Lecture 9 (Oct. 4, 2017)

9.1 Spin-$\frac{1}{2}$ in an AC Field

Consider a spin-$\frac{1}{2}$ system in a time-dependent magnetic field. The Hamiltonian is

$$H = -\frac{ge}{2m}\boldsymbol{S} \cdot \boldsymbol{B}(t)\,. \tag{9.1}$$

We will consider a particular class of time-dependent magnetic fields, which we can write in the form

$$\boldsymbol{B}(t) = V_0\hat{\boldsymbol{z}} + B_1(\cos(\omega t)\hat{\boldsymbol{x}} + \sin(\omega t)\hat{\boldsymbol{y}})\,. \tag{9.2}$$

We have previously studied the case where $B_1 = 0$.

In the case $B_1 = 0$, there are two energy eigenstates — the spin-up and spin-down states along the z-axis — that have an energy splitting of

$$\Delta E = \left|\frac{geB_0\hbar}{2m}\right|. \tag{9.3}$$

Now that we are considering a field with nonzero B_1, we expect the system to be able to absorb energy from the oscillating component of the field to transition between the two energy eigenstates of the static field.

We will analyze this problem in the interaction picture. A seemingly sensible choice is to regard the B_0 term of the Hamiltonian to be H_0, the part of the Hamiltonian for which the resulting time evolution is already understood. We write

$$H = H_0 + V(t)\,, \tag{9.4}$$

where

$$\begin{aligned} H_0 &= -\frac{ge}{2m}S^z B\,, \\ V(t) &= -\frac{ge}{2m}\boldsymbol{S}_\perp \cdot \boldsymbol{B}_\perp(t)\,. \end{aligned} \tag{9.5}$$

Here, we have defined

$$\begin{aligned} \boldsymbol{S}_\perp &:= S^x\hat{\boldsymbol{x}} + S^y\hat{\boldsymbol{y}}\,, \\ \boldsymbol{B}_\perp &:= B_1(\cos(\omega t)\hat{\boldsymbol{x}} + \sin(\omega t)\hat{\boldsymbol{y}})\,. \end{aligned} \tag{9.6}$$

We define

$$\omega_0 = \left|\frac{geB_0}{2m}\right|, \tag{9.7}$$

so that we have

$$H_0 = \omega_0 S_z\,. \tag{9.8}$$

(Note that we are taking $e < 0$ so that the signs work out here.) The time-evolution operator due to this part of the Hamiltonian is

$$U_0(t) = e^{-iH_0t/\hbar}\,. \tag{9.9}$$

The equation of motion in the interaction picture is

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi_\mathrm{I}(t)\rangle = V_\mathrm{I}(t)|\psi_\mathrm{I}(t)\rangle\,, \tag{9.10}$$

where

$$\begin{aligned} V_{\mathrm{I}}(t) &= U_0^\dagger(t)V(t)U_0(t) \\ &= e^{i\omega_0 S^z t/\hbar}V(t)e^{-i\omega_0 S^z t/\hbar} \\ &= -\frac{geB_1}{2m}e^{i\omega_0 S^z t/\hbar}(S^x\cos(\omega t)+S^y\sin(\omega t))e^{-i\omega_0 S^z t/\hbar}\,. \end{aligned} \tag{9.11}$$

Before evaluating this directly, it is convenient to define

$$S^\pm = S^x \pm iS^y\,. \tag{9.12}$$

These are called the *ladder operators*. Note that they are Hermitian conjugates of one another. We can now compute

$$\begin{aligned} e^{i\phi\sigma^z/2}S^+e^{-i\phi\sigma^z/2} &= \left(\cos\frac{\phi}{2}+i\sigma^z\sin\frac{\phi}{2}\right)S^+\left(\cos\frac{\phi}{2}-i\sigma^z\sin\frac{\phi}{2}\right) \\ &= S^+\left(\cos\frac{\phi}{2}-i\sigma^z\sin\frac{\phi}{2}\right)^2 \\ &= S^+(\cos\phi - i\sigma^z\sin\phi) \\ &= S^+e^{i\phi}\,. \end{aligned} \tag{9.13}$$

Here, in the first line we have expanded the exponentials, using the fact that the Pauli matrices square to the identity. In the second line, we have used the fact that σ^z anticommutes with both σ^x and σ^y. Note that

$$\sigma^x + i\sigma^y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}. \tag{9.14}$$

This is why these operators are called ladder operators (S^+ specifically is the *raising operator*). Taking the Hermitian conjugate of Eq. (9.13), we have

$$e^{i\phi\sigma^z/2}S^-e^{-i\phi\sigma^z/2} = S^-e^{-i\phi}\,. \tag{9.15}$$

Now we can return to the calculation of $V_{\mathrm{I}}(t)$. We have

$$\begin{aligned} V_{\mathrm{I}}(t) &= -\frac{geB_1}{2m}e^{i\omega_0 S^z t/\hbar}\left(\frac{S^+e^{i\omega t}+S^-e^{-i\omega t}}{2}\right)e^{-i\omega_0 S^z t/\hbar} \\ &= -\frac{geB_1}{4m}\left(S^+e^{i(\omega_0+\omega)t}+S^-e^{-i(\omega_0+\omega)t}\right). \end{aligned} \tag{9.16}$$

9.1.1 Resonant Drive

First, let's consider the simple case where the external frequency is $\omega = -\omega_0$. This case is called "resonant drive." In this case, the potential in Eq. (9.16) simplifies to

$$V_{\mathrm{I}} = -\frac{geB_1}{4m}\left(S^+ + S^-\right) = -\frac{geB_1}{4m}S^x\,, \tag{9.17}$$

which is time-independent. The interaction picture time-evolution operator is then

$$U_{\mathrm{I}}(t) = e^{i\frac{geB_1}{2m\hbar}S^x t}\,. \tag{9.18}$$

This operator rotates states about the x-axis of spin space by an angle $\frac{geB_1}{2m\hbar}t$. The frequency of rotation,

$$\omega_{\mathrm{R}} = \left|\frac{geB_1}{2m}\right|, \tag{9.19}$$

is called the *Rabi frequency.* Note that the time-evolution operator,

$$U_{\rm I}(t) = e^{-i\omega_{\rm R} S^x t/\hbar} = e^{-i\left(\frac{\omega_{\rm R} t}{2}\right)\sigma^x}\,, \tag{9.20}$$

does not oscillate with a frequency of $\frac{2\pi}{\omega_{\rm R}}$, but rather $\frac{4\pi}{\omega_{\rm R}}$,

$$U_{\rm I}\left(t + \frac{4\pi}{\omega_{\rm R}}\right) = U_{\rm I}(t)\,. \tag{9.21}$$

This operator has periodicity twice that of any observables in the system. What happens after a time $\frac{2\pi}{\omega_{\rm R}}$? At this time, we have

$$U_{\rm I}\left(\frac{2\pi}{\omega_{\rm R}}\right) = e^{-i\pi\sigma^x} = -\mathbb{1}\,. \tag{9.22}$$

Thus, any observables will oscillate at the Rabi frequency, but if we were able to measure the phase of a state, we would find that it oscillates at half the Rabi frequency.

Imagine that we start in a spin-up state, i.e.,

$$|\psi_{\rm I}(0)\rangle \to \begin{pmatrix}1\\0\end{pmatrix}, \tag{9.23}$$

where the arrow indicates that we are representing the state as a vector. At time t, we then have

$$\begin{aligned}
|\psi_{\rm I}(t)\rangle &\to e^{-i\omega_{\rm R} t\sigma^x/2}\begin{pmatrix}1\\0\end{pmatrix}\\
&= \left(\cos\left(\frac{\omega_{\rm R} t}{2}\right) - i\sigma^x \sin\left(\frac{\omega_{\rm R} t}{2}\right)\right)\begin{pmatrix}1\\0\end{pmatrix}\\
&= \begin{pmatrix}\cos\left(\frac{\omega_{\rm R} t}{2}\right) & -i\sin\left(\frac{\omega_{\rm R} t}{2}\right)\\ -i\sin\left(\frac{\omega_{\rm R} t}{2}\right) & \cos\left(\frac{\omega_{\rm R} t}{2}\right)\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\\
&= \begin{pmatrix}\cos\left(\frac{\omega_{\rm R} t}{2}\right)\\ -i\sin\left(\frac{\omega_{\rm R} t}{2}\right)\end{pmatrix}.
\end{aligned} \tag{9.24}$$

Thus, after a time $t = \frac{\pi}{\omega_{\rm R}}$, we have

$$|\psi_{\rm I}(\pi/\omega_{\rm R})\rangle \to \begin{pmatrix}0\\-i\end{pmatrix}, \tag{9.25}$$

and after a time $t = \frac{2\pi}{\omega_{\rm R}}$, we have

$$|\psi_{\rm I}(2\pi/\omega_{\rm R})\rangle \to \begin{pmatrix}-1\\0\end{pmatrix}, \tag{9.26}$$

where we see that the state has picked up a phase of π. We find that the state oscillates; this behavior is referred to as *Rabi oscillation.*

9.1.2 Off-Resonant Drive

Up until now, we have been working in the case of resonant drive. What happens if we're off resonance, i.e., $\omega \neq -\omega_0$? In this case, even though we went to the interaction picture, we still have a time-dependent Hamiltonian that we don't know how to solve, because $V_{\rm I}(t)$ is still time-dependent. It seems like we have gained nothing.

The issue is that we made a particular choice for which part of the Hamiltonian H would be H_0, but this was not actually the most convenient choice. Let's instead choose

$$H = \tilde{H}_0 + \tilde{V}(t)\,, \tag{9.27}$$

where

$$\begin{aligned}\tilde{H}_0 &= \omega S^z\,,\\ \tilde{V}(t) &= -\gamma(B_{\text{eff}}S^z + \boldsymbol{B}_\perp(t)\cdot\boldsymbol{S}_\perp)\,.\end{aligned} \tag{9.28}$$

Here, we have defined

$$\begin{aligned}B_{\text{eff}} &:= B_0 + \frac{\omega}{\gamma}\,,\\ \gamma &:= \frac{ge}{2m}\,.\end{aligned} \tag{9.29}$$

Now, let's work in the interaction picture with these choices. Going to the interaction picture is going to a rotating frame, that rotates with $\tilde{H}_0$. Previously, we went to a frame that rotated about the z-axis at frequency ω_0; now we are going to a frame that rotates about the z-axis at the external frequency ω.

We compute

$$\begin{aligned}\tilde{V}_{\text{I}}(t) &= \tilde{U}_0^\dagger(t)\tilde{V}(t)\tilde{U}_0(t)\\ &= e^{i\omega S^z t/\hbar}\left[-\gamma B_{\text{eff}}S^z - \frac{\gamma B_1}{2}\left(S^+e^{i\omega t} + S^-e^{-i\omega t}\right)\right]e^{-i\omega S^z t/\hbar}\,.\end{aligned} \tag{9.30}$$

The first term in brackets is unaffected by the time evolution, because S^z commutes with the exponentials outside the brackets. The second term in brackets, along with the time-evolution exponentials, looks exactly like the calculation we did in Eq. (9.16), but with the frequencies in all of the exponentials exactly matched. Thus, we find

$$\tilde{V}_{\text{I}}(t) = -\gamma B_{\text{eff}}S^z - \gamma B_1 S^x\,. \tag{9.31}$$

This is time-independent, and so we now declare the problem solved. The spin precesses about the direction of

$$\tilde{\boldsymbol{B}} = (B_1, 0, B_{\text{eff}})\,. \tag{9.32}$$

These are modified Rabi oscillations, with a frequency of

$$\omega_{\text{R}} = \gamma\sqrt{B_{\text{eff}}^2 + B_1^2}\,. \tag{9.33}$$

9.2 Path Integral Formulation of Quantum Mechanics

Time evolution in the Schrödinger picture is given by

$$|\psi(t)\rangle = \sum_{a'} c_{a'}(t)\big|a'\big\rangle\,, \tag{9.34}$$

where the $|a'\rangle$ form an energy eigenbasis, and

$$c_{a'}(t) = e^{-iE_{a'}(t-t_0)/\hbar}c_{a'}(t_0)\,. \tag{9.35}$$

We can consider these particles in position space, and define

$$u_{a'}(\boldsymbol{x}) = \big\langle\boldsymbol{x}\big|a'\big\rangle \tag{9.36}$$

to be the energy eigenfunctions. Then the position-space wavefunction is given by

$$\psi(\boldsymbol{x},t) = \sum_{a'} e^{-iE_{a'}(t-t_0)/\hbar} c_{a'}(t_0) u_{a'}(\boldsymbol{x})\,. \tag{9.37}$$

We can rewrite this as

$$\psi(\boldsymbol{x},t) = \int_{-\infty}^{\infty} \mathrm{d}^d x'\; K\big(\boldsymbol{x},t;\boldsymbol{x}',t_0\big)\psi\big(\boldsymbol{x}',t_0\big)\,, \tag{9.38}$$

with

$$K\big(\boldsymbol{x},t;\boldsymbol{x}',t_0\big) = \sum_{a'} \big\langle \boldsymbol{x}\big|a'\big\rangle e^{-iE_{a'}(t-t_0)/\hbar} \big\langle a'\big|\boldsymbol{x}'\big\rangle = \big\langle \boldsymbol{x}\big|U(t,t_0)\big|\boldsymbol{x}'\big\rangle\,. \tag{9.39}$$

The object $K(\boldsymbol{x},t;\boldsymbol{x}',t_0)$ is called the *propagator*.