcal{O}(\lambda^0)$. Because of gauge symmetry the momenta appearing in C must be momenta for collinear gauge invariant products of fields. We can write $C(\overline{\mathcal{P}},\mu)$ where the large momenta is picked out by the label operator $\overline{\mathcal{P}}$ which acts on these products of fields. For example, including this operator with our heavy-to-light current yields

$$(\bar{\xi}_n W_n)\Gamma^\mu h_v C(\overline{\mathcal{P}}^\dagger) = C(-\mathcal{P},\mu)(\bar{\xi}_n W_n)\Gamma^\mu h_v \tag{6.14}$$

(noting that $\overline{\mathcal{P}}^\dagger > 0$ so we have picked a convenient sign). We have included parentheses around $\bar{\xi}_n W_n$ because $C(-\overline{P},\mu)$ must act on this product, since only the momentum of this combination is collinear gauge invariant. It is convenient to write this result as a convolution between a real number valued Wilson coefficient and an operator depending on a new label ω

$$\begin{aligned}(\bar{\xi}_n W)\Gamma^\mu h_v C(\overline{\mathcal{P}}^\dagger) &= \int d\omega\, C(\omega,\mu)\big[(\bar{\xi}_n W_n)\delta(\omega-\overline{\mathcal{P}}^\dagger)\Gamma^\mu h_v\big] \\ &= \int d\omega\, C(\omega,\mu)\mathcal{O}(\omega,\mu)\end{aligned} \tag{6.15}$$

where $C(\omega,\mu)$ encodes the hard dynamics and $\mathcal{O}(\omega,\mu)$ encodes the collinear and ultrasoft dynamics. Thus the hard dynamics is factorized from that of collinear fields, and this in general leads to convolutions since they both have $\bar{n}\cdot p$ momenta that are $\mathcal{O}(\lambda^0)$.

We can show see that this hard-collinear factorization is a general result that can be applied to any SCET operator. Recall the following relations for W

$$i\bar{n}\cdot D_n W_n = 0\,, \qquad W_n^\dagger W_n = 1\,, \qquad i\bar{n}\cdot D_n = W_n\overline{\mathcal{P}}W_n^\dagger\,, \qquad 1/(i\bar{n}\cdot D_n) = W_n(1/\overline{\mathcal{P}})W_n^\dagger\,. \tag{6.16}$$

These conditions imply the operator equations (for any integer k)

$$(i\bar{n}\cdot D_n)^k = W_n(\overline{\mathcal{P}})^k W_n^\dagger\,. \tag{6.17}$$

and we have for a general function $f(\overline{\mathcal{P}})$ or $f(i\bar{n}\cdot D_n)$

$$\begin{aligned} f(\overline{\mathcal{P}}) &= \int d\omega\ f(\omega)\ [\delta(\omega-\overline{\mathcal{P}})]\,, \\ f(i\bar{n}\cdot D_n) = W_n f(\overline{\mathcal{P}})W^\dagger &= \int d\omega\ f(\omega)\ [W\delta(\omega-\overline{\mathcal{P}})W_n^\dagger]\,. \end{aligned} \tag{6.18}$$

If in general the hard dynamics leads to a function f of a large momentum $\overline{\mathcal{P}}$, then we have $f(\overline{\mathcal{P}})$ if it acts on a n-collinear gauge invariant product of fields, and this relation shows that we can always represent this by a convolution of a Wilson coefficient $f(\omega)$ which includes a $\delta(\omega-\overline{\mathcal{P}})$ as part of the collinear operator. (If we act on fields that transform under a collinear gauge transformation then the same is true but with

$f(i\bar{n}\cdot D_n)$ and the extra Wilson lines are included in the operator.) For example, with our current for $e^+e^- \to$ dijets we have

$$\int d\omega_1\, d\omega_2\, C(\omega_1,\omega_2)\, (\bar{\xi}_{n_1} W_{n_1})\delta(\omega_1 - \bar{n}_1\cdot\mathcal{P}^\dagger)\Gamma\delta(\omega_2 - \bar{n}_2\cdot\mathcal{P})(W^\dagger_{n_2}\xi_{n_2})\,. \tag{6.19}$$

Note that since the Y_n Wilson lines commute with $\mathcal{P}^\mu$ we can perform the ultrasoft-collinear factorization by field redefinition after having determined the most general possible Wilson coefficient, and the results will be the same as we obtained prior to discussing Wilson coefficients. In general the function $C(\omega_1,\omega_2)$ will be constrained by momentum conservation for the process under consideration, and any nontrivial dependence must be determined by matching calculations.

6.3 Operator Building Blocks

Our discussion of hard-collinear factorization in SCET in the previous section motivates setting up a more convenient notation for building operators out of products that are collinear gauge invariant. For the collinear quark field we define a "quark jet field" (SCET_I) or "quark parton field" (SCET_II)

$$\chi_n \equiv W_n^\dagger \xi_n\,, \tag{6.20}$$
$$\chi_{n,\omega} \equiv \delta(\omega - \bar{n}\cdot\mathcal{P})(W_n^\dagger\xi_n)\,,$$

where the last expression has a definite $\mathcal{O}(\lambda^0)$ momentum. With this notation our $e^+e^-\to$ dijets operator becomes

$$\int d\omega_1\, d\omega_2\, C(\omega_1,\omega_2)\; \bar{\chi}_{n,\omega_1}\Gamma\chi_{n,\omega_2}. \tag{6.21}$$

For the gluon field we define a "gluon jet field" (SCET_I) or "gluon parton field" (SCET_II) as

$$\mathcal{B}^\mu_{n\perp} \equiv \frac{1}{g}\left[\frac{1}{\bar{n}\cdot\mathcal{P}} W_n^\dagger [i\bar{n}\cdot D_n, iD^\mu_{n\,\perp}]W_n\right], \tag{6.22}$$
$$\mathcal{B}^\mu_{n\perp,\,\omega} = [\mathcal{B}^\mu_{n\perp}\delta(\omega - \bar{\mathcal{P}}^\dagger)]\,,$$

where the label operators and derivatives act only on the fields inside the outer square brackets. We can show that a complete basis of objects for building collinear operators at any order in λ is given by the three objects [14]

$$\chi_n\,, \qquad \mathcal{B}^\mu_{n\perp}\,, \qquad \mathcal{P}^\mu_{n\perp}\,. \tag{6.23}$$

Any other operators can be expressed in terms of these three objects. This basis is nice because the two gluon degrees of freedom in $\mathcal{B}^\mu_{n\perp}$ can be taken as the physical polarizations. Indeed the expansion of $\mathcal{B}^\mu_{n\perp}$ in terms of gluon fields yields

$$\mathcal{B}^\mu_{n\perp} = A^\mu_{n\perp} - \frac{q^\mu_\perp}{\bar{n}\cdot q}\,\bar{n}\cdot A_{n,q} + \ldots, \tag{6.24}$$

where the ellipses denote terms with ≥ 2 collinear gluon fields. In addition to the building blocks in (6.23), operators will also of course involve functions of $\overline{\mathcal{P}} = \bar{n}\cdot\mathcal{P}$ that appear as Wilson coefficients.

To see that Eq. (6.23) gives a complete basis we start by noting that the $\perp$ covariant derivative is redundant. If we consider it sandwiched by Wilson lines, then

$$i\mathcal{D}_n^{\perp\mu} \equiv W_n^\dagger iD^\mu_{n\perp}W_n = \mathcal{P}^\mu_{n\perp} + g\mathcal{B}^\mu_{n\perp}\,. \tag{6.25}$$

To show this we manipulate the operator as follows

$$
\begin{aligned}
W_n^\dagger iD^\mu_{n\perp} W_n &= \mathcal{P}^\mu_{n\perp} + \big[W_n^\dagger iD^\mu_{n\perp} W_n\big] \\
&= \mathcal{P}^\mu_{n\perp} + \Big[\frac{1}{\bar n\cdot\mathcal{P}}\,\bar n\cdot\mathcal{P} W_n^\dagger iD^\mu_{n\perp} W_n\Big] = \mathcal{P}^\mu_{n\perp} + \Big[\frac{1}{\bar n\cdot\mathcal{P}}\,W_n^\dagger i\bar n\cdot D_n iD^\mu_{n\perp} W_n\Big] \\
&= \mathcal{P}^\mu_{n\perp} + \Big[\frac{1}{\bar n\cdot\mathcal{P}}\,W_n^\dagger [i\bar n\cdot D_n, iD^\mu_{n\perp}] W_n\Big] = \mathcal{P}^\mu_{n\perp} + g\mathcal{B}^\mu_{n\perp}\,. \qquad (6.26)
\end{aligned}
$$

The outer square brackets indicate that deriviatives act only on objects inside. In the second line we used $\bar n\cdot\mathcal{P} = W_n^\dagger i\bar n\cdot D_n W_n$, and in the last line we used that fact that within the square brackets $[i\bar n\cdot D_n W_n] = 0$ so that we could write the result as a commutator.

We can also remove $in\cdot\partial$ derivatives by using the equations of motion for quarks and gluons. For instance the collinear quark equations of motion can be written as

$$
in\cdot\partial\chi_n = -(gn\cdot\mathcal{B}_n)\chi_n - (i\not{\mathcal{D}}_n^\perp)\frac{1}{\bar n\cdot\mathcal{P}}(i\not{\mathcal{D}}_n^\perp)\chi_n\,, \qquad (6.27)
$$

where $\mathcal{D}^\mu_{n\perp}$ is given in terms of basis objects by (6.25), and where

$$
n\cdot\mathcal{B}_n \equiv \frac{1}{g}\left[\frac{1}{\mathcal{P}} W_n^\dagger [i\bar n\cdot D_n, in\cdot D_n] W_n\right]. \qquad (6.28)
$$

The gluon equations motion allow us to elliminate $n\cdot\mathcal{B}_n$ in terms of basis objects as

$$
n\cdot\mathcal{B}_n = -\frac{2\mathcal{P}^\nu_{n\perp}}{\bar n\cdot\mathcal{P}}\mathcal{B}^{n\perp}_\nu + \frac{2}{\bar n\cdot\mathcal{P}} g^2 T^A \sum_f \big[\bar\chi_n^f T^A \not{\bar n}\chi_n^f\big] + \ldots, \qquad (6.29)
$$

where the ellipses denote a term that involves two $\mathcal{B}_{n\perp}$s. The gluon equation of motion also allow us to eliminate $in\cdot\partial\mathcal{B}^\mu_{n\perp}$ in terms of the basic building blocks, much like for the quark term. Finally, objects like $g\mathcal{B}^{\mu\nu}_{\perp\perp} \equiv [1/(\bar n\cdot\mathcal{P})\, W^\dagger [iD^\mu_{n\perp}, iD^\nu_{n\perp}] W]$ and $g\mathcal{B}^\mu_{\perp 2} \equiv [1/(\bar n\cdot\mathcal{P})\, W^\dagger [iD^\mu_{n\perp}, in\cdot D_n] W]$ can again be eliminated in terms of the building blocks with manipulations similar to those in (6.26), and with the use of (6.29).

We do still need all of the original ultrasoft fields and operators, including ultrasoft covariant derivatives and field strengths. The ultrasoft equations of motion (equivalent to the QCD equations of motion) can be used to reduce the basis for these operators. It is worth remarking about the connections between our building blocks in Eq. (6.23) and the ultrasoft operators that come from RPI and gauge covariance. Multiplying the identities in (5.32) with Wilson lines on both sides we find

$$
\begin{aligned}
iW_n^\dagger iD^\mu_\perp W_n &= i\mathcal{D}^\mu_{n\perp} + iD^\mu_{\mathrm{us}\perp} = \mathcal{P}^\mu_{n\perp} + g\mathcal{B}^\mu_{n\perp} + iD^\mu_{\mathrm{us}\perp}\,, \\
iW_n^\dagger i\bar n\cdot D\, W_n &= \bar n\cdot\mathcal{P} + i\bar n\cdot D^\mu_{\mathrm{us}}\,. \qquad (6.30)
\end{aligned}
$$

Thus factors of $\mathcal{P}^\mu_{n\perp}$ and $\bar n\cdot\mathcal{P}$ that appear in operators will be connected to higher order operators with these ultrasoft covariant derivatives.

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