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8.821 String Theory
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8.821 F2008 Lecture 17:
More on 3-point functions, the chiral anomaly, and Wilson loops

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1 Introduction

1. Finish $\langle \mathcal{O}\mathcal{O}^* J \rangle$

2. Anomalies

3. Wilson loops

2 Calculation of a 3-point function

Consider a CFT with a conserved $U(1)$ current J. The operator $\mathcal{O}_\Delta$ is an operator in the CFT with scaling dimension Δ that couples to a charged scalar field ϕ in the bulk, while the complex conjugate of this bulk scalar field couples to $\mathcal{O}_\Delta^*$. Suppose that there is a massless vector field A in the bulk that couples to J^μ. The minimal coupling of this charged scalar field to the vector field leads to a vertex $(-i\eta g^{AB}(w))$, which gives the leading contribution to the three-point function

$$
\begin{aligned}
G_{123}^\mu &\equiv \langle \mathcal{O}_\Delta(x_1) \mathcal{O}_\Delta^*(x_2) J^\mu(x_3) \rangle \\
&= \int \frac{d^D w \, dw_0}{w_0^{D+1}} (-i\eta g^{AB}(w)) \left(K^\Delta(w|\vec{x}_1) \frac{\overleftrightarrow{\partial}}{\partial w^B} K^\Delta(w|\vec{x}_2) \right) K_A^\mu(w|\vec{x}_3).
\end{aligned}
\tag{1}
$$

The double-arrow means take the derivative acting to the right minus the derivative acting to the left. K^Δ is the bulk-to-boundary propagator for the charged scalar field and $K_A^\mu(w|\vec{x}_3)$ is the bulk-to-boundary propagator for the gauge boson. The bulk-to-boundary propagator for the vector field solves the bulk Maxwell equations and goes to a delta function at the boundary (it goes to $\delta(\vec{w} - \vec{x})$ as $w_0 \to 0$). The Witten trick that we used to deduce the bulk-to-boundary propagator for the scalar field can also be used to obtain the bulk-to-boundary propagator of the vector field. We claim that it implies:

$$
K_A^B(w|\vec{x}) = C_D \frac{w_0^{D-1}}{(w - \vec{x})^{2(D-1)}} J_A^B(w - \vec{x}),
\tag{2}
$$

were C_D is some normalization constant and J_A^B is a Jacobian for inversion – it is given by

$$J_A^B(x) \equiv \delta_A^B - 2\frac{x_A x^B}{x^2},$$

(3)

which can be determined from $J_A^B = x'^2 \frac{\partial x_A}{\partial x'_B}$, with $x^A = \frac{x'^A}{(x')^2}$. This Jacobian arises because for the Witten trick, one first finds the bulk-to-boundary propagator with the singularity at infinity and then performs an inversion to put the singularity at some finite value.

Using the inversion trick and doing the w integral yields

$$G_{123}^\mu = S^\mu(x_1, x_2, x_3)\eta\frac{(\Delta - D/2)\Gamma(D/2)\Gamma(\Delta)}{\pi^{D/2}\Gamma(\Delta - D/2)}.$$

(4)

S^μ is the unique combination of x_1, x_2, and x_3 allowed by conformal symmetry and is given by

$$S^\mu(x_1, x_2, x_3) = \frac{1}{x_{12}^{2\Delta - D + 2}}\left(\frac{x_{13}^\mu}{x_{13}^2} - \frac{x_{23}^\mu}{x_{23}^2}\right)\frac{1}{x_{13}^{D-2}x_{23}^{D-2}}.$$

(5)

Using the Ward identity,

$$\frac{\partial}{\partial x_3^\mu}G_{123}^\mu = i(\delta(x_{13}) - \delta(x_{23}))\langle OO^*\rangle,$$

(6)

we can determine the normalization of the two-point function:

$$\langle OO^*\rangle = \eta\frac{2\Delta - D}{\pi^{D/2}}\frac{\Gamma(\Delta)}{\Gamma(\Delta - D/2)}\frac{1}{x_{12}^{2\Delta}}.$$

(7)

This is the k-space calcualtion of this 3-point function; it is $\frac{2\Delta - D}{\Delta} \times$ the position space answer.

Note the subtlety of the extra factor of $\frac{\Delta}{2\Delta - D} = \frac{\Delta_+}{\Delta_+ - \Delta_-}$, which occurs only for two-point functions because the $\int_\epsilon dz$ is dicey. For higher point functions, this issue does not arise. This factor comes from the following. Recall our expression for the two-point function:

$$G_z = \frac{\epsilon^\alpha z\partial_z(z^{D/2}K_\nu(kz))}{\epsilon^{D/2}K_\nu(k\epsilon)}\Big|_{z=\epsilon} = \frac{\epsilon^\alpha(\Delta_-\epsilon_-^\Delta + \Delta_+\epsilon_+^\Delta\frac{b_0}{a_0})}{\epsilon_-^\Delta + \epsilon_+^\Delta(\frac{b_0}{a_0})}$$

$$= \epsilon^\alpha(\text{diverging term} + \epsilon^{\Delta_+ - \Delta_-}\frac{b_0}{a_0}(\Delta_+ - \Delta_-)).$$

(8)

K_ν is a Bessel function and we have used $K_\nu(k\epsilon) = \epsilon^\nu + \frac{b_0}{a_0}\epsilon$ and $\Delta_\pm = \frac{D}{2} \pm \nu$. The dependence on k is absorbed into b_0/a_0. The divergence corresponds to some kind of contact interaction that we don't care about. Because of this divergence, it is possible for the subleading term to actually "matter". This is where the coefficient $\Delta_+ - \Delta_-$ comes from. Witten's position-space calculation takes the limit $z \to 0$ before taking this ratio, so this factor does not appear in his position-space calculation.

3   Gravity dual of chiral anomaly

Scale invariance and the $SU(4)_R$ symmetry of the $\mathcal{N} = 4$ Super Yang-Mills theory are exact symmetries if the theory is defined on flat space with no sources. But, if we couple the $SU(4)_R$

currents to some external gauge field A (which we can think of as the boundary value of some bulk gauge field in AdS) or if we have an external background metric (which we may think of as the boundary value of the metric in AdS), then there can be anomalies. This is because the gauginos (the λ's) are in the $4 \neq \bar{4}$ of $SU(4)_R$. So we may have anomalies in the trace of the stress tensor, T^μ_μ, or in the R-current. The parity-odd part of the three-point function of three R-currents in the $\mathcal{N} = 4$ $SU(N)$ theory is:

$$\langle J_a^\mu(x_1) J_b^\nu(x_2) J_c^\rho(x_3) \rangle_- = -\frac{N^2 - 1}{32\pi^6} i d_{abc} \frac{\text{Tr} \left(\gamma_5 \gamma^\mu \not{x}_{12} \gamma^\nu \not{x}_{23} \gamma^\rho \not{x}_{31} \right)}{x_{12}^4 x_{23}^4 x_{31}^4} \tag{9}$$

The subscript $-$ refers to the parity-odd bit of the correlator and $d_{abc} \equiv 2Tr(T_a\{T_b, T_c\})$. This three-point correlation function implies that if we couple the R-current to some background gauge field, then the divergence of the R-current will be:

$$(D_\mu J^\mu)_a = \frac{N^2 - 1}{384\pi^2} i d_{abc} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^b F_{\rho\sigma}^c. \tag{10}$$

This is one-loop exact, by the Adler-Bardeen theorem.

Now we would like to see how to compute this anomaly at strong coupling using the gravity dual. In AdS_5, J_a^μ couples to some $SO(6)$ Kaluza-Klein gauge field A_μ^a. The statement of the anomaly is that the boundary symmetry, generated by the R current J, is broken by the non-zero field strength of the background gauge field. This global R-symmetry is generated by gauge transformations in the bulk, $A^a \to A^a + (D\Lambda)^a$, where the gauge parameter Λ does not vanish at the boundary.

The effective action for Type IIB supergravity includes

$$S_{KK} \supset S_{YM}[A] + S_{CS}[A], \tag{11}$$

where S_{CS} is a Chern-Simons term:

$$S_{CS}[A] = \frac{iN^2}{96\pi^2} \int_{AdS_5} TrA \wedge F \wedge F + \cdots \tag{12}$$

Note that $TrA \wedge F \wedge F = d^5x d_{abc} \epsilon^{B_1 B_2 B_3 B_4 B_5} A_{B_1} F_{B_2 B_3} F_{B_4 B_5}$. The Chern-Simons term arises because Type IIB supergravity breaks parity. The basic reason for this parity-breaking is that the five-form field strength is self-dual, not anti-self-dual.

This Chern-Simons term is gauge invariant in the bulk; i.e. it is gauge invariant up to a total derivative.

$$\frac{\delta S_{CS}^{2p+1}}{\delta A} \sim TrF \wedge \cdots \wedge F, \tag{13}$$

so under $A \to A + D\Lambda$, the change in the action is

$$\delta S_{CS} = \frac{-iN^2}{384\pi^2} \int_{\partial AdS} d^4x \epsilon^{\mu\nu\rho\sigma} d_{abc} \Lambda^a F_{\mu\nu}^b F_{\rho\sigma}^c. \tag{14}$$

Here we have assumed $dF = 0$, or that there is no magnetic charge. Alternatively, since A couples to the R-current J,

$$\delta S_{eff} = \int d^4x D_\mu \Lambda^a J_a^\mu = -\int d^4x \Lambda^a D_\mu J_a^\mu, \tag{15}$$

so can read off the anomaly $(D_u J^\mu)_a$. Notice that the coefficent matches at large N (ignore 1 in $N^2 - 1$). The $1/N^2$ correction has also been matched.

4 Wilson Loops

Now we turn to correlation functions of non-local operators, namely the Wilson loop operator, which is an important and interesting gauge theory observable. Let C be a closed curve in spacetime and R a representation of the gauge group. The Wilson loop is:

$$W(C) \equiv Tr_R P e^{i \oint_C A}. \tag{16}$$

P denotes path ordering of the exponential:

$$W(C) = \sum_{n=0}^{\infty} (i)^n \int_{t_i \leq t_j} \prod_{i=1}^{n} dt_i A_{\mu_1}(x(t_1)) \dot{x}^{\mu_1}(t_1) \cdots A_{\mu_n}(x(t_n)) \dot{x}^{\mu_n}(t_n) \tag{17}$$

Physically, this describes the Yang-Mills contribution to the propagation amplitude of an external, heavy, charged particle in the reprsentation R. It describes a heavy particle because we have fixed the trajectory of the particle, C, and we are not considering fluctuations of this path. If C is a rectangular path that stretches a length T in the time direction and a length L in the spatial direction, we have

$$\langle W(C) \rangle \sim e^{-TV(L)}, \tag{18}$$

where V is the static quark-antiquark potential. This can be used as an order parameter for confinement – if $V(L) \sim L$, we say that the theory is confining.

Recall that in our discussion of 't Hooftology, at large N a quark has a string ending on it. Are these the strings in AdS?

To analyze this question, let us take a short detour back into the world of string theory. Consider a stack of $N + 1$ $D3$ branes sitting in some asymptotically flat space. There is therefore some $U(N+1)$ gauge theory living on the $D3$ branes. Suppose N is not so big, so that the gauge theory is at small 't Hooft coupling. Suppose we take one of the branes and separate it from the other N. This corresponds to giving an expectation value to the adjoint scalars and breaking the $U(N+1)$ down to $U(N) \times U(1)$ (Figure 1). Suppose x' is the position of the single $D3$ brane and x is the position of the N other branes. The mass of the string which extends from the single brane to the N branes is proportional to the separation: $m_w \sim |x - x'| \frac{\alpha'}{2\pi}$.

We claim that we can describe such a state at large 't Hooft coupling λ by placing a "probe" $D3$ brane in the "throat" at position $z_w = (2\pi m_w)^{-1}$ in AdS (with N units of flux). A string stretches from this probe $D3$ brane down to the stack of N branes sitting at $z \to \infty$ (Figure 2). We can see that the string stretches from z_w by matching the mass of this stretched string that falls into the horizon with the mass m_w, which we can do because it is a BPS state and therefore we can follow its mass from weak to strong coupling.

We can make our probe arbitrarily heavy, $m_w \to \infty$, by taking $z_w \to 0$ (that is, to the boundary of AdS). We do this to avoid fluctuations of the trajectory.

This string is in the N of the $U(N)$ and it couples to the adjoint scalars: the stretched string pulls on the branes so it affects the $X^a(x)$ (Figure 3). It also couples to the fermions, but we ignore this
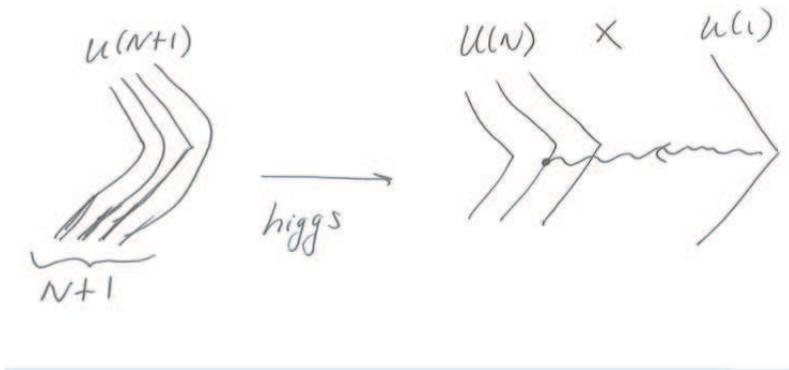
Figure 1: One of the $D3$ branes is separated from the other N, thus breaking the $U(N+1)$ down to $U(N) \times U(1)$.
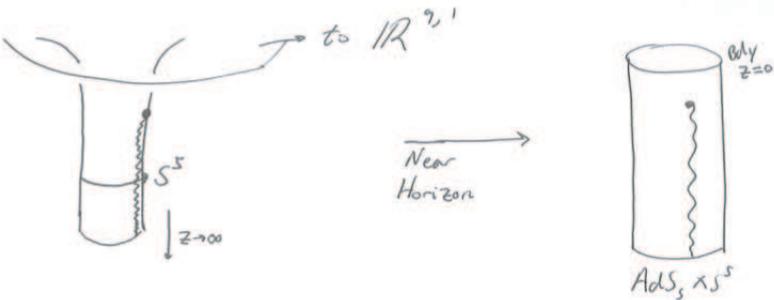
Figure 2: The string stretches into the throat from the probe $D3$ brane at z_w.
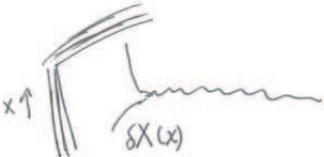
Figure 3: The string pulls on the stack of $D3$ branes, so it couples to the adjoint scalars.

to leading order in $1/\lambda$. This leads to the "Wilson-Maldacena" looop:

$$W[C] = \text{Tr } Pexp \oint \left(iA_\mu(x)\dot{x}^\mu + \Omega^I X^I(x)\sqrt{\dot{x}^2} \right) d\tau. \tag{19}$$

$x^\mu(\tau)$ is a parametrization of C. $\sum_{i=1}^{6} (\Omega^I)^2 = 1$.

Now this generalizes the GKPW prescription:

$$\langle W[C] \rangle_{CFT} = Z_{\text{string theory}}[\text{F-strings end on C}] \tag{20}$$

At small string coupling g_s, contribution will be dominated by disk ampliitude. At large λ, the world sheet path integral is dominated by saddle point. Also, at large λ, we can ignore the S^5 coordinates and coupling to the 5-form flux. So, in this approximation, the string theory partition function is approximated by:

$$Z_{\text{string theory}} \sim \int_{\partial(\text{world sheet})=C} [DX] e^{-S_{NG}[X]+\cdots} \sim e^{\sqrt{\lambda}\min(\text{Area})} \tag{21}$$

and

$$\text{Area}[x] = \int d\sigma d\tau \sqrt{\det \partial_\alpha x^A \partial_\beta x^B g_{AB}(x)} = \int d\sigma d\tau \sqrt{\frac{L^4}{z^4} \det \partial x \partial x \delta_{AB}} = L^2 \text{Area}_{L=1}[x]. \tag{22}$$

where recall $g_{AB} = \frac{L^2}{z^2} \delta_{AB}$.