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NOTES ON LIE ALGEBRAS

Lie Algebra $\mathcal{G}$. A vector space $\mathcal{G}$ with a bilinear operation $[\ ,\ ] : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ such that

(i): $[x, x] = 0$, for all $x \in \mathcal{G}$ (antisymmetry),

(ii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$, for all $x, y, z \in \mathcal{G}$ (Jacobi identity).

Typically the vector space is over the real numbers (a real vector space) or over the complex numbers (a complex vector space).

A **Lie subalgebra of** $\mathcal{G}$ is a vector subspace of $\mathcal{G}$ which is itself a Lie algebra under $[\ ,\ ]$.

The **generators** T_a of $\mathcal{G}$ with $a = 1, 2, \ldots, d$ are a set of basis vectors in $\mathcal{G}$. Here d is the dimension of $\mathcal{G}$. The Lie algebra is defined if we give the Lie brackets $[T_a, T_b]$ of all the generators. One writes

$$[T_a, T_b] = f_{ab}{}^c\, T_c\,, \tag{1}$$

where the **structure constants** $f_{ab}{}^c$ are real if the Lie algebra is a real vector space, or complex if the Lie algebra is a complex vector space.

An **ideal $\mathcal{I}$ of $\mathcal{G}$** is an invariant subalgebra of $\mathcal{G}$, namely, $[\mathcal{G}, \mathcal{I}] \subset \mathcal{I}$. An ideal is **proper** if it is not equal to $\{0\}$ nor to $\mathcal{G}$ (both of which are trivial ideals of $\mathcal{G}$). The quotient space $\mathcal{G}/\mathcal{I}$ is readily checked to be a Lie algebra.

The **derived algebra** $\mathcal{G}^{(1)}$ of $\mathcal{G}$ is the set of all linear combinations of brackets of $\mathcal{G}$. We write $\mathcal{G}^{(1)} \equiv [\mathcal{G}, \mathcal{G}]$. $\mathcal{G}^{(1)}$ is an ideal of $\mathcal{G}$. One defines $\mathcal{G}^{(i+1)} = [\mathcal{G}^{(i)}, \mathcal{G}^{(i)}]$ for $i \geq 1$. In the **derived series** of ideals $\mathcal{G}^{(1)}, \mathcal{G}^{(2)}, \ldots$ each term is an ideal of $\mathcal{G}$ (this is proven using induction and the Jacobi identity). $\mathcal{G}$ is **solvable** if its derived series ends up with $\{0\}$.

If $\mathcal{I}$ and $\mathcal{J}$ are ideals of $\mathcal{G}$ then $\mathcal{I} + \mathcal{J}$ is also an ideal of $\mathcal{G}$. If, in addition both $\mathcal{I}$ and $\mathcal{J}$ are solvable, then $\mathcal{I} + \mathcal{J}$ is also solvable (show by induction that $(\mathcal{I} + \mathcal{J})^{(n)} \subset \mathcal{I}^{(n)} + \mathcal{J}$. Since $\mathcal{I}$ is solvable, at some stage the derived series of $(\mathcal{I} + \mathcal{J})$ goes into the derived series of $\mathcal{J}$, which also terminates).

The **radical** $\mathcal{G}_r$ of a Lie algebra $\mathcal{G}$ is the maximal solvable ideal of $\mathcal{G}$, *i.e.* one enclosed in no larger solvable ideal. It follows from the above additivity property that $\mathcal{G}_r$ is unique.

The **center** $\mathcal{Z}$ of $\mathcal{G}$ is the set of all elements of $\mathcal{G}$ that have zero bracket with all of $\mathcal{G}$. The center of $\mathcal{G}$ is clearly an ideal of $\mathcal{G}$.

An **abelian** Lie algebra $\mathcal{G}$ is a Lie algebra whose derived algebra $\mathcal{G}^{\{1\}} \equiv [\mathcal{G}, \mathcal{G}]$ vanishes (the Lie bracket of any two elements of $\mathcal{G}$ is always zero). For arbitrary $\mathcal{G}$, the quotient $\mathcal{G}/\mathcal{G}^{\{1\}}$ is an abelian Lie algebra. There is a unique one-dimensional Lie algebra, the abelian algebra u_1 with a single generator T and bracket $[T, T] = 0$. Any d dimensional abelian Lie algebra is (isomorphic to) the d-fold direct sum of one-dimensional Lie algebras.

The **direct sum** $\mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \cdots$ of Lie algebras with brackets $[\,,\,]_1$, $[\,,\,]_2 \cdots$ is the sum of the vector spaces with a bracket $[\,,\,]$ defined as: (i) $[\,x\,,\,y\,] = [\,x\,,\,y\,]_i$ when $x, y \in \mathcal{G}_i$, and (ii) $[\mathcal{G}_i, \mathcal{G}_j] = 0$, for $i \neq j$. Each summand is an ideal of the direct sum.

A Lie algebra $\mathcal{G}$ is **simple** if it has no proper ideals and is not abelian. Note the following obvious consequences:

(i) the derived algebra of $\mathcal{G}$ (an ideal) must equal $\mathcal{G}$; $\mathcal{G}$ is not solvable,

(ii) being not solvable and having no proper ideals, its radical $\mathcal{G}_r$ vanishes,

(iii) the center of $\mathcal{G}$ must vanish,

(iv) $\mathcal{G}$ cannot be broken into two sets of commuting generators.

A Lie algebra $\mathcal{G}$ is **semisimple** if its radical $\mathcal{G}_r$ vanishes. Simple Lie algebras are semisimple. It can be shown that semisimple algebras are **direct sums** of simple Lie algebras.

A **reductive** Lie algebra is the direct sum of an abelian algebra and a semisimple algebra, with both nonvanishing. This is the case of interest for non-abelian gauge theory. In these algebras the radical equals the center (the abelian algebra).

<u>Comments</u>. The general Lie algebra $\mathcal{G}$ is either solvable or not solvable. The solvable algebras are not easy to classify. If the algebra $\mathcal{G}$ is not solvable then either the radical vanishes, in which case the algebra is semisimple, or the radical does not vanish, in which case the quotient $(\mathcal{G}/\mathcal{G}_r)$ is semisimple (it can be shown that it has zero radical). Any Lie algebra $\mathcal{G}$ has a Levi decomposition $\mathcal{G} = P \oplus_\sigma \Lambda$, as the **semidirect sum** of a solvable algebra P and a semisimple algebra Λ. In the semidirect sum the bracket of elements within summands are the brackets of the respective algebras, and the bracket of mixed elements are defined using a representation σ of Λ on the vector space of P.

<u>Example:</u> The Poincare algebra $\mathcal{P}$. It has two familir subalgebras spanning together $\mathcal{P}$ as a vector space; the Lorentz algebra Λ (semisimple) and the translation algebra P (abelian). The Poincare algebra is not solvable since Λ is not, is not semisimple since its radical equals the non-vanishing algebra P, and is not reductive since it is cannot be written as a direct sum of a semisimple and an abelian part. One has $\mathcal{P} = P \oplus_\sigma \Lambda$.

THE CLASSICAL LIE ALGEBRAS

Let V be a vector space over a field F ($\mathbb{R}$ or $\mathbb{C}$). The **general linear** algebra $gl(V)$ is the algebra of endomorphisms (linear transformations, not necessarily invertible) of V. As a vector space over F, End(V) has dimension $(\dim(V))^2$. The Lie bracket is just the commutator of the linear transformations. More concretely, when V is a vector space of dimension n over F, one can think of $gl(V)$ as the algebra $gl(n,F)$ of $n \times n$ matrices with entries in F. The classical algebras fall into four families $\mathbf{A}_\ell, \mathbf{B}_\ell, \mathbf{C}_\ell$ and $\mathbf{D}_\ell$. These are all subalgebras of $gl(V)$. Any subalgebra of $gl(V)$ is called a **linear** Lie algebra.

$\mathbf{A}_\ell$: Let dim $V = \ell + 1$, the algebra is called $sl(\ell+1, F)$ (for special linear) and is that of endomorphisms of zero trace (a basis independent restriction). Its dimension is $(\ell+1)^2 - 1$.

To describe the next three families of algebras we will use bilinear forms $f(v,w)$ on an n-dimensional vector space V:

$$f(v,w) = v^t s\, w\ , \tag{2}$$

where, for a chosen basis, s is a fixed invertible $n \times n$ matrix and $v, w \in V$. We then consider V endomorphisms x such that for all v and w

$$f(x(v), w) = -f(v, x(w)) \quad \rightarrow \quad sx = -x^t s\,. \tag{3}$$

We claim that the set of endomorphisms that satisfy (3) form a Lie algebra under commutation. Indeed, if x_1 and x_2 satisfy $f(x_i(v), w) = -f(v, x_i(w))$ so does $[x_1, x_2]$. Additionally, tracing the relation $sxs^{-1} = -x^t$ we deduce that x has zero trace. The Lie algebra in question is thus a subalgebra of $sl(n,F)$. The bilinear form $f(v,w)$ is symmetric (antisymmetric) under the exchange of v and w if s is a symmetric (antisymmetric) matrix.

$\mathbf{C}_\ell$: Let dim $V = 2\ell$ and choose some specific basis in which

$$s = \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix}\ . \tag{4}$$

The Lie algebra of endomorphisms that satisfy (3) is called $sp(2\ell, F)$ (for symplectic). Also, $sp(2\ell, F) \subset sl(2\ell, F)$. For a general matrix $x \in sp(2\ell, F)$ we find that the $\ell \times \ell$ blocks take the form

$$x = \begin{pmatrix} m & n \\ p & q \end{pmatrix} \quad \rightarrow \quad n = n^t,\ p = p^t,\ m^t = -q\,. \tag{5}$$

The dimension is readily found; we need two symmetric $\ell \times \ell$ matrices p and n giving $\ell(\ell+1)$, and one arbitrary matrix m (that determines q) giving ℓ^2. Thus dim $sp(2\ell, F) = \ell(2\ell+1)$.

$\mathbf{B}_\ell$: Let dim $V = 2\ell + 1$. This time

$$s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_\ell \\ 0 & I_\ell & 0 \end{pmatrix} . \tag{6}$$

The Lie algebra of V endomorphisms x that satisfy (3) is called $o(2\ell+1, F)$ (for orthogonal). Note that $o(2\ell + 1, F) \subset sl(2\ell + 1, F)$. Moreover, dim $o(2\ell + 1, F) = \ell(2\ell + 1)$ (the same dimension as that of $sp(2\ell, F)$).
$\mathbf{D}_\ell(\ell \geq 2)$: Let dim $V = 2\ell$. This time

$$s = \begin{pmatrix} 0 & I_\ell \\ I_\ell & 0 \end{pmatrix} . \tag{7}$$

The Lie algebra of V endomorphisms x that satisfy (3) is called $o(2\ell, F)$. Note that $o(2\ell, F) \subset sl(2\ell, F)$ and that dim $o(2\ell, F) = \ell(2\ell - 1)$.
The above description given for the orthogonal algebras actually correspond to the maximal noncompact forms. The real orthogonal algebra $o(\ell)$ is obtained with a real vector space V with dim$V = \ell$ and $s = I_\ell$. This is the algebra of real antisymmetric matrices $x = -x^t$.

COMPLEX AND REAL LIE ALGEBRAS

The relevant issues are clarified with an example involving Lie algebras with three generators. Consider the Lie algebra $sl(2, C)$ of complex traceless 2×2 matrices. The Lie bracket is commutator (preserves tracelessness). This is naturally a vector space over the complex numbers as traceless matrices remain traceless by multiplication by a complex number. The algebra is therefore a complex Lie algebra. We can choose a basis of generators

$$J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \quad J_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{8}$$

The brackets are given

$$[J_+ \, , J_- \,] = J_3, \quad [\, J_3 \, , \, J_\pm \,] = \pm 2 J_\pm \, . \tag{9}$$

It is possible to show this complex Lie algebra is the unique simple complex Lie algebra with three generators. Even though the algebra is complex we can easily get a real algebra since the brackets in (9) have only real numbers. We can declare the vector space to be real and say that the abstract basis vectors (J_+, J_-, J_3) have the brackets in (9). This is now a real Lie algebra. More concretely we can define the real algebra as the algebra $sl(2,R)$ of traceless 2×2 real matrices (naturally a real vector space). The generators in (8) are good ones for this algebra.

There is a way to construct another real algebra from $sl(2,C)$, this time take

$$\begin{gathered} 2\,S_1 = -i(J_+ + J_-) = -\,i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad 2\,S_2 = J_- - J_+ = -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \\ 2\,S_3 = -iJ_3 = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \end{gathered} \tag{10}$$

We recognize $S_k = -i\sigma_k/2$, with σ_k the (hermitian and traceless) Pauli matrices. The brackets have real structure constants

$$[\,S_i\,,\,S_j\,] = \epsilon_{ijk}\,S_k\,, \tag{11}$$

Declaring the S_k to be basis vectors of a real vector space we get the familiar simple real Lie algebra $su(2)$, described as the algebra of 2×2 traceless *antihermitian* matrices. Although the matrices have complex entries, the vector space is naturally real – complex multiplication ruins antihermiticity. The real algebras $sl(2,R)$ and $su(2)$ are *not* isomorphic over the reals, they are the two real forms associated with $sl(2,C)$.

The real forms of $sl(\ell+1,C)$ are $sl(\ell+1,R)$ and $su(\ell+1)$, defined as the algebra of traceless antihermitian $(\ell+1)\times(\ell+1)$ matrices (a compact subalgebra of $sl(\ell+1,C)$).

The real Lie algebra $u(\ell)$ is defined as the algebra of $\ell\times\ell$ antihermitian (complex) matrices. It has real dimension ℓ^2 (the associated Lie group is the unitary matrix group $U(\ell)$).

The real Lie algebra $usp(2\ell)$ is the algebra of antihermitian matrices in $sp(2\ell,C)$. Using (5) the antihermiticity gives

$$\begin{pmatrix} m & n \\ -n^\dagger & -m^t \end{pmatrix} \quad \text{with} \quad m^\dagger = -m,\; n^t = n\,. \tag{12}$$

The real dimension of $usp(2\ell)$ is $\ell(2\ell+1)$. The algebras $usp(2\ell)$ and $sp(2\ell,R)$ are two real forms of $sp(2\ell,C)$. Actually $usp(2\ell)$ is a compact real form while $sp(2\ell,R)$ is a non-compact real form.

REPRESENTATIONS

A **representation** of a Lie algebra $\mathcal{G}$ on a vector space V is a linear map $\psi : \mathcal{G} \to gl(V)$ which is a homomorphism of Lie-algebras ($\psi([x_1, x_2]) = [\psi(x_1), \psi(x_2)]$, for all $x_1, x_2 \in \mathcal{G}$). It is an **irreducible** representation if the image of $\mathcal{G}$ in $gl(V)$ acts on V without a proper invariant subspace.

Schur's lemma. Let $\psi : \mathcal{G} \to gl(V)$ be an irreducible representation on a finite dimensional complex vector space V. Let $\alpha : V \to V$ be a linear mapping commuting with $\psi(x)$ for all $x \in \mathcal{G}$. Then α is proportional to the identity map: $\alpha = aI$ for some number a. *Proof*: α must have one eigenvector in V with some eigenvalue a since $\det(\alpha - \lambda I) = 0$ must have at least one (complex) root $\lambda = a$). Then show that the set of vectors in V with α eigenvalue a form a nonvanishing invariant subspace. Conclude that the invariant subspace must be the whole V, so $\alpha(v) = av$ for any vector in V, which means α is a multiple of the identity.).

The **adjoint** representation of the Lie algebra is a representation where the vector space in question is precisely the Lie algebra: $V = \mathcal{G}$. We write ad : $\mathcal{G} \to gl(\mathcal{G})$ and define ad x as the linear map

$$\text{ad } x : y \to [x, y\,] \quad \text{or} \quad \text{ad } x(y) = [x, y\,]\,. \tag{13}$$

To verify it is a representation we must check that the linear maps satisfy

$$[\,\text{ad } x\,, \text{ad } y\,] = \text{ad } [x, y\,]\ . \tag{14}$$

This is verified acting on an element of the algebra and using the Jacobi identity.

The adjoint representation of a **simple** Lie algebra is irreducible for otherwise, by (13), the invariant subspace would be an ideal. For semisimple algebras the adjoint representation is reducible.

The **kernel** of ad is formed by the elements x of the algebra that generate the zero map, and therefore must have zero brackets with everything. So the kernel of ad coincides with the center $\mathcal{Z}$ of the Lie algebra. If $\mathcal{G}$ is simple its center vanishes and the map ad : $\mathcal{G} \to gl(\mathcal{G})$ is one to one; any simple Lie algebra $\mathcal{G}$ is isomorphic to a subalgebra of $gl(\mathcal{G})$.

Looking at the generators we have

$$\text{ad } T_a\,(T_b) = [T_a, T_b] = T_c\, f_{ab}{}^c\,. \tag{15}$$

To find the explicit matrix form of the adjoint action we view T_b as the column vector with all entries equal to zero except for a 1 in the b-th entry. Then $\mathrm{ad}\, T_a\,(T_b) = T_c(\mathrm{ad}\, T_a)_{cb}$ and we thus conclude that

$$(\mathrm{ad}\ T_a)_{cb} = f_{ab}{}^{c}\,. \tag{16}$$

An arbitrary representation r of $\mathcal{G}$ of dimension d_r is defined by $d_r \times d_r$ matrices t^r_a that represent the generators:

$$[\, t^r_a \,, t^r_b \,] = f_{ab}{}^{c}\, t^r_c\,. \tag{17}$$

On the Lie algebra there is a well defined symmetric bilinear form $\kappa\,(\cdot\,,\,\cdot)$ called the **Killing metric** and defined as (with $\circ$ denoting composition of linear maps)

$$\kappa\,(x, y) = -\mathrm{Tr}\,\Big(\mathrm{ad}\, x \circ \mathrm{ad}\, y\Big)\,. \tag{18}$$

$\kappa(x, y)$ is simply minus the trace of the operator $[\, x, [\, y, \cdot\,]\,]$. This metric is uniquely defined, up to a multiplicative constant, by two properties:
(i) invariance under automorphisms σ of the Lie algebra ($\sigma : \mathcal{G} \to \mathcal{G}$ is an automorphism if $[x, y] = z$ implies that $[\sigma(x), \sigma(y)] = \sigma(z)$. σ defines an element in $gl(\mathcal{G})$.)

$$\kappa(\sigma(x), \sigma(y)) = \kappa(x, y)\ , \tag{19}$$

(ii) associativity, in the sense that

$$\kappa(\, [x, y]\, , z\,) = \kappa\,(\, x, [y, z]\,)\,. \tag{20}$$

One readily verifies that (18) satisfies (19) (note that $\mathrm{ad}\,\sigma(x) = \sigma \circ \mathrm{ad}\, x \circ \sigma^{-1}$) and (20).

A fundamental result of Killing and Cartan is that a Lie algebra is **semisimple** if and only if its Killing form is **nondegenerate**. A bilinear form $\kappa\,(\cdot\,,\,\cdot)$ is degenerate if there is a non-zero vector x such that $\kappa\,(\,x\,, \cdot\,) = 0$. A particular case is readily verified; algebras with *abelian* ideals (thus not semisimple) have degenerate Killing forms. To see this split the algebra generators into two groups : those in the abelian ideal I and those outside I. With $x \in I$, $\kappa(x, y) = 0$ for all y or, equivalently, the trace of the operator $[x, [y, \cdot\,]\,]$ vanishes. This is clear because acting on elements of I this operator gives zero, and acting on elements outside I it gives elements inside I. Thus κ is degenerate.

A Lie algebra is said to be **compact semisimple** if the Killing form is **positive definite**, that is $\kappa(x, x) > 0$ for $x \neq 0$. The corresponding Lie group is a compact manifold.

While the product of elements $x, y \in \mathcal{G}$ is not defined, having representations $\psi(x), \psi(y)$ we can multiply them together and the Lie bracket becomes a commutator. This suggests the definition of the **universal enveloping algebra** $U(\mathcal{G})$ associated with the Lie algebra $\mathcal{G}$. It is the associative algebra spanned by monomials in the generators of $\mathcal{G}$, (say, $x, y, x^2, xy, yx, \cdots$) where monomials are identified upon use of the brackets as commutators (if $[x, y] = z$, we can say that in the enveloping algebra $xy = yx + z$).

A representation of a Lie **group** G on a vector space V is a map $G \to GL(V)$ which is a homomorphism of Lie groups. Here the **general linear group** $GL(V)$ is the group of *invertible* linear maps $V \to V$. Concretely, a representation r of dimension d_r a Lie group G is defined by an invertible $d_r \times d_r$ matrix $D^r(g)$ for each $g \in G$ such that

$$D^r(g_1)D^r(g_2) = D^r(g_1 g_2), \qquad \text{for all} \;\; g_1, g_2 \in \mathcal{G}\,. \tag{21}$$

Given a Lie group G with Lie algebra $\mathcal{G}$ the **adjoint representation of the group** is a map Ad: $G \to \text{Aut}(\mathcal{G})$ that associates to each element $g \in G$ an invertible linear transformation Ad g which is an automorphism of the Lie algebra $\mathcal{G}$:

$$\text{Ad}\, g\,[\,x\,,\,y\,] = [\,\text{Ad}\, g(x), \text{Ad}\, g\,(y)\,]\,. \tag{22}$$

It follows immediately from this and (19) that the Killing form is Ad-invariant:

$$\kappa\,(\,\text{Ad}\, g(x), \text{Ad}\, g(y)\,) = \kappa\,(x, y\,)\,. \tag{23}$$

As befits a representation, one must have

$$\text{Ad}\,(g_1 g_2) = \text{Ad}\,(g_1)\text{Ad}\,(g_2)\,. \tag{24}$$

The representation can be described concretely as (with repeated indices summed)

$$\text{Ad}\, g : T_a \;\; \to \;\; \text{Ad}\, g\,(T_a) = \; T_b\, \bar{D}_{ba}(g)\,, \tag{25}$$

where $\bar{D}_{ab}(g)$ denotes $\dim(\mathcal{G})\times\dim(\mathcal{G})$ matrix representation of $\text{Ad}\, g$. One readily checks that the above two equations imply that, as expected, $\bar{D}_{ab}(g_1 g_2) = \bar{D}_{ac}(g_1)\bar{D}_{cb}(g_2)$.

Geometrically one understands the adjoint group action on the algebra as follows. Recall that the Lie algebra $\mathcal{G}$ associated with a Lie group G can be identified with the tangent space of G at the identity. For any fixed $g \in G$ the transformation $h \to ghg^{-1}$ (conjugation by g)

is a map of G to itself leaving the identity element fixed and inducing a linear transformation from the tangent space at h to the tangent space at ghg^{-1}. Therefore, this map induces a linear transformation on the tangent space at the identity, a linear transformation on the Lie algebra[1].

The adjoint action can be described using any $\mathcal{G}$ representation and a G representation of the same dimensionality. Let t_a^r be matrices representing the generators T_a in a representation r of $\mathcal{G}$ and let $D^r(g)$ be matrices of the same dimension in the representation r of G. The action of Ad g is via conjugation with $D^r(g)$, and (25) gives

$$D^r(g)\, t_a^r\, D^r(g)^{-1} = t_b^r\, \bar{D}_{ba}(g)\ . \tag{26}$$

If the representation r of $\mathcal{G}$ is the adjoint, $D^r(g) = \bar{D}(g)$. Note that the $\bar{D}_{ba}(g)$ on the above right-hand side are numbers, not matrices.

For a matrix group G whose elements $V \in G$ are $k \times k$ matrices, the generators T_a of the associated Lie algebra $\mathcal{G}$ are themselves $k \times k$ matrices (this is called the fundamental representation) and (25) and (26) read

$$\mathrm{Ad}\, V(T_a) = V\, T_a V^{-1} = T_b\, \bar{D}_{ba}(V)\,. \tag{27}$$

COMPACT SEMISIMPLE LIE ALGEBRAS

Consider again the Killing metric

$$\kappa_{ab} \equiv \kappa\,(T_a, T_b) = -\mathrm{Tr}\,(\,\mathrm{ad}\, T_a \circ \mathrm{ad}\, T_b\,) = -\mathrm{Tr}\,(\,\bar{t}_a\, \bar{t}_b\,)\,. \tag{28}$$

where $\bar{t}_a$ denotes the matrix representation of the adjoint action (ad T_a). By (16) we have $(\bar{t}_a)_{cb} = f_{ab}{}^c$. The metric is real since the structure constants are real. We note that

$$-\mathrm{Tr}\,(\,[\,\bar{t}_a\,,\, \bar{t}_b\,]\; \bar{t}_c\,) = f_{ab}{}^e\, \kappa_{ec}\,. \tag{29}$$

Cyclicity of the trace implies that the $f_{ab}{}^e\, \kappa_{ec}$ is totally antisymmetric in a, b and c.

[1]More precisely the Lie Algebra $\mathcal{G}$ associated with a group G is defined as the set of all left-invariant vector fields on the group manifold. The bracket is the Lie bracket of vector fields, viewed as differential operators. As a vector space, the set of left-invariant vector fields is isomorphic to the tangent space to the group at the identity (each tangent vector at the identity can be extended to a left-invariant vector field). By left-invariant one means invariant under diffeomorphisms of the group induced by left multiplication: $g \to ag$, $\forall g \in G$.

Since the matrix κ_{ab} is symmetric, real, and positive definite ($\mathcal{G}$ is compact semisimple)[2] there is an real orthogonal $\mathcal{O}$ matrix that diagonalizes κ, namely $\mathcal{O}\kappa\,\mathcal{O}^T = \mathcal{D}$, with $\mathcal{D}$ a diagonal matrix with positive entries. We can define new generators $T'_a = \mathcal{O}_{ac}T_c$, and then we verify that

$$-\text{Tr}\,(\ \bar{t}'_a\,\bar{t}'_b\ ) = \mathcal{D}_{ab}\,.$$

By a further (real) scaling all diagonal elements can be made equal to a constant denoted as $C(\mathcal{G})$. In the resulting basis, which we call again T_a, we have

$$-\text{Tr}\,(\,\bar{t}_a\,\bar{t}_b\,) = C(\mathcal{G})\,\delta_{ab}\ = \kappa_{ab}\,. \tag{30}$$

In this basis we write $[T_a, T_b] = f_{ab}{}^c\,T_c$. Since $f_{ab}{}^e\kappa_{ec}$ is totally antisymmetric, (30) implies that $f_{ab}{}^e\delta_{ec} = f_{ab}{}^c$ is totally antisymmetric. We thus define a totally antisymmetric symbol

$$f_{abc} \equiv f_{ab}{}^c\,, \tag{31}$$

and write

$$[T_a, T_b] = f_{abc}\,T_c\,, \qquad (\text{ad}\ T_a)_{bc} = -f_{abc}\,. \tag{32}$$

We have shown that for compact semisimple real Lie algebras there is a basis in which equations (30) and (32) hold.

We now verify that the adjoint representation of G acts on $\mathcal{G}$ via real orthogonal matrices. For this we begin with (30), insert $\bar{D}$ matrices, and use (26) applied to the adjoint:

$$\begin{aligned} -C(\mathcal{G})\delta_{ab} &= \text{Tr}(\bar{t}_a\,\bar{t}_b) \\ &= \text{Tr}\,\Big(\bar{D}(g)\,\bar{t}_a\,\bar{D}^{-1}(g)\bar{D}(g)\,\bar{t}_b\bar{D}^{-1}(g)\,\Big) \\ &= \text{Tr}\,\Big(\bar{t}_e\,\bar{t}_f\,\Big)\bar{D}_{ea}(g)\bar{D}_{fb}(g) \\ &= -C(\mathcal{G})\bar{D}_{ea}(g)\bar{D}_{eb}(g)\,, \end{aligned} \tag{33}$$

showing that the matrix $\bar{D}(g)$ is indeed orthogonal ($\bar{D}^T\bar{D} = 1$).

For arbitrary representations of the Lie algebra we define the matrix $\kappa\,(r)$

$$\kappa_{ab}(r) \equiv\ -\text{Tr}\,(\,t^r_a\,t^r_b\,)\,. \tag{34}$$

[2]Positivity also follows if the matrices in the adjoint representation of $\mathcal{G}$ are antihermitian: any $x \in \mathcal{G}$ is represented in the adjoint by an $X(= -X^\dagger)$ and $\kappa(x, x) = -\text{Tr}(XX) = \text{Tr}(X^\dagger X) \geq 0$.

Inserting group matrices $D^r(g)$ as in (33) and using (26) we see that

$$\kappa_{ab}(r) = \kappa_{cd}(r)\, \bar{D}_{ca}(g)\bar{D}_{db}(g) \quad \rightarrow \quad \kappa(r) = \bar{D}^T(g)\, \kappa(r)\bar{D}(g)\,.$$

Since $\bar{D}(g)$ is orthogonal it follows that $\kappa(r)$ commutes with all matrices $\bar{D}(g)$. For simple algebras the adjoint representation of the corresponding group acts irreducibly. By Schur's lemma this means that $\kappa(r)$ must be proportional to the identity. We thus define

$$-\mathrm{Tr}\,(\, t^r_a\, t^r_b\,) = C(r)\, \delta_{ab}\,. \tag{35}$$

In the chosen basis, the element T_aT_a (a summed!) of the enveloping algebra is called a **Casimir operator**. It commutes with all T_a's and therefore with all elements of the enveloping algebra. Indeed, with repeated indices summed,

$$[\, T_aT_a\,,\, T_c] = T_a\,(f_{acb}T_b) + \,(f_{acb}\,T_b)T_a = (f_{acb} - f_{bca})T_aT_b = 0\,. \tag{36}$$

More geometrically, for any semisimple algebra $\kappa^{ab}T_aT_b$ is a Casimir, where the inverse κ^{ab} of the Killing metric κ_{ab} exists because of semisimplicity.

Since T_aT_a is a Casimir, in any irreducible representation r the Casimir matrix $t^r_a\, t^r_a$ commutes with all the matrices t^r_a representing generators. The Casimir matrix, by Schur's lemma, must be a multiple of the identity in *any* irreducible representation:

$$-t^r_a\, t^r_a = C_2(r)\, I_{d(r)}\,, \tag{37}$$

where $d(r)$ denotes the dimension of the representation r. The constants $C(r)$ and $C_2(r)$ are simply related; taking the trace of (37) and the contraction of (30) one finds

$$C_2(r)\,\dim(r) = C(r)\,\dim(\mathcal{G})\,. \tag{38}$$

For the adjoint representation (37) gives

$$f_{acd}\, f_{bcd} = C_2(\mathcal{G})\delta_{ab} \;\rightarrow\; C_2(\mathcal{G}) = \frac{1}{\dim\,(\mathcal{G})}\; f_{abc}\, f_{abc}\,, \tag{39}$$

and (38) implies that $C_2(\mathcal{G}) = C(\mathcal{G})$. As defined, the constants C and C_2 are basis dependent.

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