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**HONG LIU:**

OK, let us start. So let me first remind you what we did at the end of last lecture. So we start talking about how to calculate the Euclidean correlation functions or how to relate the Euclidean correlation functions in the boundary field series to the bulk quantity.

So the question we are always interested in-- so the question we are always interested in in the field theory is the generating functional. And so this is defined, say, in the Euclidean series to be-- we insert this. You insert this information into your Lagrangian. And then you just calculate the correlation function of this exponential, OK?

So even if I write down a single field  $\phi$ , single source  $\phi$ , and a single operator-- so you should really understand this as the collapsing of all possible operators-- all possible operators and all possible corresponding sources you can put in. And those are a lot of important things-- is that you should imagine this exponential as a power series, OK? So this expression should-- for the purpose of the generating functional.

And this exponential should be understood as a power series in  $\phi$ . So we expand it in  $\phi$ . And then the term proportion to  $\phi$ , then gives you the one-point function proportion to  $\phi$  squared, give you the two-point function of  $O$ . And the cubic term would be the three-point function of  $O$ , et cetera.

And so this should-- yeah, this exponential does not need to make sense non-perturbatively. You only need to make sense of it as a power series of  $\phi$ , OK? You only need to make sense of it perturbatively and then just given by the correlation functions of  $O$ .

OK, so this is a power series of  $\phi$ , which is the coefficient given by the correlation functions of  $O$ , OK? So this clear? And then we argued last time that given the relation  $\phi(x)$  is related to some operator to some field  $\phi$ , to the boundary value of  $\phi$ .

Yeah, given that, say, the operator  $O$  is related to some bulk field  $\phi$  and the nominal [INAUDIBLE] modes, the boundary value of  $\phi$  should be related to the source corresponding to this  $o$ , then you can in some sense immediately write down what this generating function

should correspond to on the gravity side. This can be conceivably just a partition function with this deformation-- with this deformation. And then this should be the same as the partition function on the gravity side.

Then we should expect that CFT  $\phi$  should be the same as the gravity partition function reaches the boundary condition, OK? So we expect this to be true, OK? And so this is just a heuristic way of writing the more precise relation. For example, for scalar, the way to understand this is that  $\phi(x, z)$  at  $z \rightarrow 0$  should go to  $\phi(x, z) e^{-\Delta z}$  OK? So this is a precise mathematical meaning of this relation.

So the boundary value of capital  $\phi$  should be given by the small  $\phi$ . And of course, depending on different fields, then this power may be different. So last time, we talked about the, say, for conserved current, we just do it to vector fields. And then this power will be 0.

And then also for the [INAUDIBLE] perturbation will be a different power, et cetera, OK? So any questions regarding this? Good.

So now, the thing is that in general, we cannot really verify this relation because the right-hand side is very hard to compute. Actually, we don't even know how to define the right-hand side in the full quantum gravitational regime. And it's just a formal expression.

Say for quantum gravity exist, then maybe there should exist a partition function, OK? But there is a regime-- we do know how to calculate this side. So this is so-called classical gravity regime or also called semi classical regime.

In the semi classical regime, which is the limit which  $\alpha'$  goes to 0 and the  $G$  newton goes to 0 or similarly  $g_s$  go to-- string coupling goes to zero. And in this limit, one can formally write this  $z$  bulk just as a Euclidean path integral of all the bulk fields. OK, so now I use the convention for the Euclidean path integral without the minus sign.

So we just integrate, say-- so this is your action. Then you integrate over all possible gravity fields. Again, this should be integrated as a collection of all the fields on the gravity side, including whatever scalar field, gravity, vector field-- you catch everything, OK?

And there'll be some appropriate boundary condition-- OK, so appropriate boundary condition. And so this SE-- so this SE, just the Euclidean action-- standard Euclidean action. Say the Einstein term and the plus-- say the matter terms, OK. And then the leading order in this limit--

and the meeting order in that limit, you can just perform this integral by saddle point of approximation, which is say,  $\alpha$  prime goes to 0 limit,  $\kappa$  goes to 0 limit.

OK,  $\kappa$  is essentially the G newton. So you can calculate this path integral by saddle point approximation. Just solve the equation motion following from this action, OK? I assume you all know what's saddle point approximation.

So the saddle point approximation you just solve. So  $z$  bulk-- that's given by SE. Now you evaluate it at the solution of the classical equation motion.

So  $\phi_c$  is just a classical solution. It's a classic solution which satisfies the right boundary condition. Correct boundary condition, OK? So in this limit, we can really calculate this quantity.

And this is very good news. This is very good news because this limit-- because this limit is the-- what did we say before?  $N$  equal to infinity.

And the  $\lambda$  goes to infinity limit of the boundary theory of the super Yang-Mills theory. OK, so we can actually-- so given this, then essentially we know the generating functional. Then we can actually calculate the correlation functions in the strongly coupled field series, OK?

And it's just by solving the classical gravity equation, just by solving the classical gravity equations. So we conclude-- this is an important equation. So let me write it here.

Yes, so let me just make a small remark-- make a small remark. So to [ $\epsilon$  read in  $\epsilon$ ] order, this correlation function essentially is controlled by this  $1$  over  $2$   $\kappa$  square. So this is  $1$  over  $2$  newton.

So remember that previously, we-- when we talk about relation between the G newton and the field theory relation, this is proportional to  $N$  square,  $N$  square of the [INAUDIBLE] theory. And this is indeed consistent. So in the saddle point approximation, the SE, this is just controlled by  $1$  over  $2$   $\kappa$  square when we just [ $\epsilon$  put it  $\epsilon$ ] to  $N$  square. And this is indeed consistent with our expectation from [INAUDIBLE] theory that the leading order, the generating functional, should be proportional to  $N$  square, OK? We support for normalization.

OK, so now we have obtained something powerful. So now we have obtained something very powerful. It said in the  $N$  equal to infinity and the  $\lambda$  goes to infinity limit, the logarithm of CFT, the generating functional, is related to-- we said the boundary condition, the boundary

value. Yeah, this is the boundary condition.

Yeah, maybe. Just, we have this, which is the boundary value which  $\phi$  goes to  $z$  to some power of  $\phi$ . And thus,  $z$  goes to 0, OK?

OK? And this is the part of the [? non-normalizable ?] mode for this bulk field  $\phi$ , OK? And this  $a$ -- so this precise number  $a$  depends on-- so this exponent  $a$  depends on, say, dimension of the operator. Dimension, spin-- say, possible other quantum numbers of operator.

So for scalar, we just have this guy, which we just wrote, OK? And so let me remind you for the vector, for the conserved current, which is due to a vector field, to a gauge field, to a massless vector field-- and then this gives you  $a$  equal to 0. Yeah, maybe this just gives you  $a$  equal to 0 in this case.

So the leading, non-normalizable term-- so just the boundary value of the vector field is just given by the source to the current. And for the metric, we discussed last time, is that for the metric perturbation, we just do it to the stress tensor. So this is corresponding to  $a$  equal to 2, OK?

So where this comes from is that, remember-- and last time, we say that  $A_\mu$ -- say if you have a gauge field, say a [? mass ?] term, when you go to the boundary, then you get  $a_\mu$  then plus some  $b_\mu Z^d$  minus 2. So you see, there's no power here,  $Z$  equal to 0 here, OK? Similarly for the metric, component of the metric, go to 0, goes to  $1/Z^2$ , say, the boundary value.

And the boundaries-- yeah,  $\delta g_\mu$  is the source to the-- for [?  $T_\mu$ . ?] So here, the  $a$  is actually equal to minus 2. Or  $a$  equal to 2, yeah--  $a$  equal to minus 2.

That's right. OK? Any questions on this?

Now let me make some remarks. So the first remark is what I said a little bit earlier. It's that the  $\phi$  in this generating functional should be considered as infinitesimal. OK?

So the reason we want to consider same term is that what we said before. We said we should consider the generating functional simply just as a power series. Because we don't want to take on perturbation to deform your regional series, to actually change your regional series. We just use it as a generating function.

So similarly, the same thing should be considered the gravity side, OK? So similarly on the gravity side-- so this  $\phi$  is due to the non-normalizable mode. So this non-normalizable mode should also be infinitesimal.

OK, so don't disturb your asymptotic ADS geometry. OK, any questions? So this is first remark.

The second remark-- now let me call this equation. Let me give the equation name. It said that both sides of star are actually divergent, OK? Are actually divergent.

So the left-hand side, for a very simple reason, we are considering a quantum field theory. We are considering a generating function of composite operators in a quantum field theory. They're always short distance divergences when the two operators come together, OK? So this is just the usual uv divergences of a QFT.

On the right-hand side, we will see soon in the example this is also divergent. And the reason it's divergent is also we have seen it before. So remember the ADS metric is like this. OK, so I write all boundary directions together in terms of  $dx^2$ . So because when  $Z$ -- when you go to the boundary,  $Z$  goes to 0. And then the whole metric blows up.

So essentially, you have infinite volume. And yeah, essentially-- so roughly you can say this is a volume divergence near boundary of ADS, OK? So the relation between them, of course, is precisely what we called IR uv connection.

So the IR-- so the volume divergences associated, say, with the space time have infinite volume is related to the uv divergences in the field series sites, OK? So in order to actually obtain the finite answer, we need to renormalize them. We need to renormalize them.

So the idea to role match them from the field theory point of view, essentially, you need to add some counterterms to cancel the divergences, OK? You need to add the counterterms to cancel the divergences. So the similar logic we do on the gravity side.

So that means that in order to make sense of this expression, we introduce a renormalized action which is defined by your original action-- say, the [INAUDIBLE] action. But this is divergence.

So we first need to regularize it. So we normally do regularization. So instead of evaluating all the way to  $z$  equal to 0, which will be divergent, we cut it off at some  $z$  equal to  $\epsilon$ . So this is your full space time.

So  $z$  equal to 0 is, again, your boundary. So you integrate all the way to  $z$  equal to 0. You will get a divergence.

So we just cut it off at a  $z$  equal to  $\epsilon$ , OK? And then we added the counterterm, which can be expressed in terms of the  $\phi_c$  evaluated at this  $\epsilon$ , OK? And this [? displaced ?] row should cancel the divergence here, OK? Should cancel divergence here.

But this  $S_t$  is much-- so  $S_t$  should satisfy some simple criterion. In principle, this  $S_t$  can be arbitrary, but should satisfy some criterion. First, this must be local functionals of  $\phi_c$ , OK? The reason it must be local functionals of  $\phi_c$  is because this should come from short distance physics.

And short distance physics is not something we can really control even in the renormalizable theories. Even in the renormalizable theories, we just look at the quantity which are insensitive to those short distance physics. So the short distance physics, not something we can control.

But one important thing about short distance physics is that they all must come from a single point, OK? So it means the counterterm you act must be local, OK? And of course must be covariant, et cetera.

So local, covariant-- satisfy those basic conditions and the covariant, et cetera, with all the right symmetry. Essentially, this is the only condition those counterterms should satisfy. And then you can just choose your counterterms to cancel the divergent terms here, OK?

But now the key thing is that this is different from just throwing away the divergent term here. Because this counterterm also contain some finite terms. So that's normally, in the [INAUDIBLE] field series, that may change the finite terms here. And those finite terms are constrained by the locality and the covariant and [? those ?] symmetries.

And actually, there's no ambiguity in them, OK? Even though the divergent term, in some sense, there's ambiguity. But in the finite term, there's no ambiguity. And so we will see an example at the end of this class.

OK, so we need to renormalize them. And this is the criterion to normalize it, OK? So once we have renormalized it, then we can just obtain the  $n$ -point function.

So the general  $n$ -point function-- say,  $\phi(x_1, \dots, x_n)$ -- if you look at this  $n$ -point function, so let's say connected, then we'll be just corresponding to the standard story. You just take  $n$

derivatives on the partition function, not the normalized one. OK, so you take derivative over  $\phi_1$  corresponding to  $\phi_1$  and then  $\phi_2$  corresponding to  $\phi_2$ , et cetera, and the  $\phi_n$  corresponding to  $\phi_n$ .

And then at the end of the calculation, after you have done the derivative, you set all the  $\phi$  to 0, OK? So then you have extracted n-point function, OK? So this is the same that you expand that exponential in power series and then only extract terms power n in  $\phi$ , OK? So same thing.

And now because this is now equal to that-- so we can write this in terms of gravity quantities. So writing as n-point function of SE will normalize SE as a function  $\phi_c$ , now  $\phi_1 \times 1$ ,  $\phi_n \times \phi_n$ . Then at the end, you take  $\phi$  equal to 0, OK? All the  $\phi$  to 0, OK?

So now you in principle, by calculating the gravity action with the appropriate boundary condition, then you would have obtained any n-point functions from the field theory. Any questions about this? Yes.

**AUDIENCE:** Can we check that relations start on the lattice?

**HONG LIU:** Can we check that? Yeah, in principle, you can. But it's just not easy to do.

**AUDIENCE:** Have people tried?

**HONG LIU:** So people have tried, not for super Yang-Mills series. So there's some version of these for some matrix quantum mechanics. People have tried.

In matrix quantum mechanics on the left-hand side, you can actually calculate using the lattice or using Monte Carlo. So maybe not lattice. It's just a matrix integral.

You can calculate it using a Monte Carlo and then can compare with the gravity. Then they turned out to agree with very high accuracy. But for super Yang-Mills series, the left-hand side is not doable yet using lattice calculation.

So-- and the one particularly important thing is the one-point function. So maybe just separate it to mention elliptics. So one-point function is important because one-point function just expectation value, which of course we often are interested in.

So if you have a symmetry breaking, whether you develop expectation value or if you want to

extract the expectation value of a stress tensor, et cetera. So the one-point function is of very special importance in many cases. And also, if you know the one-point function in the presence of the source, then essentially, you actually know the full n-point function.

Because you can just take the derivatives here. Then you can get to the higher point functions because of the-- yeah, if you know the source here, if you take one derivative, now you get two-point function, et cetera. OK?

So one-point function-- so one-point function we can write as the following. So you just take one derivative,  $\delta \phi(x)$ , due to this operator, OK? So now let's do the scalar case.

But similar can be applied to other case, OK? So you can just replace this  $\phi(x)$  by the boundary value of capital  $\phi$ . So this is for the scalar case.

And you could write it as  $\delta S$  of  $\phi_c$ . On the right I said, derivative over  $\phi_c(z, x)$ . And as  $z$  goes to 0. Because  $\phi_c$  has to approach  $\phi$  with this power. And essentially, you just replace it, OK?

But the reason we want to write this form-- because this form have a very nice geometric interpretation. This has a very nice geometric interpretation because  $\delta S$ ,  $\delta \phi_c$  evaluated at  $z, x$  should be equal to  $\phi_c(z, x)$ .

And this  $\pi$ , which is the canonical momentum conjugate to  $\phi$  treating the  $z$  direction as time. This  $z$  direction treating as time. And so this  $\pi_c$  just means it's the classical-- means that this is evaluated to the classical solution-- classical solution  $\phi_c$ . So this expression-- is this expression obvious to you?

**AUDIENCE:** No. Just from the other graphs.

**HONG LIU:** Hm?

**AUDIENCE:** It's just [INAUDIBLE] conjugation, essentially, right?

**HONG LIU:** Yeah, this is the-- yeah, actually, I forgot the name of this equation. Maybe it's-- normally this equation is taught in the so-called Hamilton-Jacobi theory. It's normally called in the Hamilton-Jacobi theory.

But this is a well-known fact in the classical mechanics. Let me just tell you one thing. So consider you have a, say, classical mechanical system. So you move from  $t_0$  to  $t_1$ , OK?



And then suppose you have already found your classical solution. So this is a classical trajectory. And then suppose you evaluate your classical action.

Say so this is  $t_0$  to  $t_1$   $dt$ , all evaluated at  $X_c$ ,  $\dot{X}_c$ , et cetera, OK?  $X_c$  is already classical solution. And so this function-- and let's suppose that  $X_c$  at  $t_0$  is equal to  $X_0$ ,  $X_c$  at  $t_1$  equal to  $X_1$ .

So this is the initial value and the final value of your trajectory, OK? And then-- so clearly, because of this boundary condition, the  $S$  depends on  $X_0$  and  $X_1$ . Then this is an important statement in classical mechanics-- is that  $\delta S$ , you do the valuation over the final value of your location.

This gives you the momentum at that point at  $P_1$ . And if you do the valuation over  $\delta X_0$ , then that gives you the initial momentum, the [INAUDIBLE] or the initial momentum, at this point, OK? So if you're not familiar with this fact, try to derive it yourself when you go home.

Take two minutes to derive it. If you find the right the way to do it, it's actually very simple. Yeah, but it's also in the chapter 1 of Landau, Lifshitz. And it's very-- essentially the first few pages of Landau, Lifshitz. You can also take a look at that.

And now this situation we have here is very similar to that. So we are evaluating the classical action all the way to some value of  $z$ . And  $z$  essentially is the boundary-- is the time here. And the  $\phi_c$  is the value of  $\phi$  at this value of  $z$ , which is the analog of our  $X_1$  here, OK?

Then you take the derivative-- so this is a function, a version with that. Then you take the derivative over  $\phi_c$ , which is the value-- because [? one you ?] take derivative over  $X_1$ . Then here, you get the  $P_1$ .

There, you get equals 1 in momentum conjugate to  $\phi$ , OK? So this is just the functional version of that. Any questions on this?

So now we have a very elegant expression. We said the one-point function. We said now, if you look at this-- now, the one-point function-- so even in the presence of the source, you just  $z$  equal to 0 limit--  $z$  to the power  $d$  minus  $\delta$ ,  $\phi_c z, x$ . And of course, this should also be renormalized because of here, we renormalized  $R$ .

So here, it should also be renormalized. And so here-- yeah, OK? Yeah, so this is a nice

expression.

So it tells you that the one-point function you see essentially just goes one into a limit of the canonical momentum to that field evaluated at the boundary, OK? So now, with a little bit of effort, we will show-- so you remember  $\phi(x, z)$ , when you go to  $z$  equal to 0, as we said last time equal to  $Ax$ , you have two asymptotic behavior.

So this  $A$  is related to the source. So  $A$  is related to the  $\phi$ . So we discussed last time  $A$  is related to the  $\phi$ , OK? So now you can show, using this expression, that the one-point function is actually exactly equal to  $2\nu$  times  $B$ .

And the  $\nu$ -- yeah, let me just introduce my notation, remind you of which we had been used last time. You said the  $\delta$ . So last time, we derived this  $\delta$  equal to  $d$  divided by  $2 + \nu$ . And the  $\nu$  is equal to  $d^2$  over  $4 + m^2 R^2$  for scalar fields, OK?

So you can show-- so we will do that. We will reach this point at the end of this class, OK? So you show that actually, the one-point function precisely is given by  $B$ .

So this is very nice. This is very nice. So you have two asymptotic behavior.

And one of them is related to the source. And the other is related to the expectation value, OK? And so I urge you to do a self-consistency check yourself as we did last time that the [INAUDIBLE] you can determine the dimension of this operator, which is precisely  $\delta$ .

And from this relation, you can also check that this relation is compatible. We said  $O$  have dimension  $\delta$ , OK? And  $\delta$  are given by that  $\delta$ . Again, this you can check just by the scaling the two side, the how two sides scale under the scaling, OK?

**AUDIENCE:** So usually in QFT there's no relation between the conjugate momentum and one-point function? There's no kind of a--

**HONG LIU:** No, no. No, this is conjugate momentum in the gravity side. This is not--

**AUDIENCE:** So within QFT, there isn't any--

**HONG LIU:** No, this is the gravity canonical momentum related to the field theory one-point function. This has nothing to do with canonical momentum on the field theory side. So this is the canonical momentum on the gravity side.

The boundary value of the canonical momentum on the gravity side with the rate of direction treated as the time, which gives you the one-point function in the field theory. Good. Good, OK.

So now let's look at them more explicitly. So let's now give you an example to compute the two-point function for scalar operator. And then we'll also derive this relation.

We'll also derive this relation, OK? So before we do that, do you have any questions? Yeah.

**AUDIENCE:** I guess usually in QFT, when you have Gaussian integrals and one-point function is 0, in what case does this happen here?

**HONG LIU:** Say-- in many cases, in QFT-- so if you have any state with a non-zero [? energy, ?] any state above the ground state where you have a non-zero stress tensor. And then the stress tensor will have a one-point function. And so for example, if you have a symmetry breaking in the scalar fields, we also have expectation value, yeah.

And also, if you have charge density, then the current will have expectation value. Yeah. Any other questions?

So yeah, actually, before we do the example, let me also make a brief remark why we actually first considered the Euclidean correlation functions. So now first, can I ask the reason why we want to do the Euclidean correlation function first? Hm?

**AUDIENCE:** It's easier?

**HONG LIU:** It's true. It's true. And why it's easier? Hm?

**AUDIENCE:** You can--

**HONG LIU:** Sorry?

**AUDIENCE:** Usually Euclidean takes less time.

**HONG LIU:** It's true. Yeah, so there's several reasons maybe to save time. Actually, we don't have much time.

Several reasons-- first Euclidean interval, you can more easily define it using the Euclidean path integral. And so there's no ordering issues, et cetera. And we just have Euclidean correlation functions. But the Lorentzian correlation functions, there are many of them.

So it depends on the orderings, et cetera. And the typical Lorentzian function cannot be defined straightforwardly using path integral. So except the Feynman correlation functions, say time-ordered.

So if you want to look at other ordered correlation functions, they cannot be defined straightforward. They cannot be defined straightforward using path integral. And so it's much more complicated.

Actually, for many applications-- in particular, say, for the linear response for the many body system-- say for condensed matter or for QCD, et cetera, mostly we're interested in the retarded Green function. And that is not easy to define using path integral. And so you need to develop some other tricks. So that's why we are doing the Euclidean first. OK?

So now let's look at example two-point function of a scalar, OK? So this is essentially the simplest example. And for a two-point function, they only need to have the quadratic term in  $\phi$ , OK?

And that means here, you only need to include the quadratic term in capital  $\phi$ . So that means it's enough just to consider the action to the quadratic reliable, OK? So that means that if we are interested in the two-point function to leading order, then you can just look at the quadratic action.

And for scalar, then, for scalar operator  $O$ , then you are just [INAUDIBLE] massive scalar fields. So we can write down Euclidean action. OK?

You don't have to worry about the higher order corrections. And then the metric is the Euclidean version of this. It's just the Euclidean metric. And this is  $dx^2$ -- so  $dx^2$ , Euclidean metric,  $dx^2$  will be just  $\delta_{\mu\nu} dx^\mu dx^\nu$ . It's really just the standard Euclidean metric.

So essentially, we take the-- so here, where previously was  $-dt^2 + dx^2$  plus the vector  $x$  square. Then we take that  $dt^2$  into Euclidean signature. Then it just becomes a pure Euclidean. All right, I hope this is clear.

And then our goal is just find renormalized  $\phi_c$  using this action, which satisfies the right boundary condition, OK? OK, so this is our goal. And if we find this guy, then you take the derivative. Then you get two-point functions.

OK, first, let me-- first, you can write down the canonical momentum with the  $z$  as the time direction. So this is just-- OK, so this is canonical momentum. Under the equation motion-- OK, equation motion.

And again, because of a translation symmetry in those  $X$  in the boundary directions, we can write it in the Fourier transform. You can write  $x, z$  in the form of  $k, z$ -- so [? expression ?]  $ikx$ , just as we did last time when we solved the [? free ?] theory. Then this gives you-- so you plug this in when you put in the explicit metric here.

Then you get  $z d + 1 \partial_z z - 1 - d \partial_z \phi - k^2 z^2 \phi$  and then minus  $m^2 R^2 \phi = 0$ , OK? So this is very similar to what we did before, what we did last time when we talked about normalizable modes and non-normalizable modes. This is the same equation. The only difference is now we are in the Euclidean signature.

So this equation can be solved exactly-- so in terms of Bessel functions. But actually, we will not need the explicit solution here at the end. So for now, we will not try to solve it explicitly, OK? So we only do that at the end.

So now let's first look at the on-shell action, OK? So there's a simple trick. Because this is a quadratic, you can actually do an integration by part in the action.

So let's say plug in the-- let's plug in the classical solution to this equation, to the classic equation, into this action. So let's do an integration by parts in this term.

Then when you do integration by parts, then you get two terms. You get a boundary term come from integration by parts. Then you get the bulk term. So the bulk term can be written in the following way.

So let me first copy the  $m^2 \phi$  term. And then it's the term which you obtained by integration by part of that term. So that term can be written as  $\partial m$  and then plus the boundary term plus the total derivative term. Just did a simple manipulation.

I did integration by parts here. So now you recognize this guy. It's precisely the combination which appeared in that equation motion.

This is always the case when you have a quadratic action, OK? So this is just 0. So we don't

need to worry about this. So this term just said  $[? N ?]$  equal to 0 on shell.

So this is just equal to 0. And so we only have a boundary term. So this is a total derivative. But the boundary-- so we always see the boundary term in, say, the directions along the boundary.

So you only need to worry about the boundary term in the  $z$  direction. So then this just translated into the  $z$  equal to 0 and the  $z$  equal to infinity, OK? So this is translating to minus  $1/2 d dx$ , then square root  $g$ ,  $g_{zz}$ . Only the  $z$  direction matters.

Then partial  $z$   $\phi$  times  $\pi c$ . Then  $z$  equal to 0 and the infinity. So now you recognize this guy is precisely our  $\pi$  with some minus signs-- precisely our  $\pi$ .

So this can also be written as  $1/2 d dx \pi c$  and the  $\pi c$  you write it as a 0, infinity-- is equal to 0, is equal to infinity. So sometimes, we also write it in momentum space, which is convenient because we can solve it in the momentum space. So it can also be written as  $d dk 2 \pi d \phi c$   $kz$  and the  $\pi c$  minus  $k, z$ , then 0, infinity.

So we have now reduced it to evaluate this on-shell action to just two boundary terms at  $z$  equal to infinity and  $z$  equal to 0, OK? So now, we will later see that at infinity, with  $\pi c$  times  $\phi c$  at  $z$  equal to infinity just equal to 0, OK. We will see, OK?

So we will see explicitly that in the right solution, that both  $\pi c$  and the  $\phi c$ , they go to 0. But actually, there's also a simple way to argue without looking at the [INAUDIBLE] solution that this is actually generically also 0 for the following reason. It's that at infinity, of course, we want the full solution to be well-defined.

So we should also require  $\phi c$  or partial  $z$   $\phi c$  to be finite, OK? We don't want them to diverge. Then you have a singular solution, OK? So we want them to be finite.

And now you see the  $\pi$ . The  $\pi$ -- you work out the fact of the  $\pi$  when the  $z$  goes to infinity is proportional to  $1$  minus  $d$  partial  $z$   $\phi$ , OK? Because of the square root  $g$ -- square root  $g$  there take the determinant. It's  $1$  over  $z$  to the power  $d, d$  plus  $1$ .

And the  $g_{zz}$  inverse is the  $z$  square. So you get this power, OK? You get this power. So this-- we generically go to 0 for  $z$  goes to infinity when  $d$  is greater than 1, OK?

So for any dimension greater than 1, if we impose this kind of regularity condition, then this is

always 0. But in fact, for this case, if you solve those equations explicitly, you find actually even for  $d$  equal to 1, this goes to 0, OK? So this is actually generically 0. So we don't need to worry about them.

So now let's just focus on the  $z$  equal to 0 side. Now let's focus on  $z$  equal to 0 side. OK, so we have this condition for  $\phi$  which we derived last time. It says  $\phi$  should satisfy some  $A z$  to the power  $d$  minus  $\delta$  plus some  $B z$ , the leading term. You know, because there are sub leading terms, this is a leading term. And so from that expression, you can also obtain the  $\pi$  from that expression--  $\pi$ .

And then you can easily work out. And that's proportional to, say, minus  $A d$  minus  $\delta$ ,  $z$  to the minus  $\delta$  minus  $\delta$   $B z$  delta minus  $d$ , OK? So you just take the derivative, and you find that.

So just intuitively, you have the same thing, this  $z$  minus  $d$  factor multiplying the derivative of  $\phi$ . So intuitively, now we go to  $z$  equal to 0. Then this factor actually is bad. Because this factor goes to infinity, OK? This factor goes to infinity.

So you see that the  $\pi$  indeed is generically divergent because of this term, OK? Generically divergent because of this term. So now let's look at the product between them because this involves the product between them. So the leading term product between them,  $\phi \pi$ , is proportional to  $z$  to the power of  $d$  minus  $2\delta$ , OK?

So this is all the expression for the delta. And  $\nu$  is always greater than or equal to 0. So we see that this term was always divergent. It's always divergent for  $\nu$  greater than 0.

And for  $\nu$  equal to 0, it actually requires a little bit special treatment. Because for  $\nu$  equal to 0, then  $\delta$  equal to  $d$  divided by 2. Then these two terms have the same power, then become degenerated in this logarithm term. And again, it will be divergence. But I will not go into that, OK?

So the story is that this is always divergent. This is always divergent. So this will confirm what we said earlier.

So this tells you that  $\phi$  is divergent. So we need to renormalize it. We need to renormalize it. So we need to add the counterterm to it, OK? We need to add the counterterm to it.

So the choice of the counterterm, as we said before, should be local on the covariant. So here, it's very simple. We only work out the quadratic order.

We only need to write down things quadratic in  $\phi$ . So the counterterm should also be quadratic in  $\phi$ . So we can write counterterm as the following.

So it's easier to write in the momentum space, say, evaluated at this  $\epsilon$ . So we need to introduce our cut-off equal to  $\epsilon$  and then introduce a counterterm. So the counterterm introduced should be local. So the local term-- so at a quadratic level, all possible local terms can be written as follows.

We can write it in momentum space as the following. Say, as something which is a function of  $k^2$ --  $\phi^2$ ,  $k^2 \phi^2$ , say,  $\phi^2$ ,  $k^2 \phi^2$ , OK? So the criterion is said  $f(k^2)$ .

So the locality means that  $f(k^2)$  must be analytic in  $k^2$ . So let me just give you some intuition. So this is a compact expression using momentum space.

But in coordinate space, it's easy to understand. So this is analytic in  $k^2$ . So you expand it in  $k^2$ .

So  $k^2$  is this Lorentz covariant  $k^2$ -- using this Lorentz covariant  $k^2$ . Yeah, the same  $k^2$  is here, which is a Lorentz covariant  $k^2$ . So you can expand this in  $k^2$ .

So the 0-th order term just a constant. And the constant, when you go back to the coordinate space, this is just  $\phi^2$ , OK? This is simple. It's a local term. You can write in terms of two  $\phi$ s. And the next order term will be  $k^2 \phi^2$ .  $k^2 \phi^2$ , when you go back to coordinate space, this will be just partial  $\phi^2$  and et cetera.

And indeed, these are the only local terms you can write down. And so essentially, you can summarize in the momentum space just as something which is analytic in  $k^2$ , OK? So now, we want to choose the  $k^2$  to cancel the divergence, OK? So want to choose the case-- I chose this to cancel the divergence, OK?

So now, in order to do a-- so now I need to introduce a little bit of notation. Because if we want to worry about the locality-- and it's not enough just to look at this leading term, OK? Look at leading term-- so remember, those things-- so those behavior obtained solving this equation by setting this to be 0. So when  $z$  goes to 0 limit to leading order, the  $kz$  just goes to 0



compared to this term.

So that expression, which we did last time, are obtained by solving this equation without this  $kz$  squared term. So we include this  $kz$  squared term so that we include some higher order corrections, OK? We include some higher order corrections. But now since we worry about whether this  $fk$  squared is analytic-- so we have to worry about those  $k$  dependence.

OK, so now let me introduce some notation, OK? So let's consider the basis of solution-- the basis of solutions to this equation,  $\phi_1$ , which I call  $\phi_1$ , and  $\phi_2$ . So  $\phi_1$  and  $\phi_2$  precisely are defined by these two asymptotic behavior.

So  $\phi_1$  has the behavior that goes to  $d$  minus  $\delta$ . And the  $\phi_2$  then have the behavior when you go to  $z$  equal to 0, it goes to  $z$  to the power  $\delta$  as  $z$  goes to 0, OK? So this is the leading term for  $\phi$ .

But for small  $z$ , this can be written as a power series. And then the leading term will be just like that. Then you will have, say, some  $a_1 k^2 z^2$  term come from treating the next order list term and also higher order terms-- say,  $k^4, z^4$ , et cetera.

So similarly,  $\phi_2$  would be start with  $z$  to the power of  $\delta$ . Then will be also have higher order terms. OK?

So the key is that you can work out those coefficients, OK? Because this is local expansion of infinity, and you can work out those coefficients explicitly. And all the dependents on  $k$  are local in the sense that they all are dependant only on  $k^2$ . It's because the  $k^2$  is what  $k^2$  appears in the original equations, OK?

And similarly, you can work out the  $\pi$ . So the corresponding  $\pi_1$  and the  $\pi_2$  corresponding canonical momentum for them equal  $\pi_1, \pi_2$ , OK? Then the  $\pi_1, \pi_2$ , each of them will have these two behaviors, OK? Each of them will have these two behaviors and along with these higher order terms, OK?

Good? So now-- so now let's look at this on-shell action. So this on-shell action, before we regularize it, evaluate it at  $z$  equal to  $\epsilon$ . So this would be-- oh right, there's one more step. Sorry.

So using this notation, we can just write our  $\phi$  equal to  $A\phi_1 + Bk$  same momentum case--  $\phi_2$ . So now these are exact expression because the  $\phi_1, \phi_2$  are two bases and

$A_k$  and  $AB$  are just linear coefficients of that basis, OK? And the  $\pi^c$  would be similarly  $A_k$ , where  $\pi^1 + B_k \pi^2$ , OK?

So under the SE  $\pi^c$  would be, if you plug this in, just have  $A^2 \pi^1$  and  $\pi^1 + B^2 \pi^2 + AB \pi^1 \pi^2 + \pi^1 \pi^2$ , OK? And this is the one that's divergent.

So this is the one which is divergent coming from the expression we said before. OK, this is divergent. And so this is  $\pi^1$ , this is  $\pi^1$ , OK? So this is divergent. And the other term you can all check are finite, OK?

So this is divergent. So we need to cancel this term. But you cannot just abstract this term. You cannot just abstract this term because this term is not covariant. Because  $A$ -- because this term is not covariant.

This term does not have the form of the  $\pi^2$  et cetera, does not have this covariant form, OK? So this term by itself is not covariant. So you cannot just subtract this term.

But now, you can easily-- so now, we have  $\pi^2$  here. So now you can easily pick a  $\pi^1$  just to cancel this term, OK? Just by comparing these two, we can just pick an  $f$  to cancel that term. So it turns out that  $f$  can be written as  $\pi^1 \pi^1$  times  $\pi^2$ , OK?

So essentially, this is our  $f$ . It's the ratio between  $\pi^1$  and  $\pi^1$ . So now if you expand it, since I'm out of time-- so let me not do it here. If you expand it, you can easily see this term cancels that term, OK?

But there are other terms from the covariant. So there are also finite terms brought by this term. But there's also finite terms brought by this term. And this is local because this is a ratio of  $\pi^1$  and  $\pi^1$ . And both  $\pi^1$  and  $\pi^1$  are obtained by solving this equation locally at the infinity just by power series and dependence on  $k^2$ . It's always analytic, OK?

So this is guaranteed. So this is analytic function of  $k^2$ . OK, so you take a little bit of exercise.

You work out what is  $a$  here,  $a^1$  here, et cetera. Then you can write this term more explicitly in terms of standard term like  $\pi^2$ , partial  $\pi^2$ , et cetera, OK? So now you add these two together.

So now you add these two together. You get a renormalized counterterm, OK? Then you find

the renormalized expression.

Then you find the renormalized expression is  $S_R \phi_c$  is given by  $1/2$ , again in momentum space,  $2\nu A$ , minus  $k$ ,  $B_k$ . So you find everything depends on  $\phi_1$ ,  $\pi$ . They all disappear in the end. You just get something like this.

You get a finite term like this. So now this is ambiguous. It's all finite and ambiguous. So this is your on-shell action.

So normally, what we do is that we solve this equation. Then we can find the explicit  $A$  and  $B$ . So we solve this equation with the following boundary conditions.

It said we first impose the boundary condition. You have  $A_k$  equal to  $\phi_k$  because  $A$  is the source.  $A$  is identified with the source.

But we also need to impose regularity at  $z$  equal to infinity. So we wanted the solution to be well-defined at  $z$  equal to infinity. So it turns out these are the two conditions which uniquely determine the solution.

So that means these two conditions together which will determine  $B$  in terms of  $A$ . OK, so this will determine  $B$  in terms of  $A$ . So in particular, this is a linear solution. So  $B$  will have the form which-- some number  $\chi$  times  $A$ . So  $\chi$  is the thing which you determine by solving the equation explicitly, OK?

This is a form--  $\chi$  you obtain from solving the equation. Just short a couple minutes. It's just-- so now you plug this into that. You plug this into that.

Plug this into that. Unfortunately, I erased my equation. OK, so you plug this into that.

So  $\chi$  you assume is some known function which you obtain after you are solving the equation explicitly. Then what you get-- so without that page. So what you get is you will get  $S$  renormalized action  $\phi_c$  equal to  $1/2$  the integration.

So  $2\nu A_k$ --  $2\nu \lambda$ -- or we here call  $\chi$ --  $A_k$  and  $A$ , minus  $k$ , OK? So you remember this is just all  $\phi_k$ . This is just all  $\phi_k$ .

So now you can obtain the one-point function. Just take the derivative over  $\phi$ -- say, minus  $k$ . You get the Ok.

And this is precisely just  $2\nu\chi$  times  $A_k$ . And this is precisely  $2\nu B$ , OK? And you can write it in the coordinate space also,  $O_x$  equal to  $2\nu B$ .

So this we derive in the linear order by looking at the quadratic equation motions. But actually, this turns out you can prove it actually to all order. You work with nonlinear equations, et cetera, you can show this is always true that the one-point function is always given by this  $B$ , OK? One-point function is always given by this  $B$ .

So let me say one thing. So now the two-point function-- so the two-point function, you just take two derivatives. You take two derivatives.

So that just gives you  $2\nu\chi$ , OK? Just the  $2\nu\chi$ . So this can also be written as  $2\nu B$  divided by  $A$  because the  $\chi$  is equal to-- and this makes perfect sense physically because this is our Euclidean correlation function, OK?

And the  $2\nu$  divided by  $A$  can also be written using what we write there. So this is the expectation value divided by  $\phi_k$  divided by the source, so  $A$  because it's  $\phi$  and  $2\nu B$  is expectation value.

And then we see this expression is precisely a so-called linear response. The expectation value in the presence of the source is the correlation function times the source-- the correlation times the source. So again, this equation is completely general.

So for any two-point function case, we'll always find this  $B$  divided by  $A$  times  $2\nu$ . And you can also alternatively written it as the ratio  $O$  divided by  $\phi$ , OK? So just the final minute.

Now you can solve the equation explicitly. So now you can solve the equation explicitly. You can solve the equation explicitly.

And you can solve your [? eom ?] using the Bessel function. And then you find that actually, the function which is no [INAUDIBLE] the infinity at  $z$  equal to infinity is given by this form.  $k$ -- so this Bessel  $k$  function with index  $\nu$ .  $\nu$  is the same as this  $\nu$ .

And then from here, you just expand in small  $z$ . Then you can find the  $B$  divided by  $A$ . Then you find  $B$  divided by  $A$  as  $\Gamma(\nu)$ , minus  $\nu$ ,  $\Gamma(\nu)$ .

So from here, you just expand in small  $z$ . And then you extract different power. You obtain the ratio of  $B$  divided by  $A$ , OK?

So as you get  $\gamma_{\nu} - \nu$ ,  $\gamma_{\nu} k$  divided by  $2$  to the power  $2\nu$ . So you find the two-point function. So since the two-point function is just the  $2\nu$  B, A-- so you find the Euclidean two-point function in momentum space. It's just  $2\nu \gamma_{\nu}$ , minus  $\nu$ ,  $\gamma_{\nu}$ ,  $k$  divided by  $2$  to the  $2\nu$ .

So in some sense, this is precisely what you expect because we are working with the conformal field theory. And the two-point function can only depend on  $k$  as a power because there's no other scale. There's no other scale.

And the power is determined by the dimension of the operator. And for example, if you go to the coordinate space-- so you Fourier transform, go to coordinate space. So that just gives you the  $X$  to the power  $2\Delta$ . Sorry-- I will stop here.