Assignment #4

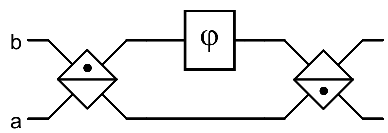
Due: Monday, March 18, 2013

1. **Heisenberg-limited interferometry with the Yurke state**

   The Yurke state $|\psi\rangle = (|n\rangle|n-1\rangle + |n-1\rangle|n\rangle)/\sqrt{2}$ allows one to obtain a measurement of an unknown phase ϕ with uncertainty $\langle\Delta\phi\rangle = \frac{\sqrt{2}}{n}$, using a Mach-Zehnder interferometer. Methods for experimentally realizing these states have been proposed, for example, using Bose-Einstein condenstates [Castin & Dalibard, Phys. Rev. A vol. 55, p. 4330, 1997]. For this problem, use the following definition for the beamsplitter:

$$
\begin{aligned}
BaB^\dagger &= \frac{1}{\sqrt{2}}(a+ib) \\
BbB^\dagger &= \frac{1}{\sqrt{2}}(b+ia)
\end{aligned}
\tag{1}
$$

   a) Let us now analyze the Mach-Zehnder interferometer, fed with a Yurke state as input. Use this setup:



   and work in the Schrodinger picture, by doing the following. Let the input be the Yurke state, $|\phi_0\rangle = |\psi\rangle$, let the state after the first 50/50 beamsplitter be $|\phi_1\rangle = B|\phi_0\rangle$, the state after the phase shifter be $|\phi_2\rangle = P|\phi_1\rangle$, and the state after the final 50/50 beamsplitter be $|\phi_3\rangle = B^\dagger|\phi_2\rangle$. Give expressions for $|\phi_1\rangle$, $|\phi_2\rangle$, and $|\phi_3\rangle$. Note that the transform of the phase shifter P is $PaP^\dagger = ae^{i\phi}$. Double-check that when $\phi = 0$, the output is the same as the input $|\phi_3\rangle = |\phi_0\rangle$. Hint: write these states in terms of operators acting on the vacuum.

   b) What is the uncertainty with which you can determine ϕ using the Yurke state input? This is

$$
\langle\Delta\phi^2\rangle = \frac{\langle\Delta M^2\rangle}{\left|\frac{\partial\langle M\rangle}{\partial\phi}\right|^2}\,,
\tag{2}
$$

   where $M = a^\dagger a - b^\dagger b$ is the difference in the photon numbers measured at the outputs of the interferometer. Compute $\langle\Delta\phi^2\rangle$, evaluated at $\phi = 0$ (the point at which the interferometer is balanced), using the $|\phi_3\rangle$ you obtained above. You should find $\langle\Delta\phi\rangle = \frac{\sqrt{2}}{n}$.

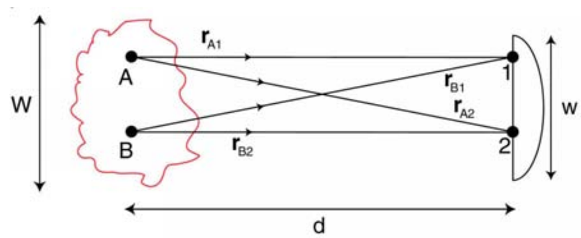
   c) In lecture, we used the Heisenberg picture to compute statistics about interferometer performance with coherent state inputs. For comparison with the Yurke state, let's now work out what happens with coherent states in the Schrodinger picture. Using the same diagram as above, let the input now be a coherent state and a vacuum state, $|\psi_0\rangle = |\alpha\rangle|0\rangle$. Just as above, let the state after the first 50/50 beamsplitter be $|\psi_1\rangle = B|\psi_0\rangle$, the state after the phase shifter be $|\psi_2\rangle = P|\psi_1\rangle$, and the state after the final 50/50 beamsplitter be $|\psi_3\rangle = B^\dagger|\psi_2\rangle$. Give expressions for $|\psi_1\rangle$, $|\psi_2\rangle$, and $|\psi_3\rangle$.

d) Given the quantum fluctuations of the coherent state, use uncertainty propagation to determne the uncertainty with which you can determine ϕ using the coherent state input, as a function of $\bar{n} = |\alpha|^2$ and the phase shift angle ϕ.

2. **Hanbury Brown and Twiss Experiment with Atoms**

This problem illustrates the coherence and collimation requirements for performing a Hanbury Brown and Twiss (HBT) experiment with atoms. In fact the HBT experiment was done for both bosons (^{4}He) and fermions (^{3}He) by Jeltes and company in 2007 (T. Jeltes et al., Nature **445**, 402 (2007)). (Note: Ignore gravity in this problem.)

If a free particle starts at point A at time $t = 0$ with an amplitude (wavefunction) ψ_A, then the amplitude at another point 1 and time $t = \tau$ is proportional to $\psi_A e^{i(\mathbf{k}\cdot\mathbf{r}_{A1} - \omega\tau)}$, where $\mathbf{r}_{A1}$ is the vector from A to 1, $\mathbf{k}$ is the particle's wavevector, and $\hbar\omega$ is its total energy. This can be regarded as Huygen's principle for matter waves, and is a special case of the Feynman path integral formulation of quantum mechanics.



(Based on figure 19-5, in G. Baym, *Lectures on Quantum Mechanics*)

(a) Correlation function

Assume we have a particle at A with amplitude ψ_A and one at B with amplitude ψ_B. The joint probability P of finding one particle at 1 and one at 2 is

$$P = \left|\psi_A e^{i\phi_{A1}} \psi_B e^{i\phi_{B2}} \pm \psi_A e^{i\phi_{A2}} \psi_B e^{i\phi_{B1}}\right|^2 \tag{3}$$

and is proportional to the second-order coherence function $g^{(2)}(1,2)$. The $\pm$ is for bosons/fermions and makes the two-particle wavefunction symmetric/antisymmetric under the exchange of particles. Here, $\phi_{A1} = \mathbf{k}_A \cdot \mathbf{r}_{A1} - \omega\tau$ is the phase factor for the path from point A to detector 1, etc. Calculate P as a function of $\mathbf{r}_{21}$, the vector from point 2 to point 1 on the detector.

(b) Transverse Collimation

Assume you are given a source (e.g. a ball of trapped atoms) with transverse dimension W and detector with transverse dimension w where $|\mathbf{r}_{21}| \leq w$. The distance between source and detector d is much greater than all other distances. The transverse component of the phase factor in part (a) can be written: $\phi_t = (\mathbf{k}_A - \mathbf{k}_B)_t \cdot (\mathbf{r}_{21})_t$. Assume that the signal at the detector is mainly due to atoms with wavevectors distributed around $\mathbf{k}_0$. Argue that the transverse collimation required to see second order correlation effects can be expressed as $Ww \ll d\lambda_{dB}$, where λ_{dB} is the deBroglie wavelength corresponding to $\mathbf{k}_0$. (Hint: How does ϕ_t vary for atoms originating at different points in the source and being detected at different points on the detector?)

Consider a ^{6}Li MOT at 500 μK. Calculate the deBroglie wavelength. Assuming a MOT and detector of approximately equal size ($W \approx w$), estimate an upper bound on the MOT and detector size using $d = 10$ cm.

(c) Longitudinal Collimation

(*i*) The longitudinal component of the phase factor in part (a) can be written: $\phi_l = (\mathbf{k}_A - \mathbf{k}_B)_l \cdot (\mathbf{r}_{21})_l$. Assume a Gaussian distribution of wavevector differences $p(\mathbf{k}_A - \mathbf{k}_B) = e^{-|\mathbf{k}_A - \mathbf{k}_B|^2\gamma^2}$ where the width γ is related to the temperature of the atoms. Calculate $\langle P \rangle$ using this distribution and your result from part (a). Sketch $\langle P \rangle$ for both fermions and bosons, indicating the extent of $(\mathbf{r}_{21})_l$ over which the second order correlation effect can be seen. (Hint: Use the fact that $\phi_t \ll 2\pi$ from part (b) to simplify the integral.)

(*ii*) Now assume you have a pulsed source of atoms with longitudinal dimension L. Atoms are released at time $t = 0$ and detected at some later time $t = \tau$. Give geometric arguments to show that the wavevectors of detected atoms must obey $|(\mathbf{k}_A - \mathbf{k}_B)_l| \leq \frac{mvL}{\hbar d}$, where the velocity $v = \frac{d}{\tau}$. This implies that the different velocity groups separate during the expansion, narrowing (by a factor $\frac{L}{d}$) the velocity distribution of atoms detected at any particular time.

Consider again the ^{6}Li MOT from part (b). Assuming $\tau = 0.1$s and $L \approx W$, estimate the necessary timing resolution of the detector in order to see second order correlation effects?

(d) Phase-Space Volume Enhancement

We now pull all the pieces together. The peak in $g^{(2)}(1,2)$ is visible for $(\mathbf{k}_A - \mathbf{k}_B) \cdot \mathbf{r}_{21} \leq 2\pi$. This is equivalent to saying that we must detect atoms from within a single phase space cell, defined by $\delta p_x \delta x \leq h$ (and likewise for y and z). In our trapped atom sample, the 3D volume of a phase space cell is $\delta x \delta y \delta z = (\lambda_{dB})^3$. Liouville's theorem says that as our ball of atoms expands, the number of phase space cells remains constant. Verify that, by using this pulsed source, the volume of a coherent phase space cell is increased by a factor d^3/W^2L by the time atoms reach the detector. What is the order of magnitude of this increase (assuming $L \approx W$)?

Estimate the average occupation of a cell of phase space for the ^{6}Li MOT from parts (b) and (c). How does this compare with the average occupation of a BEC or a degenerate Fermi cloud?

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