Chapter 3: Duality Toolbox

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Lecture 18

3.1: GENERAL ASPECTS

3.1.1: IR/UV CONNECTION

As seen before, equipped with holographic principle, we can deduce $\mathcal{N} = 4$ super Yang-Mills theory in (3+1) dimension from AdS gravity. However, from field theory perspective, where does the extra dimension come from? The answer lies already in the way we take the low energy limit. As we see before, as we approach the center of D3 brane, namely $r \to 0$, we get the low energy limit of boundary theory. In other words, the extra dimension can be considered as representing the energy scale of the boundary theory!

Since this is a very important point, let us go over it again using AdS metric:

$$ds^2 = \frac{R^2}{z^2}(-dt^2 + d\vec{x}^2 + dz^2) \qquad t, \vec{x}: \text{ defined in boundary units} \tag{1}$$

Local proper time and proper length at same z are

$$d\tau = \frac{R}{z}dt \qquad dl = \frac{R}{z}dx \tag{2}$$

which implies the relations between the energies and distances in boundary and bulk:

$$E_{YM} = \frac{R}{z}E_{loc} \qquad d_{YM} = \frac{z}{R}d_{loc} \tag{3}$$

We can interpret these relations as follows: for the same bulk processes at different z, i.e. with the same E_{loc} and d_{loc}, the corresponding processes in the field on the boundary are

$$E_{YM} = \frac{1}{z}(E_{loc}R) \propto \frac{1}{z} \qquad d_{YM} = z\frac{d_{loc}}{R} \propto z \tag{4}$$

In particular, we have

$$z \to 0 \implies E_{YM} \to \infty,\ d_{YM} \to 0 \text{ (UV processes of SYM)} \tag{5}$$

$$z \to \infty \implies E_{YM} \to 0,\ d_{YM} \to \infty \text{ (IR processes of SYM)} \tag{6}$$

Generally, a typical bulk process has an energy scale $E_{loc} \sim \frac{1}{R}$, which implies the energy scale of corresponding boundary process $E_{YM} \sim \frac{1}{z}$. This relation connecting IR and UV in boundary theory by the "depth" of corresponding process in bulk is called IR/UV connection.

Remarks:

1. Putting an IR cuttoff in AdS at $z = \epsilon \Longleftrightarrow$ In the boundary introducing a short-distance (UV) cutoff at $\Delta x \sim \epsilon$ or energy cutoff at $E \propto \frac{1}{\epsilon}$.

2. In a conformal theory on $\mathbb{R}^{3,1}$, there exists arbitrarily low energy excitations, corresponding to $z \to \infty$ region in the bulk (Poincare patch). If a theory has an energy gap, bulk spacetime has to "end" at a finite proper distance. An example of CFT on $S^3 \times \mathbb{R} \Longleftrightarrow$ gravity in global AdS is given in problem set.

3.1.2: MATCHING OF SYMMETRIES

We have the following table showing the symmetries of $\mathcal{N}=4$ SYM and IIB string theory in $\mathrm{AdS}_5 \times S_5$:

$\mathcal{N}=4$ SYM		IIB string theory in $\mathrm{AdS}_5 \times S_5$
conformal $SO(4,2)$	$\Longleftrightarrow$	isometry of AdS_5: $SO(4,2)$
global (internal) $SO(6)$	$\Longleftrightarrow$	isometry of S^5: $SO(6)$
SUSY (global): 4 ($\mathcal{N}$)+4 (CFT SUSY) Weyl spinors: 32 components of real supercharges	$\Longleftrightarrow$	same amount of local SUSY

Table 1: Symmetries of $\mathcal{N}=4$ SYM and IIB string theory in $\mathrm{AdS}_5 \times S_5$

Remarks:

1. Isometry of $\mathrm{AdS}_5 \times S_5$ is a subgroup of diffeomorphisms (coordinate transformations), which are *local* symmetries on gravity side. We thus have

$$\text{Global Symmetries} \Longleftrightarrow \text{Local Symmetries (gauge)}$$

2. What is special about isometry?
This is subgroup which leaves the asymptotic form of the metric invariant, which can be regarded as those gauge transformations (diffeomorphism) falling off sufficiently fast at infinity. In other words, isometries are *large gauge transformations* , which can be considered as global part of the diffeomorphisms.

3. The story works in general:

CFT in Mink_d		AdS_{d+1} gravity
conformal $SO(d,2)$	$\Longleftrightarrow$	isometry $SO(d,2)$
global $U(1)$	$\Longleftrightarrow$	local $U(1)$
global SUSY	$\Longleftrightarrow$	local SUSY

Table 2: General symmetries matching

3.1.3: MATCHING OF PARAMETERS

According to our discussion in Chapter 2, we summarize in the following table to show the relations between parameters of SYM and gravity:

$\mathcal{N}=4$ SYM		IIB in $\mathrm{AdS}_5 \times S_5$
g_{YM}^2	$=$	$4\pi g_s$
$\lambda \equiv g_{YM}^2 N$	$=$	$\frac{R^4}{\alpha'^2}$
$\frac{\pi^4}{2N^2}$	$=$	$\frac{G_N}{R^8}$

Table 3: Parameters of $\mathcal{N}=4$ SYM and IIB string theory in $\mathrm{AdS}_5 \times S_5$

We often consider dimensional reduction on S^5 to get the effective gravitational constant in AdS_5. This can be done by simply integrating out the volume of S^5 in our action:

$$\frac{1}{G_5} = \frac{V_5}{G_N} \implies \frac{G_5}{R^3} = \frac{\pi}{2N^2} \tag{7}$$

where we used the volume of S^5 is $V_5 = \pi^3 R^5$. We are specially interested in following two limits:

1. Semi-classical gravity limit. In this limit we set $\hbar = 1$ and treat gravity as classical background field and restrict length scales much larger than string length. Hence, we equivalently have quantum field theory in curved spacetime. Our parameters take the limit of $G_N/R^8 \to 0$ and $\alpha'/R^2 \to 0$, which corresponds

to $N \to \infty$ and $\lambda \to \infty$ respectively by table 3. This result has a remarkable physical interpretation: *strong coupling limit is described by classical gravity!* Furthermore, we may find

$$\frac{1}{N^2} \text{ corrections in CFT} \iff \text{quantum gravity corrections}$$
$$\frac{1}{\sqrt{\lambda}} \text{ corrections in CFT} \iff \frac{\alpha'}{R^2} \text{ corrections (string has a finite size)}$$

2. Classical string limit. In this limit we release the restriction of α'/R^2 above, which can be considered as classical string theory in curved spacetime. Similarly, in this limit, we have

$$N \to \infty \iff \frac{G_N}{R^8} \to 0 \ (g_s \to 0)$$
$$\lambda \text{ arbitrary} \iff \frac{\alpha'}{R^2} \text{ arbitrary}$$

which is exactly 't Hooft limit.

3.1.4: MATCHING OF THE SPECTRUM

From now on, we will consider semi-classical gravity regime in the bulk (*i.e.* QFT in a curved spacetime). The following table shows the matching of Hilbert space in the general correspondence, not necessary to be $\mathcal{N} = 4$ SYM:

boundary theory		bulk theory
representations of conformal group $SO(2,4)$	$\iff$	representations of isometry $SO(2,4)$
conformal local operators	$\iff$	bulk fields
scalar operators $\mathcal{O}$	$\iff$	scalar fields ϕ
vector operators J_μ	$\iff$	vector fields A_M
tensor operators $T_{\mu\nu}$	$\iff$	tensor fields h_{MN}

Table 4: Correspondence of Hilbert space

If there are other symmetries, quantum numbers and representations under them, they should also match. Actually, spectrum of IIB supergravity on S^5 has been worked out long ago and for all of them counter parts have been found in $\mathcal{N} = 4$ SYM. The most important correspondence in their spectra are as follows:

$\mathcal{N} = 4$ SYM		IIB supergravity
Lagrangian: $\mathcal{L}_{\mathcal{N}=4}$	$\iff$	dilaton: Φ
$SO(6)$: J^a_μ	$\iff$	gauge field reduced from S^5: A^a_μ
stress tensor: $T_{\mu\nu}$	$\iff$	metric perturbations: h_{MN}

Table 5: Correspondence of Hilbert space

Given an operator $\mathcal{O}(x)$ in a filed theory, a natural thing to is to deform the theory by adding a source term for $\mathcal{O}(x)$:

$$\int d^d x \phi_0(x) \mathcal{O}(x) \tag{8}$$

where if $\phi_0(x)$ is a constant and generally $\mathcal{O} \sim \mathrm{Tr}(\phi_1\phi_2\cdots))$, this is equivalent to changing the coupling in $\mathcal{O}(x)$. Now the question raises: what does this operation mean in the bulk? Let us consider it in a specific way. We know in IIB supergravity, $g_s = e^{\langle\Phi\rangle}$, where Φ is dilaton. For a spacetime with boundary like AdS, we can identify

$$\langle\Phi\rangle = \Phi_\infty \qquad \text{value of } \Phi \text{ at the boundary} \tag{9}$$

because we may naturally assume there is no fluctuation of Φ approaching boundary. Since in the correspondence of parameters, $g^2_{YM} = 4\pi g_s$, we will get a relation $g^2_{YM} = 4\pi e^{\Phi_\infty}$. Note the coupling constant $1/g^2_{YM}$ can be regarded as the (uniform) source C for $\mathcal{L}_{\mathcal{N}=4}$, and the deformation of form of (8) of the boundary Lagrangian is $\int \delta C \mathcal{L}_{\mathcal{N}=4}$, we have

$$\int \delta C \mathcal{L}_{\mathcal{N}=4} = \frac{1}{4\pi} e^{-\Phi_\infty}(-\delta\Phi_\infty) \int \mathcal{L}_{\mathcal{N}=4} \tag{10}$$

We conclude that this deformation corresponds to changing the boundary value Φ_∞ of dilaton Φ, or in this way we say the bulk field Φ corresponds to boundary operator $\mathcal{L}_{\mathcal{N}=4}$. Indeed, one can extend this duality to more general cases:

$$\int d^dx\phi_0(x)\mathcal{O}(x) \text{ in boundary theory } \iff \text{ bulk field } \phi(x) \text{ corresponding to } \mathcal{O}(x) \text{ with boundary value } \phi_0(x) \tag{11}$$

up to some (re)normalization factor because of mass-dimension relation that we will explain in the next lecture.

One can use this identification to argue: (a) any conserved current J^μ in boundary is dual to a bulk gauge field, *i.e.* global boundary symmetry $\iff$ bulk gauge symmetry. (b) stress tensor is always dual to metric perturbations. To see (a), consider deforming the boundary theory by

$$\int d^dx a_\mu(x)J^\mu(x) \tag{12}$$

where we work in Poincare patch and based on our duality: $a_\mu(x) = A_\mu(x,z)|_{z=0}$, where A_μ is the bulk dual vector field. Since (12) is invariant under

$$a_\mu \to a_\mu + \partial_\mu\Lambda(x) \tag{13}$$

for any $\Lambda(x)$ as long as a_μ vanishes fast enough in boundary and J^μ is conserved by boundary global symmetry, we expect bulk dynamics to be invariant under some gauge symmetry of which (13) is a subset. This is exactly the case that we endow A_μ $U(1)$ gauge symmetry, which has transformation of

$$A_M(x,z) \to A_M(x,z) + \partial_M\tilde{\Lambda}(x,z) \tag{14}$$

and approaching boundary the transverse components will be

$$A_\mu(x,z)|_{z=0} \to A_\mu(x,z)|_{z=0} + \partial_\mu\tilde{\Lambda}(x,z)|_{z=0} \qquad \Lambda(x) \equiv \tilde{\Lambda}(x,z)|_{z=0} \tag{15}$$

which coincides with (13). To see (b), note adding $\int d^dx h_{\mu\nu}T^{\mu\nu}$ to boundary theory can be considered as deforming the boundary metric as

$$\eta_{\mu\nu} \to \eta_{\mu\nu} + h_{\mu\nu} \equiv g^{(b)}_{\mu\nu} \tag{16}$$

From AdS metric

$$ds^2 = \frac{R^2}{z^2}(-dt^2 + d\vec{x}^2 + dz^2) = \frac{R^2}{z^2}dz^2 + g_{\mu\nu}dx^\mu dx^\nu \tag{17}$$

where

$$g_{\mu\nu}(z,x^\mu)|_{z\to 0} = \frac{R^2}{z^2}\eta_{\mu\nu} \tag{18}$$

we may expect that

$$g_{\mu\nu}(z,x^\mu)|_{z\to 0} = \frac{R^2}{z^2}g^{(b)}_{\mu\nu} \tag{19}$$

which implies $T_{\mu\nu}$ should correspond to metric perturbations, *i.e.* the bulk theory must involve gravity.