Review L15

Turing-Gierer-Meinhardt models

Local excitation, global inhibition

$$\frac{\partial a}{\partial t} = r_a + k_a \frac{a^2}{i} - \gamma_a a + D_a \frac{\partial^2 a}{\partial x^2}$$

$$\frac{\partial i}{\partial t} = k_i a^2 - \gamma_i i + D_i \frac{\partial^2 i}{\partial x^2}$$

variables

a: concentration activator

i: concentration inhibitor

t: time

x: position

constants (parameters)

r_a: basal activator synthesis rate

k_a, k_i: rate constant for synthesis

γ_a, γ_i : decay rates

D_a, D_i: diffusion constants

$$\frac{\partial a}{\partial t} = r_a + k_a \frac{a^2}{i} - \gamma_a a + D_a \frac{\partial^2 a}{\partial x^2}$$

$$\frac{\partial i}{\partial t} = k_i a^2 - \gamma_i i + D_i \frac{\partial^2 i}{\partial x^2}$$

choose dimensionless variable

normalize 4 variables

$$\frac{\partial A}{\partial \tau} = 1 + R \frac{A^2}{I} - A + \frac{\partial^2 A}{\partial s^2}$$

$$\frac{\partial I}{\partial \tau} = Q\left(A^2 - I\right) + P \frac{\partial^2 I}{\partial s^2}$$

homogeneous solution

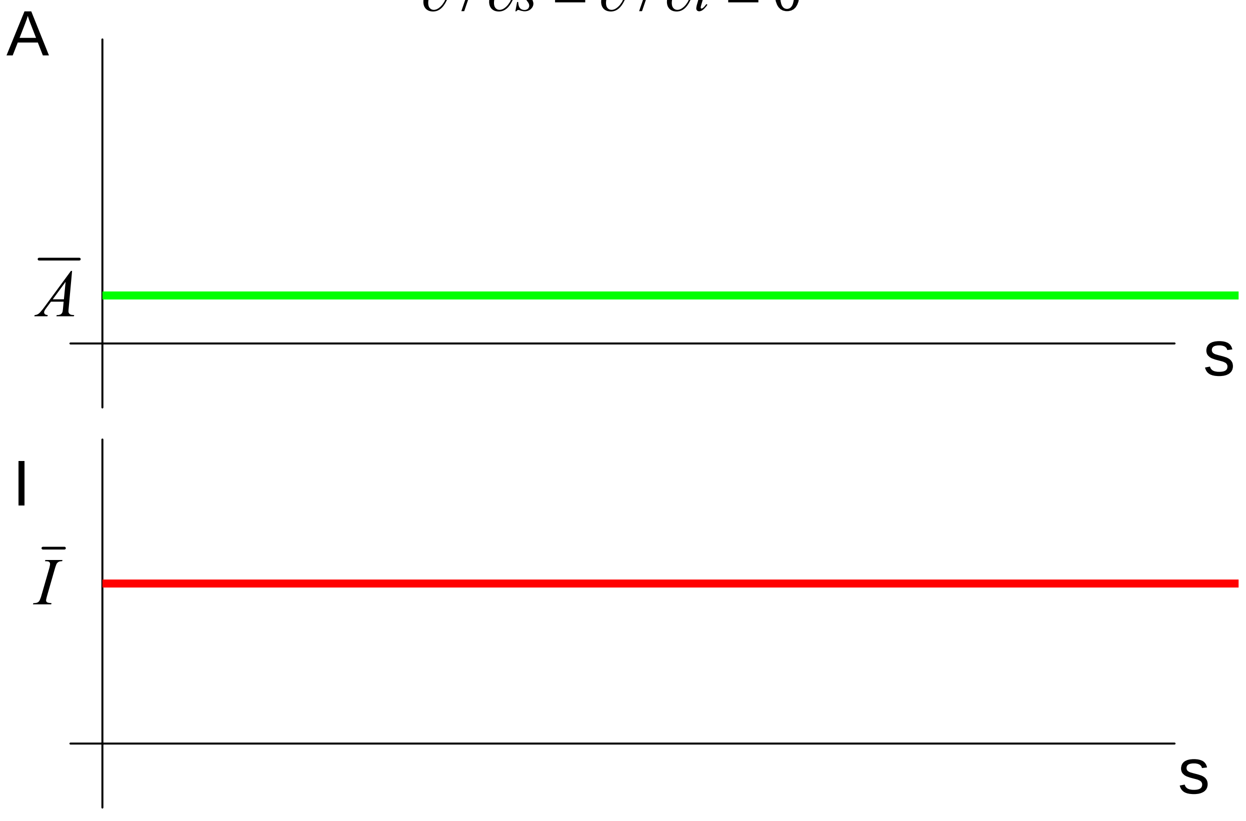
$\partial / \partial s = \partial / \partial t = 0$

$$\overline{A} = R + 1$$

$$\bar{I} = (R+1)^2$$

homogeneous solution

$$\partial / \partial s = \partial / \partial t = 0$$



stability of homogeneous solution

$$\begin{bmatrix} \frac{2R\overline{A}}{\overline{I}} - 1 & -\frac{R\overline{A}^2}{\overline{I}^2} \\ 2\overline{A}Q & -Q \end{bmatrix} = \begin{bmatrix} \frac{R-1}{R+1} & -\frac{R}{(R+1)^2} \\ 2(R+1)Q & -Q \end{bmatrix}$$

trace < 0
det > 0

$$\frac{R-1}{R+1} < Q$$

$$Q > 0$$

inhomogeneous solution:

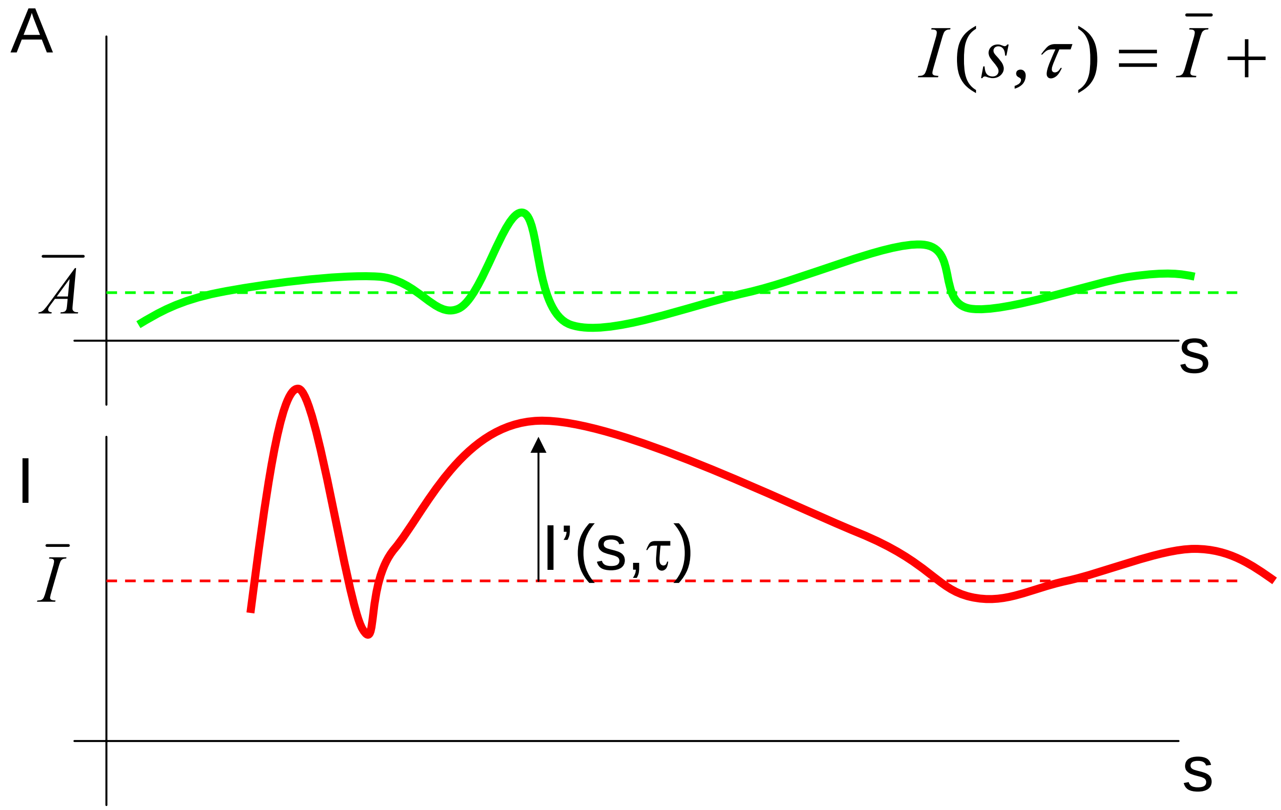
$$A(s,\tau) = \overline{A} + A'(s,\tau)$$

$$I(s,\tau) = \overline{I} + I'(s,\tau)$$

inhomogeneous solution

$$A(s,\tau) = \overline{A} + A'(s,\tau)$$

$$I(s,\tau) = \overline{I} + I'(s,\tau)$$



$$A(s,\tau) = \overline{A} + A'(s,\tau)$$
$$I(s,\tau) = \bar{I} + I'(s,\tau)$$

$\longrightarrow$

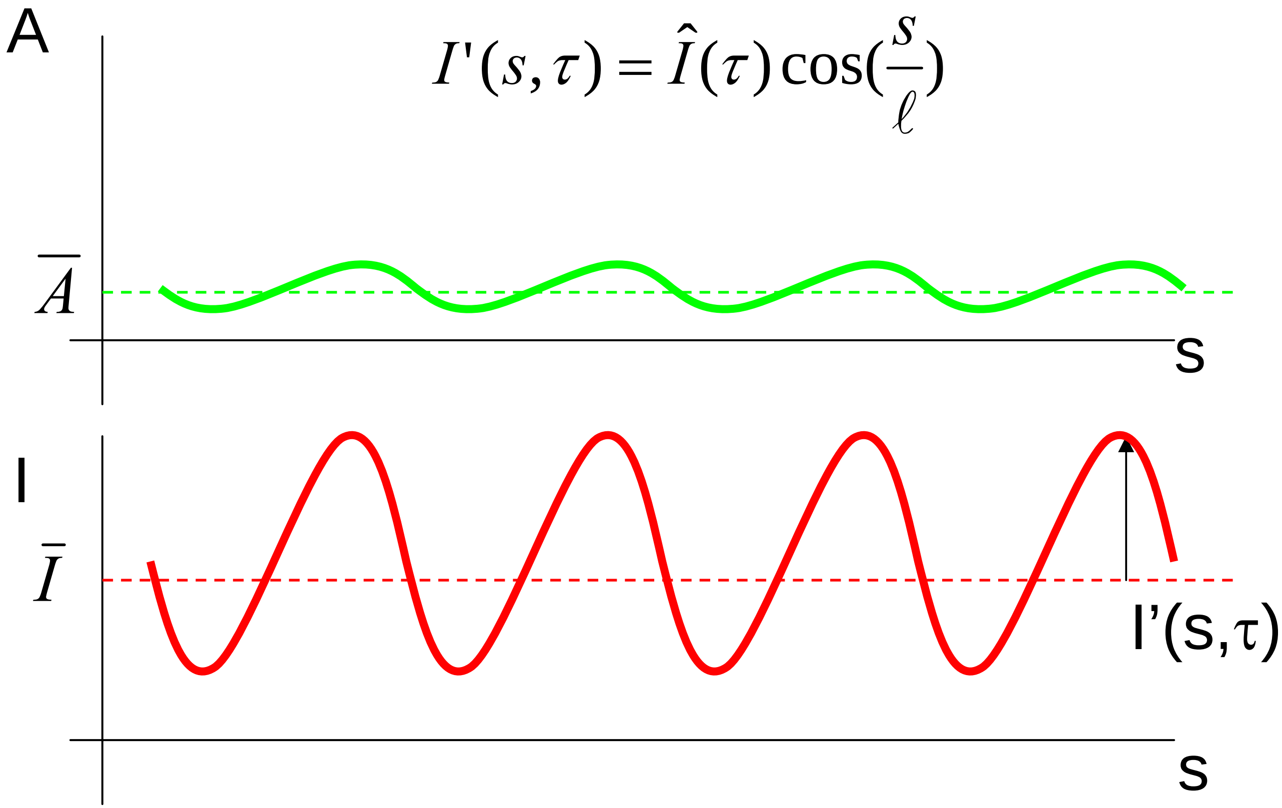
$$\frac{\partial A'}{\partial \tau} = \frac{R-1}{R+1} A' - \frac{R}{(1+R)^2} I' + \frac{\partial^2 A'}{\partial s^2}$$
$$\frac{\partial I'}{\partial \tau} = 2Q(1+R) A' - QI' + P \frac{\partial^2 I'}{\partial s^2}$$

trial solution:

$$A'(s,\tau) = \hat{A}(\tau) \cos(\frac{s}{\ell})$$
$$I'(s,\tau) = \hat{I}(\tau) \cos(\frac{s}{\ell})$$

$$A'(s,\tau) = \hat{A}(\tau)\cos(\frac{s}{\ell})$$

$$I'(s,\tau) = \hat{I}(\tau)\cos(\frac{s}{\ell})$$



$$A(s,\tau) = \overline{A} + A'(s,\tau)$$

$$I(s,\tau) = \overline{I} + I'(s,\tau)$$

$$A'(s,\tau) = \hat{A}(\tau)\cos(\frac{s}{\ell})$$

$$I'(s,\tau) = \hat{I}(\tau)\cos(\frac{s}{\ell})$$

$\longrightarrow$

$$\frac{d\hat{A}}{d\tau} = \left(\frac{R-1}{R+1} - \frac{1}{\ell^2}\right)\hat{A} - \frac{R}{(1+R)^2}\hat{I}$$

$$\frac{d\hat{I}}{d\tau} = 2Q(1+R)\hat{A} - \left(Q + \frac{P}{\ell^2}\right)\hat{I}$$

stability
inhomogeneous
solution

$$-\left(\frac{R-1}{R+1} - \frac{1}{\ell^2}\right)\left(Q + \frac{P}{\ell^2}\right) + \frac{2QR}{1+R} > 0$$

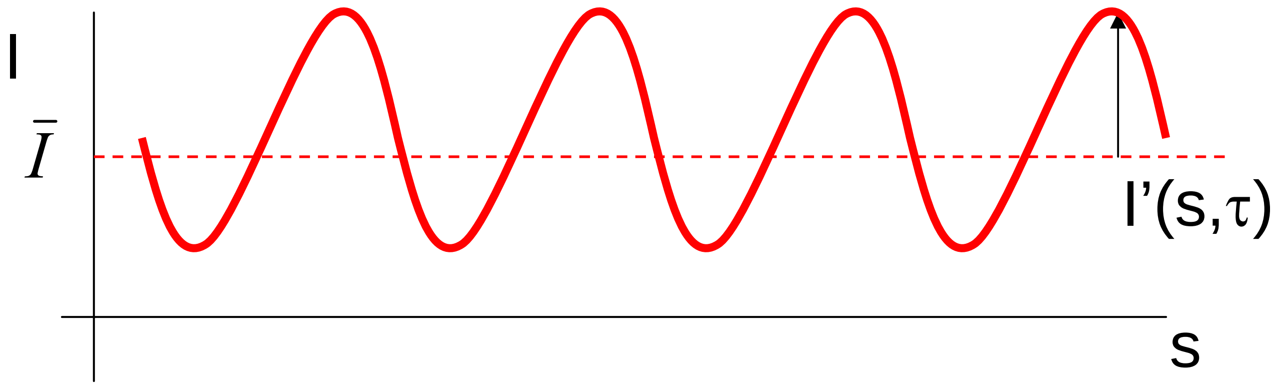
$$Q + \frac{P}{\ell^2} - \left(\frac{R-1}{R+1} - \frac{1}{\ell^2}\right) < 0$$

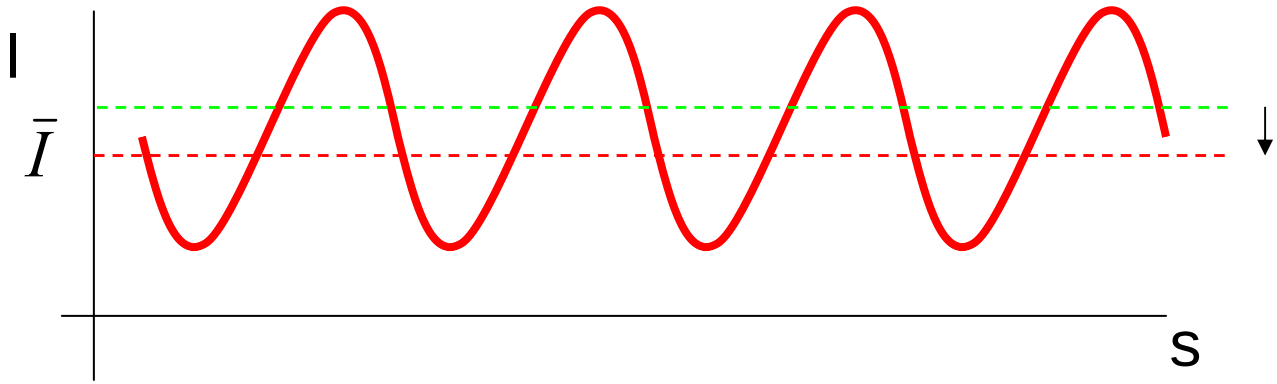
$\longrightarrow$

$$\frac{Q}{P} > \frac{R-1}{R+1}$$

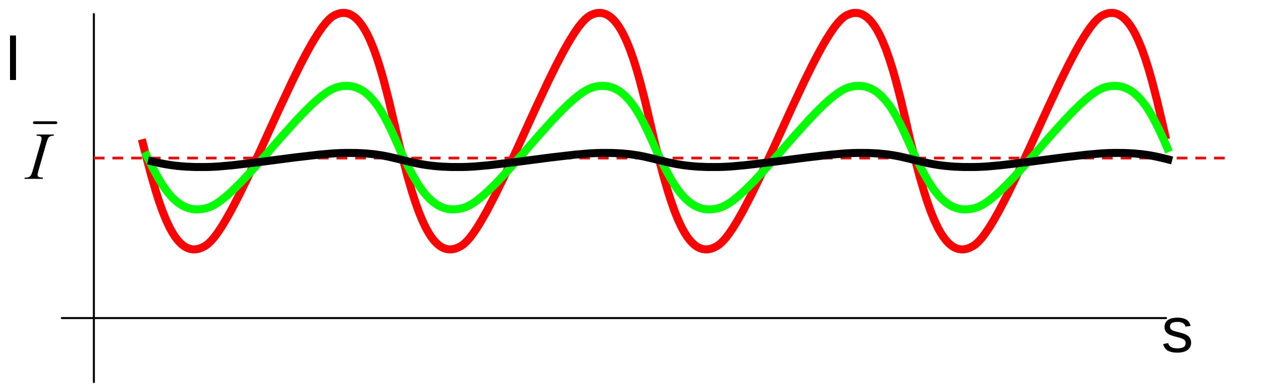
homogeneous stability: $Q > \frac{R-1}{R+1}$

inhomogeneous stability: $\frac{Q}{P} > \frac{R-1}{R+1}$





homogeneously stable: $\bar{I}$ relaxes back to previous value after small uniform disturbance



inhomogeneously stable: I’ relaxes back to $\bar{I}$ after small spatial disturbance