Quantum antiferromagnets:

Heisenberg Hamiltonian on d-dim. hypercubic lattice

$$H = J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j \qquad \text{with } J > 0.$$

Assume ground state has at least short range Neel order & derive effective theory for ~~___~~ configurations where Neel vector varies slowly on scale of lattice spacing.

This is in the same spirit as the hydrodynamic description of superfluids.

As in that discussion must keep the order parameter $\hat{N} \sim \epsilon_i \vec{S}_i$ & the conserved spin (= Fourier component of $\vec{S}_i$ near $q = 0$).

$\therefore$ Write $\hat{n}(i,\tau) = \epsilon_i \hat{N}(i,\tau)$ (norm.) $+ a^d \vec{L}(i,\tau)$.

where (norm.) $= \sqrt{1 - a^{2d} \vec{L}^2}$, $\hat{N}^2 = 1$ & $\hat{N} \cdot \vec{L} = 0$

$\vec{L}$ = local magnetization density

(a = lattice spacing).

Expand to lowest order in $\vec{L}$ & $\partial\hat{N}$.

$$H \simeq JS^2 \sum_{\langle ij\rangle} \hat{n}_i \cdot \hat{n}_j$$

$$\simeq JS^2 \sum_{\langle ij\rangle} \left[\left(-\hat{N}_i \cdot \hat{N}_j\right) + a^{2d}\, \vec{L}_i \cdot \vec{L}_j \right]$$

$$\simeq JS^2 \sum_{\langle ij\rangle} \left[\frac{\left(\hat{N}_i - \hat{N}_j\right)^2}{2} + a^{2d}\, \vec{L}_i \cdot \vec{L}_j \right] + \underbrace{\text{const.}}_{\text{drop}}$$

$$= \frac{JS^2}{2} \int \frac{d^dx}{a^d} \left(\nabla\hat{N}\right)^2 a^2 + d\, a^d JS^2 \int d^dx\, \vec{L}^2 .$$

$$= \int d^dx \left[\frac{JS^2}{2a^{d-2}} \left(\nabla\hat{N}\right)^2 + dJS^2 a^d\, \vec{L}^2 \right]$$

$$\equiv \int d^dx \left[\frac{\rho_s}{2} \left(\nabla\hat{N}\right)^2 + \frac{S^2 \vec{L}^2}{2\chi_\perp} \right]$$

with "spin stiffness" $\rho_s = JS^2 a^{2-d}$

& transverse spin susceptibility (per unit volume)

$$\chi_\perp = \frac{1}{2dJa^d} .$$

Berry phase term

$$S_B = iS \int_0^1 du \int_0^\beta d\tau \Big[\sum_i \epsilon_i \, \hat{N}_i(\tau,u) \cdot \partial_\tau \hat{N}_i(\tau,u) \times \partial_u \hat{N}_i(\tau,u)$$

$$+ a^d \sum_i \vec{L}_i(\tau,u) \cdot \partial_\tau \hat{N}_i(\tau u) \times \partial_u \hat{N}_i(\tau u)$$

$$+ \hat{N}_i(\tau u) \cdot \partial_\tau \hat{N}_i(\tau,u) \times \partial_u \vec{L}_i(\tau,u)$$

$$+ \hat{N}_i(\tau u) \cdot \left(\partial_\tau \vec{L}(\tau u) \times \partial_u \hat{N}_i(\tau u)\right)\Big]$$

2^{nd} term $= 0$ as $\vec{L}$, $\partial_\tau \hat{N}$ & $\partial_u \hat{N}$ are all $\perp r$ to $\hat{N}$ & hence lie in the same plane.

Call 1^{st} term $= S_B'$

$$4^{th} \text{ term} = iS \int_0^1 du \int_0^\beta d\tau \; \hat{N}_i \cdot \partial_\tau \vec{L} \times \partial_u \vec{N}_i$$

$$= -iS \int_0^1 du \int_0^\beta d\tau \; \hat{N}_i \cdot \vec{L} \times \partial_\tau \partial_u \vec{N}_i$$

after integrating by parts & throwing out surface term (due to periodic bdry conditions).

$$\therefore\; S_B = S_B' + iSa^d \int_0^1 du \int_0^\beta d\tau \sum_i \Big(\hat{N}_i \cdot \partial_\tau \hat{N}_i \times \partial_u \vec{L}_i + \hat{N}_i \cdot \partial_u \partial_\tau \hat{N}_i \times \vec{L}_i \Big)$$

$$= S_B' + iSa^d \int_0^\beta d\tau \int_0^1 \partial_u \sum_i \left[\hat{N}_i(u) \cdot \partial_\tau \hat{N}_i(u) \times \vec{L}_i(u) \right] du$$

$$= S_B' + iSa^d \sum_i \int_0^\beta d\tau \left(\hat{N}_i(\tau) \times \partial_\tau \hat{N}_i(\tau) \right) \cdot \vec{L}_i(\tau)$$

($u=0$ does not contribute as $\partial_\tau \hat{N} = 0$ at $u=0$)

$\therefore$ Partition function

$$Z = \int [D\hat{N}\, D\vec{L}]\; \delta(\hat{N}^2 - 1)\; \delta(\hat{N}\cdot\vec{L})\; e^{-S}$$

$$S = S_B' + iS \int d^dx\, d\tau\; \vec{L} \cdot (\hat{N} \times \partial_\tau \hat{N}) + \int d^dx\, d\tau \left[\frac{\rho_s}{2} (\nabla \hat{N})^2 + \frac{S^2 \vec{L}^2}{2\chi_\perp} \right]$$

For a given configuration of $\vec{N}$ at any given point, $\vec{L}$ has 2 independent components.

Integrating them out

$$Z = \int [D\hat{N}]\, \delta(\hat{N}^2 - 1)\, e^{-S_B' - \int d^d x \int d\tau \underbrace{\left[\frac{\rho_s}{2}(\nabla \hat{N})^2 + \frac{\chi_\perp}{2}(\partial_\tau \hat{N})^2\right]}}$$

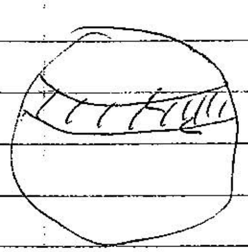
~~Ca~~ Write as $\rho_s/2 \left[(\nabla \hat{N})^2 + \frac{1}{c^2}(\partial_\tau \hat{N})^2\right]$

where $c = (\text{spin wave-velocity}) = \sqrt{\rho_s/\chi_\perp}$

First evaluate S_B' in $d=1$.

$$S_B' = iS \sum_i \epsilon_i A_i$$

$$= iS \sum_{i\ \text{even}} (A_i - A_{i-1})$$



$A_i - A_{i-1}$ = area contained between the 2 loops traced by $\hat{N}_i$ & $\hat{N}_{i-1}$

$$= iS \int \frac{dx\, d\tau}{2a}\, a\, \hat{n} \cdot \partial_\tau \hat{n} \times \partial_x \hat{n}$$

$$= (2\pi i S) \int \frac{dx\, d\tau}{4\pi} \left(\hat{n} \cdot \partial_\tau \hat{n} \times \partial_x \hat{n} \right)$$

Let $Q = \int \frac{dx\, d\tau}{4\pi} \; \hat{n} \cdot \left(\partial_\tau \hat{n} \times \partial_x \hat{n} \right)$.

With identification $\vec{N}(x = -\infty) = \hat{N}(x = +\infty)$

Q counts the # of times ~~$\hat{N}(x,\tau)$~~ ~~sw~~ $\hat{N}$-sphere is covered by mapping from $x-\tau$ space

$\Rightarrow$ Q is an integer.

$$\therefore \; Z = \int [D\hat{N}] \; \delta(\hat{N}^2 - 1) \; e^{-\left[\int d x \, d\tau \, \frac{\rho_s}{2} \left[(\nabla \hat{N})^2 + \frac{1}{c^2} (\partial_\tau \hat{N})^2 \right] + 2\pi i S Q \right]}$$

For S = integer, $e^{2\pi i S Q} = 1$

$$\Rightarrow \; Z = \int [D\hat{N}] \; \delta(\hat{N}^2 - 1) \; e^{-\int dx\, d\tau \, \frac{\rho_s}{2} \left[(\nabla \hat{N})^2 + \frac{1}{c^2} (\partial_\tau \hat{N})^2 \right]}$$

But for $S = \frac{1}{2}$-integer, $e^{2\pi i S Q} = (-1)^{Q}$

so there is destructive interference between paths belonging to different topological sectors in the path integral.

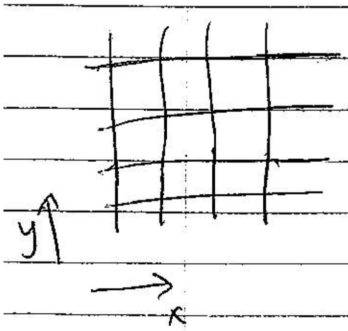
(Different topological sectors are indexed by Q).

$\Rightarrow$ There is a qualitative difference between $\frac{1}{2}$-integer & integer spin.

~~Integer spin magnets are~~ easier to ~~treat within this~~ approach - so focus on those.

What about higher dimension?

Consider $d = 2$ for instance.



$$\text{Now } \quad S_B' = iS \sum_{\vec{r}} \epsilon_r \, \cancel{Q} \cdot A_r$$

$$= 2\pi i S \sum_{y} (-1)^y Q_y$$

where $$Q_y = \frac{1}{4\pi} \int dx \, d\tau \; \hat{n}(x, y, \tau) \cdot \partial_x \hat{n}(x, y, \tau) \times \partial_\tau \hat{n}(x, y, \tau)$$

~~For~~ Q_y is an integer

For smooth configuration of the $\hat{N}$-field that vary slowly along both $\hat{x}$ & $\hat{y}$ directions, this quantized integer # Q_y must be independent of y.

$\Rightarrow$ In the thermodynamic limit,

$$S'_B = 0 \quad \text{in } d=2 \quad \text{and IIIly for } d>2.$$

(However S'_B may be non-zero for configuration in which $\hat{N}$ is not smooth).

Smooth configurations are sufficient to consider to describe ordered phases (should they exist).

$\therefore$ Can study ordered phases in $d \geq 2$ & their low-T properties with field theory

$$S = \int d\tau\, d^d x \;\; \frac{\rho_s}{2} \left[(\nabla \hat{N})^2 + \frac{1}{c^2} (\partial_\tau \hat{N})^2 \right]$$

Return to $d=1$ integer spin chains.

To get some idea of what might happen, generalize to N-components & study in large-N limit.

When $N \to \infty$, model is exactly solvable - can use this to build up a systematic expansion in $1/N$

(Case of interest has $N=3$).

$\therefore$ Consider $Z = \int [D\hat{n}]\, \delta(\hat{n}^2 - 1)\, e^{-\int dx\, d\tau\, \frac{\rho_s}{2} (\partial_\mu \hat{n})^2}$

(have rescaled $\tau \to \tau c$, $\partial_\mu = (\partial/\partial x, \partial/\partial(c\tau))$).

Also convenient to rescale $\hat{n} \to \hat{n}' = \sqrt{N}\, \hat{n}$

and define $\frac{1}{g} = \rho_s/N$.

$$Z = \int [D\hat{n}]\, \delta(\hat{n}^2 - N)\, e^{-\frac{1}{2g}\int dx\, d\tau\, (\partial_\mu \hat{n})^2}$$

(drop $'$ on $\hat{n}$). Study large-N limit with g fixed as $N \to \infty$.

$$Z = \int [D\hat{n}\, D\lambda]\; e^{-\frac{1}{2g}\int dx\, d\tau \left[(\partial_\mu \hat{n})^2 + i\lambda (\hat{n}^2 - N) \right]}$$

Each term is of $O(N)$ $\Rightarrow$ can do λ integral by saddle point.

Look for translation / time reversal invariant saddle points $i\langle \lambda(x\tau) \rangle = m^2$ independent of $(x\tau)$.

$$Z_{S.P} = \int [D\vec{n}]\; e^{-\frac{1}{2g}\int dx\, d\tau \left[(\partial_\mu \vec{n})^2 + m^2 \vec{n}^2 \right] + \frac{NL\beta}{2g} m^2}$$

$\vec{n}$-integral is quadratic $\Rightarrow$ can be done exactly

$$Z_{S.P} = (\text{const.}) \left[\frac{1}{\det(-\partial_\mu^2 + m^2)} \right]^{N/2} e^{\frac{NL\beta}{2g} m^2}$$

$$= (\text{Const.})\; e^{-\frac{N}{2} \operatorname{tr} \ln(-\partial_\mu^2 + m^2) + \frac{NL\beta}{2g} m^2}$$

$$= (\text{Const.})\; e^{-\frac{N}{2} \sum_{k\omega_n} \ln(k^2 + \omega_n^2 + m^2) + \frac{NL\beta}{2g} m^2}.$$

Saddle point condition: Minimize w.r.t m^2.

$$\Rightarrow \quad \frac{1}{2}\sum_{k\omega_n} \frac{1}{k^2+\omega_n^2+m^2} = L\beta/2g$$

Limit as $L\to\infty$, $\beta\to\infty$

$$\Rightarrow \quad g\int\frac{dk\,d\omega}{(2\pi)^2}\frac{1}{k^2+\omega^2+m^2} = 1.$$

Integral is UV divergent - introduce cutoff $\Lambda \sim 1/a$ where a is lattice spacing of original spin model.

$$LHS = g/2\pi \int_0^\Lambda dK \frac{K}{K^2+m^2} \qquad \left(K = \sqrt{k^2+\omega^2}\right)$$

$$= g/2\pi \ln\frac{\Lambda}{m} = 1$$

$$\Rightarrow \quad m = \Lambda e^{-2\pi/g}.$$

$\Rightarrow$ Solution with m non-zero & real for any non-zero value of g.

Saddle point action describes a free N-component particle with dispersion $\omega^2 = k^2 + m^2$.

This is the dispersion of a massive relativistic particle.

$\Rightarrow$ Excitations of the system have an energy gap m.

(which is the minimum energy needed to create these particles).

This gap goes hand in hand with short ranged antiferromagnetic spin correlations

$$\langle \vec{S}_i \cdot \vec{S}_j \rangle \simeq (-1)^{j-i} \underbrace{\langle \hat{n}_i \cdot \hat{n}_j \rangle} \rightarrow \frac{(-1)^{j-i}}{N} \langle \hat{n}_i \cdot \hat{n}_j \rangle$$

after the rescaling

Calculate $\sim$ in the large-N approximation

$$\cancel{\langle \hat{n}_i \cdot \hat{n}_j \rangle = \frac{1}{L} \sum_k e^{ik\cdot(x_i - x_j)} \langle \vec{n}(k) \cdot \vec{n}(-k) \rangle}$$

As the $\vec{n}$-integral is Gaussian in this limit, get

$$\langle \hat{n}(x_i) \cdot \hat{n}(x_j) \rangle = N g \langle x_i | \frac{1}{-\partial_\mu^2 + m^2} | x_j \rangle$$

$$\Rightarrow \langle \vec{S}_i \cdot \vec{S}_j \rangle = e^{i\pi(x_j - x_i)} \, g \int \frac{dk\, d\omega}{(2\pi)^2} \frac{e^{ik(x_j - x_i)}}{k^2 + \omega^2 + m^2}$$

$$= \frac{g}{4\pi} e^{i\pi(x_j - x_i)} \int \frac{dk}{\sqrt{k^2 + m^2}} e^{ik(x_j - x_i)}$$

$$= \frac{g}{4\pi} e^{i\pi(x_j - x_i)} K_0\left(m|x_j - x_i|\right)$$

K_0 = modified Bessel function.

For large $|x_j - x_i|$, $K_0(m|x_j - x_i|) \sim \left(\sqrt{\frac{\pi}{2|x_j - x_i| m}}\right) e^{-m|x_j - x_i|}$

so that $\langle \vec{S}_i \cdot \vec{S}_j \rangle \sim (\text{Const.}) \dfrac{e^{i\pi(x_j - x_i)} e^{-m|x_j - x_i|}}{\sqrt{|x_j - x_i|}}$

as $|x_j - x_i| \to \infty$.

$\therefore$ Spin correlations decay ~~expontially~~ exponentially with "correlation length" $\xi = 1/m$.

Does the large-N result also describe $N = 3$?

Yes - to support this conclusion study the theory in some other ways

(i) RG calculation at small g:

g controls the strength of the fluctuations (effective "temperature"). We now study how the value of g changes as we trade the original theory for a different coarse-grained one obtained by partially performing the functional integral over short-scale fluctuations within a certain block size.

Let Λ be the UV cut-off that defines the theory.

Integrate out fluctuations between Λ & Λe^{-l}.

Write $$n_\alpha(x\tau) = n^0_\alpha(x\tau)\sqrt{1-\varphi^a\varphi^a} + \sum_{a=1}^{N-1} \varphi^a(x\tau)\, e^a_\alpha(x\tau)$$

where $n^0_\alpha(x\tau)$ is a "background field" with wave vector $< \Lambda e^{-l}$, $\varphi^a(x\tau)$ are "spin wave"-like fluctuations orthogonal to $n^0_\alpha(x\tau)$ & have components

$\Lambda e^{-l} < |k| < \Lambda$

e^a_α is a basis orthogonal to n^0_α.

Impose the constraints $n_\alpha n_\alpha = 1$, $n^0_\alpha n^0_\alpha = 1$

$$n^0_\alpha e^a_\alpha = 0 \; ; \; e^a_\alpha e^b_\alpha = \delta^{ab}.$$

$$e^a_\alpha e^a_\beta + n^0_\alpha n^0_\beta = \delta_{\alpha\beta}$$

Note that $\partial_\mu n^0_\alpha = B^a_\mu e^a_\alpha$ (as $\partial_\mu n^0$ is $\perp r$ n^0)

$$\Rightarrow B^a_\mu = e^a_\alpha \partial_\mu n^0_\alpha = - n^0_\alpha \partial_\mu e^a_\alpha.$$

Also $\partial_\mu e^a_\alpha = A^{ab}_\mu e^b_\alpha - B^a_\mu n^0_\alpha$.

with $A^{ab}_\mu = e^b_\alpha \partial_\mu e^a_\alpha = - e^a_\alpha \partial_\mu e^b_\alpha = - A^{ba}_\mu$.

Put in this decomposition of $\hat{n}$ into the action & expand to quadratic order in φ. Then integrate out φ to get effective action for theory ~~defn~~ with cutoff Λe^{-l}.

(This method of RG was introduced by Polyakov & is known as the "background field" method):

$$S = \frac{1}{2g}\int d^2x \, \left(\partial_\mu n_\alpha\right)^2$$

$$= \frac{1}{2g}\int d^2x \left[\left(\partial_\mu \sqrt{1-\varphi^a\varphi^a}\right) n^0_\alpha + \sqrt{1-\varphi^a\varphi^a}\, B^a_\mu e^a_\alpha \right.$$

$$\left. + \left(\partial_\mu \varphi_a\right) e^a_\alpha + \varphi_a A^{ab}_\mu e^b_\alpha - B^a_\mu n^0_\alpha \varphi_a \right]^2$$

$$= \frac{1}{2g}\int d^2x \left[\left(\partial_\mu n^0_\alpha\right)^2 + \left(\partial_\mu \varphi_a - A^{ab}_\mu \varphi_b\right)^2 \right.$$

$$\left. + B^a_\mu B^b_\mu \left(\varphi_a\varphi_b - \varphi^c\varphi^c \delta_{ab}\right)\right]$$

All other x-terms vanish due to orthogonality of n^0 & e^a or due to momentum conservation (as $\hat{n}_0$ & φ_a have momenta that lie in different ranges).

Integrate out φ:

$$S_{eff} = \frac{1}{2g}\int d^2x \left[\left(\partial_\mu n^0_\alpha\right)^2 + B^a_\mu B^b_\mu \left\langle \varphi_a \varphi_b - \varphi^c \varphi^c \delta_{ab} \right\rangle \right.$$

$$\left. + A^{ab}_\mu A^{ac}_\mu \left\langle \varphi_b \varphi_c \right\rangle \right]$$

where $\langle \dots \rangle$ is calculated ~~w.r.t.~~ using the φ-action

$$S_\varphi = \frac{1}{2g}\int d^2x \left(\partial_\mu \varphi_a\right)^2 .$$

The last term depends only on e^a_α & not on n^0_α.

~~The~~ In the 2^{nd} term

$$\left\langle \varphi_a \varphi_b - \varphi^c \varphi^c \delta_{ab} \right\rangle = g \int_{\Lambda e^{-l}}^{\Lambda} \frac{d^2p}{4\pi^2} \frac{1}{p^2} \left(\delta^{ab} - (N-1)\delta^{ab} \right)$$

$$= -g \frac{(N-2)}{2\pi} \delta^{ab} \int_{\Lambda e^{-l}}^{\Lambda} \frac{dp}{p} = -g l \frac{(N-2)}{2\pi} \delta^{ab}$$

$\therefore$ We need $B^a_\mu B^a_\mu = \left(e^a_\alpha \partial_\mu n^0_\alpha\right)\left(e^a_\beta \partial_\mu n^0_\beta\right)$

$$= \left(\partial_\mu n^0_\alpha \partial_\mu n^0_\beta\right)\left(\delta_{\alpha\beta} - n^0_\alpha n^0_\beta\right)$$

$$= \left(\partial_\mu n^0_\alpha\right)^2 - \cancel{\left(n^0_\alpha \partial_\mu n^0_\alpha\right)^2}$$

$$= \left(\partial_\mu n^0_\alpha\right)^2 .$$

$\therefore$ Get $S_{eff} = \frac{1}{2g'}\int d^2x \left(\partial_\mu n^0_\alpha\right)^2$ + terms independent of $\hat{n}_0$ $\therefore$

Finally we need to rescale (x, τ) to keep the cut-off the same as before (i.e we change the units of measuring (x, τ) so that the cutoff in the new units is the same as before).

In the present problem we should let $x \to x' = x e^{-l}$

$$\tau \to \tau' = \tau e^{-l}$$

So that $k \to k' = k e^{l}$ $(k = (k, \omega))$ so that $\Lambda' = (\Lambda e^{-l}) e^{l} = \Lambda$.

This rescaling has no effect on the action here as ~~because~~ the integrals over x, τ are compensated by ∂_μ^2.

~~Now we can compare the two theories defined at different~~ cut-off scales.

$$\frac{1}{g(\Lambda e^{-l})} = \frac{1}{g(\Lambda)} - \frac{(N-2)\,l}{2\pi}.$$

~~For infinitesimal l, $g(\Lambda e^{-l})$~~.

~~Write~~ Write $g = g(l)$

$$\frac{1}{g(l)} - \frac{1}{g(0)} = -\frac{(N-2)}{2\pi}l$$

Consider l infinitesimal & convert this into a differential eqn

$$\frac{dg}{dl} = \frac{(N-2)}{2\pi}g^2.$$

~~$\Rightarrow$ With increasing l, g becomes bigger & bigger~~

$\Rightarrow$ Even if "effective temperature" is small ~~as~~ on lattice scale on coarse graining, it increases

Suggests that the long distance/time physics is that of a theory with large-g which is expected to be disordered.

The eqn $\frac{dg}{dl} = \frac{(N-2)g^2}{2\pi}$ is an example of ~~an~~ a renormalization group flow eqn.

~~At what scales~~ This flow eqn has been derived assuming small g so that short-scale fluctuations are weakly interacting.

At what scale does g becomes of $O(1)$?

Integrating the flow eqn $g(l) = \dfrac{g(0)}{1 - \frac{(N-2)}{2\pi} l\, g(0)}$

Let $g(l^*) \simeq 1 \Rightarrow 1 - \frac{(N-2)}{2\pi} l^* g(0) \approx g(0)$

$$\Rightarrow \frac{1}{g(0)} - \frac{(N-2)\,l^*}{2\pi} \simeq 1$$

For small $g(0)$, $1/g(0) \gg 1 \Rightarrow l^* \simeq \dfrac{2\pi\, g(0)}{N-2}$.

When $g(l^*) \sim O(1)$, expect that correlations decay on the scale of the cut-off $\sim \Lambda^{-1}$

In the original units the cut off is $\Lambda^{-1} e^{l^*}$

$$\therefore \text{ Expect: } \xi^{-1} \approx \Lambda^{-1} e^{-\frac{2\pi}{(N-2)g}}$$

This exponential dependence of ξ on $1/g$ agrees with earlier large-N calculation so long as $N \geq 3$.

Thus expect large-N is qualitatively correct to describe $N \geq 3$.

Finally we can study the theory at strong coupling g by using a lattice version.

Consider $H = \sum_i \frac{g L_i^2}{2} - \frac{1}{g} \sum_i \hat{n}_i \cdot \hat{n}_{i+1}$

which is a lattice Hamiltonian whose continuum limit is the σ-model field theory.

For large g, can first diagonalize $g L_i^2/2$ & then include $\frac{1}{g} \hat{n}_i \cdot \hat{n}_{i+1}$ as a perturbation.

$H_0 = g \sum_i L_i^2/2$ is easily diagonalised.

At each site gd state has $L_i = 0$, excited states have $L_i = 1, 2, \cdots$ & are separated from gd state by gap $\sim g$.

Including $\frac{1}{g} \hat{n}_i \cdot \hat{n}_{i+1}$ as a perturbation, it is ~~clear that gd state will be a spin singlet but~~ excited $L=1$ states can now hop with amplitude $\sim \frac{1}{g}$.

~~$\Rightarrow$ These states will be gapped~~ ..

$\Rightarrow$ Excitation gap will persist - lowest excitations are triplets with gap $\sim g$.

This is again consistent with large-N & weak coupling RG approaches.

What about spin-$\frac{1}{2}$ AF chains in $d=1$?

The nearest neighbour spin-$\frac{1}{2}$ chain $H = J\sum_i \vec{S}_i \cdot \vec{S}_{i+1}$ was solved exactly first by Bethe & since then by others.

It is known that though there is no AF LRO, the spin-spin correlations are power-law at the AF wavevector $k=\pi$.

This goes hand in hand with excitations that are gapless (& are linear dispersing) – however due to the absence of true LRO these gapless excitations are not to be thought of as Goldstone bosons/spin waves.

To get some feeling for the physics, study a much simpler model – the spin-$\frac{1}{2}$ quantum XX chain

$$H = J\sum_i \left(S_{ix} S_{i+1x} + S_{iy} S_{i+1y} \right)$$

Let $S_i^{\pm} = S_{ix} \pm i S_{iy}$

$$H = \frac{J}{2} \sum_i \left(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+ \right)$$

Write $\vec{S}_i = \vec{\sigma}_i/2$, $\sigma_i^{\pm} \equiv S_i^{\pm}$

$$H = \frac{J}{2} \sum_i \left(\sigma_i^+ \sigma_{i+1}^- + h.c \right).$$

The Hilbert space at each site is 2-dimensional.

This is the same size as the Hilbert space of a (spinless) fermion - however the $\vec{\sigma}_i$ for different i commute ~~rather than~~ & hence are not fermionic operators.

This difficulty is overcome by a trick due to Jordan & Wigner.

Define
$$\boxed{\begin{aligned} c_i^+ &= \left(\prod_{j<i} \sigma_j^z \right) \sigma_i^+ \\ c_i &= \left(\prod_{j<i} \sigma_j^z \right) \sigma_i^- \end{aligned}}$$

This change of variables is known as the Jordan-Wigner transformation.

~~Equivalently~~ Note $c_i^\dagger c_i = \sigma_i^+ \sigma_i^-$

$$= \tfrac{1}{4}\left(\sigma_{ix} + i\sigma_{iy}\right)\left(\sigma_{ix} - i\sigma_{iy}\right)$$

$$= \frac{1}{4}\left(2 - i\left[\sigma_{ix}, \sigma_{iy}\right]\right)$$

$$\Rightarrow c_i^\dagger c_i = \frac{1}{4}\left(2 - 2\sigma_{iz}\right) = \tfrac{1}{2}\left(1 - \sigma_{iz}\right)$$

$$\Rightarrow \sigma_{iz} = 1 - 2c_i^\dagger c_i$$

Note that $c_i^\dagger c_i = 0$ or 1 so that

$$\sigma_{iz} = (-1)^{c_i^\dagger c_i}$$

Thus we may also invert the transformation to write

$$\boxed{\begin{aligned} \sigma_i^+ &= \prod_{j<i} (-1)^{c_j^\dagger c_j}\, c_i^\dagger \\ \sigma_i^- &= \prod_{j<i} (-1)^{c_j^\dagger c_j}\, c_i \end{aligned}}$$

It is readily checked that the $\{c_i, c_i^\dagger\}$ satisfy <u>fermion</u> anticommutation relations.

Now rewrite the Hamiltonian in terms of c.

$$\sigma_i^+ \sigma_{i+1}^- = \left(\prod_{j<i} (-1)^{c_j^\dagger c_j}\right) c_i^\dagger \left(\prod_{k<i+1} (-1)^{c_k^\dagger c_k}\right) c_{i+1}$$

$$= c_i^\dagger (-1)^{c_i^\dagger c_i} c_{i+1}$$

$$= c_i^\dagger \left(1 - 2 c_i^\dagger c_i\right) c_{i+1}$$

$$= c_i^\dagger c_{i+1}$$

$$\therefore H = J/2 \sum_i \left(c_i^\dagger c_{i+1} + h.c\right).$$

This is a simple tight-binding model for the spinless fermions c_i & hence can be exactly diagonalized.

Go to k-space

$$c_i = \frac{1}{\sqrt{N}} \sum_k e^{ikx_i} c_k \quad \text{where } N = \text{total \# of sites}$$

$$k \in [-\pi, \pi) = \text{momentum in } 1^{st} \text{ BZ}.$$

$$H = \frac{J}{2N} \sum_i \sum_{kk'} \left(e^{-ikx_i + ik'x_{i+1}} c_k^\dagger c_{k'} + h.c \right)$$

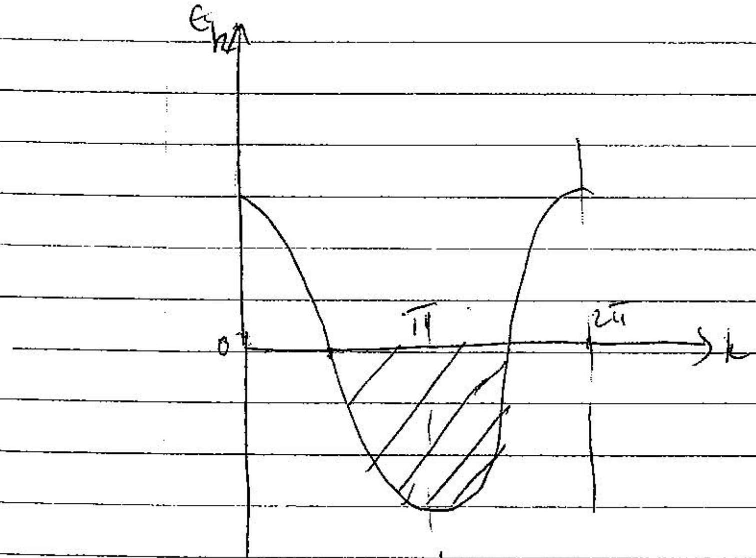
$$= \frac{J}{2N} \sum_i \sum_{kk'} e^{-ix_i(k-k')} e^{ik'a} c_k^\dagger c_{k'} + h.c$$

(a = lattice spacing)

$$\therefore H = \frac{J}{2} \sum_{kk'} \delta_{kk'} e^{ik'a} c_k^\dagger c_{k'} + h.c$$

$$= J \sum_k (\cos ka) c_k^\dagger c_k \equiv \sum_k \epsilon_k (\cos ka).$$

Fill up all $-$ve energy levels.



$\therefore$ In ground state, get $\frac{1}{2}$-filled band with Fermi points at $\pi \pm \pi/2$.

Clearly there are gapless excitations ~~corresp~~

- for instance can add a single fermion with momentum near the Fermi level with arbitrarily low energy cost (in thermodynamic limit).

Adding single fermion changes total fermion # by 1

As $c_i^\dagger c_i = \frac{1-\sigma_{iz}}{2}$, it also changes $S^z_{tot} = \sum_i S_i^z$

$= \frac{1}{2} - S_{iz}$ by -1 in the original spin chain.

Thus the spin-½ chain - at least in the XX limit - is gapless.

If we include a $J_z \sum_i S_{iz} S_{i+1\,z}$ term, then a 4-fermion interaction is generated in the fermion description. The model is no longer trivially soluble - nevertheless a great deal can be understood quite generally & simply using "bosonization" techniques.

This may be used to show that the gaplessness persists upto $J_z = J$ which is the isotropic spin-$1/2$ AF chain.

What happens for $J_z > J$?

To understand this, consider the large J_z limit.

First ignore J completely.

Then the J_z term is minimized if the spins follow

$\uparrow\downarrow\uparrow \cdots \downarrow$ pattern. There are 2 such

ground states which are related by a global spin-flip.

These states break ~~the~~ a discrete Ising symmetry

$(S_i^z \to -S_i^z,\ S_{ix} \to -S_{ix},\ S_{iy} \to S_{iy}\ \forall i)$.

In contrast to broken continuous symmetry, broken discrete symmetry is stable in $d=1$ at $T=0$