Suppl. notes, part I $\qquad$ 8.311, 2004

Field oscillators

$$S = \int d^4x \; \frac{1}{8\pi}(E^2 - B^2) + \frac{1}{c}\vec{j}\cdot\vec{A} - \rho\varphi$$

$$S = \int d^4x \frac{1}{8\pi}\left[\left(-\frac{1}{c}\frac{\partial A}{\partial t} - \nabla\varphi\right)^2 - (\nabla\times A)^2\right] + \frac{1}{c}jA - \rho\varphi$$

$$\frac{\delta S}{\delta A} = 0, \; \frac{\delta S}{\delta\varphi} = 0 \rightarrow \text{Maxwell eqs.}$$

to construct field oscillators, go to Coulomb gauge

$$A' = A + \nabla\chi \qquad \text{want } \underline{\vec{\nabla}\cdot\vec{A}' = 0}$$

$$\varphi' = \varphi - \frac{1}{c}\frac{\partial\chi}{\partial t} \qquad \text{use } \chi = -\Delta^{-1}(\nabla\cdot A), \quad \Delta = \nabla^2$$

Now, $\int d^4x \left(-\frac{1}{c}\frac{\partial A}{\partial t} - \nabla\varphi\right)^2 = \int d^4x \left(\left(\frac{\partial A}{c\partial t}\right)^2 + (\nabla\varphi)^2\right)$

(no cross-term in Coulomb gauge)

$$S = \int d^4x \left\{\left[\frac{1}{8\pi c^2}\left(\frac{\partial A}{\partial t}\right)^2 - \frac{1}{8\pi}(\nabla\times A)^2 + \frac{1}{c}\vec{j}\vec{A}\right] + \left[\frac{1}{8\pi}(\nabla\varphi)^2 - \rho\varphi\right]\right\}$$

$\underline{\varphi\text{-part}}$ $\quad \frac{\delta S}{\delta\varphi} = 0 \rightarrow \nabla^2\varphi = -4\pi\rho$ (no time delay!)

$$\int dx^4\left(\frac{1}{8\pi}(\nabla\varphi)^2 - \rho\varphi\right) \rightarrow -\frac{1}{2}\int dt \int d^3r_1 \, d^3r_2 \, \frac{\rho_1\rho_2}{|r_1 - r_2|}$$

$\underline{A\text{-part}}$ $\quad$ Expand $\vec{A}(r,t)$ in orthogonal functions:

(i) $$\vec{A}(r,t) = \sum_\lambda q_\lambda(t)\frac{\sqrt{8\pi}\,c}{\sqrt{V}}\vec{e}_\lambda \begin{cases}\cos \vec{k}_\lambda\cdot\vec{r} \\ \sin \vec{k}_\lambda\cdot\vec{r}\end{cases}$$

$\vec{\nabla}\cdot\vec{A} = 0 \rightarrow \vec{e}_\lambda\cdot\vec{k}_\lambda = 0$ (two $\vec{e}_\lambda$ for each $\vec{k}_\lambda$)

Volume $V = L\times L\times L$ (Assume $\underline{\text{very}}$ large L!)

periodic boundary conds. $\vec{k}_\lambda = \frac{2\pi}{L}(n_x, n_y, n_z)$ integers

$\frac{1}{\sqrt{V}}$ is the normalization factor

$$\sum_\lambda = \sum_{n_x, n_x, n_z, \{\cos, \sin\}, \vec{e}_\lambda}$$

This is expansion of the field $\vec{A}$ in normal modes:

$$\vec{A}_{(r,t)} = \sum_{\lambda, i} q_{\lambda i}(t)\, \vec{A}_{\lambda i}(r) \qquad i = 1, 2$$

$$\vec{A}_{\lambda,1}(r) = \frac{\sqrt{8\pi c}\,\vec{e}}{\sqrt{V}} \cos(\vec{k}_\lambda \vec{r})$$

$$\vec{A}_{\lambda,2}(r) = \frac{\sqrt{8\pi c}\,\vec{e}_\lambda}{\sqrt{V}} \sin(\vec{k}_\lambda \vec{r})$$

$$\int \vec{A}_{\lambda, i} \cdot \vec{A}_{\mu, j}\, d^3 r = 4\pi c^2 \delta_{\lambda\mu} \delta_{ij} \qquad \text{(orthogonality)}$$

$$\int d^3 r \left(\frac{1}{8\pi c^2} \left(\frac{\partial A}{\partial t}\right)^2 - \frac{1}{8\pi} (\nabla \times A) \right) = \sum_{\lambda, i} \left(\frac{1}{2} \dot{q}_{\lambda i}^2 - \frac{\omega_\lambda^2}{2} q_{\lambda i}^2 \right), \qquad \omega_\lambda = c|k_\lambda|$$

$$\int d^3 r\, \frac{1}{c} \vec{j} \vec{A} = \sum_{\lambda i} q_{\lambda i}(t)\, \frac{e\sqrt{8\pi}}{\sqrt{V}}\, \vec{e}_\lambda \cdot \vec{v}(t) \begin{cases} \cos k_\lambda r(t) \\ \sin k_\lambda r(t) \end{cases}$$

$$\vec{j}(r,t) = e\vec{v}(t)\, \delta^3(r - r(t)) \leftarrow$$ relativistic current of point charge

The action takes form:

Oscillator representation

$$S = \sum_{\lambda i} \int dt \left[\frac{1}{2} \dot{q}_{\lambda i}^2 - \frac{\omega_\lambda^2}{2} q_{\lambda i}^2 + q_{\lambda i} f_{\lambda i}(t) \right]$$

$$f_{\lambda i} = e\sqrt{\frac{8\pi}{V}}\, \vec{e}_\lambda \cdot \vec{v}(t) \begin{cases} \cos \vec{k}_\lambda \cdot \vec{r}(t) \\ \sin \vec{k}_\lambda \cdot \vec{r}(t) \end{cases}, \qquad \omega_\lambda = c|k_\lambda|$$

Dynamics of the field:

$$\frac{\delta S}{\delta q_{\lambda i}} = 0 \quad \rightarrow \quad \ddot{q}_{\lambda i} + \omega_{\lambda i}^2 q_{\lambda i} = f_{\lambda i}(t)$$

Green's function for oscillator:

$$f(t) = \delta(t - \tau)$$

$$q(t) = \begin{cases} \frac{1}{\omega} \sin \omega(t - \tau), & t > \tau \\ 0, & t < \tau \end{cases}$$

Any $f(t) \rightarrow q(t) = \int_{-\infty}^{t} dt' \frac{1}{\omega} \sin \omega(t - t')\, f(t')$

Solved all radiation problems! (in principle...)

given motion of charge → find $f_{\lambda i}(t) \to q_{\lambda i}(t) \to \vec{A}(r,t)$

- Approach ignores back effect of radiated fields on the charge

- Could have used other normal mode representations

e.g., $A(r,t) = \sum_\lambda q_\lambda(t) \sqrt{\frac{4\pi c^2}{V}}\, \vec{e}_\lambda\, e^{i k_\lambda r}$

then $q_\lambda(t)$ are complex: $q_\lambda(t) = q^*_{\lambda'}(t)$, $\vec{k}_{\lambda'} = -\vec{k}_\lambda$

Energy of the field: $E = \int d^3r \frac{1}{8\pi}(E^2 + B^2)$

in repres. (i), $E = \sum_{\lambda,i}\left(\frac{1}{2}\dot{q}_{\lambda i}^2 + \frac{\omega_\lambda^2}{2} q_{\lambda i}^2\right)$

Momentum of the field $\vec{P} = \int d^3r \frac{1}{4\pi c}\, \vec{E}\times\vec{B}$

$\vec{P} = \sum_\lambda \# \vec{k}_\lambda\; q_{\lambda i}(t)\, \dot{q}_{\lambda j}(t)\, \epsilon_{ij}$

Example Non-relativistic dipole radiation

point charge $r(t) = a\hat{z}\sin\omega t \Rightarrow \cos k_\lambda r(t) = 1$

$V(t) = a\omega\hat{z}\cos\omega t \quad \uparrow \quad \sin k_\lambda r(t) = 0$

$a\omega \ll c$

$\ddot{q}_\lambda + \omega_\lambda^2 q_\lambda = f_\lambda \begin{cases} \cos\omega t, & t>0 \\ 0, & t<0 \end{cases}$

$f_\lambda = ea\omega\sqrt{\frac{8\pi}{V}} \begin{cases} 0, & \vec{e}_\lambda \parallel \vec{k}\times\hat{z} \\ \sin\theta_\lambda \end{cases}$

$\vec{e}_\lambda \cdot \hat{z}$

$q_\lambda(t) = \frac{f_\lambda}{\omega_\lambda^2 - \omega^2}(\cos\omega t - \cos\omega_\lambda t)$

radiated energy $E_{osc} = \sum_\lambda\left(\frac{1}{2}\dot{q}_\lambda^2 + \frac{\omega_\lambda^2}{2} q_\lambda^2\right) \underset{\uparrow}{=} \sum_\lambda \int_0^t f_\lambda(t')\,\dot{q}_\lambda\, dt'$

work by f_λ

$E_{osc} = \sum_\lambda \frac{f_\lambda^2}{\omega_\lambda^2 - \omega^2} \int_0^t \cos\omega t'(\omega_\lambda \sin\omega_\lambda t' - \omega\sin\omega t')\,dt'$

$$E_{osc} = \sum_\lambda \frac{f_\lambda^2}{\omega_\lambda^2 - \omega^2}\left(\frac{\omega_\lambda}{2}\left[\underbrace{\frac{1-\cos(\omega_\lambda-\omega)t}{\omega_\lambda - \omega}}_{\text{resonance}} + \underbrace{\frac{1-\cos(\omega_\lambda+\omega)t}{\omega_\lambda+\omega}\right] + \frac{1}{2}(\cos^2\omega t - 1)}_{\text{non. resonance}}\right)$$

resonance ≡ diverges @ $\omega_\lambda \to \omega$, gives rise to $E(t)$ growing in time, i.e. to radiation

non. resonance ≡ no divergence when $\omega_\lambda \to \omega$, describes transition (field settling down) in the near zone

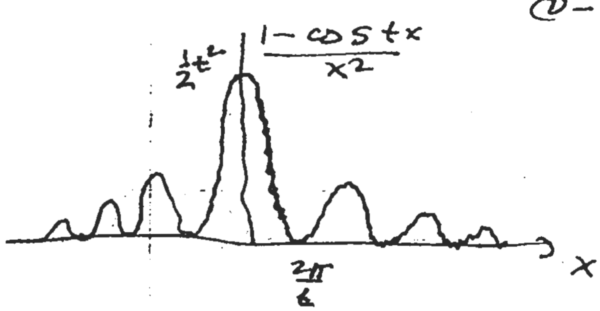
Thus $E^{rad}(t) = \sum_\lambda E_\lambda(t)$, $E_\lambda^{rad} = \dfrac{f_\lambda^2 \omega_\lambda}{2(\omega_\lambda^2 - \omega^2)} \dfrac{1-\cos(\omega_\lambda-\omega)t}{\omega_\lambda - \omega}$

$$\sum_\lambda \ldots = \frac{1}{2}\int_\pi \frac{V d^3k}{(2\pi)^3} \ldots \underset{\omega_\lambda = \nu = c|k_\lambda|}{=} V \int \frac{\nu^2 d\nu\, d\Omega}{2(2\pi c)^3} \ldots$$

$k_x \geq 0$ because modes are cos & sin

$$E^{rad}(t) = \frac{V}{2(2\pi c)^3} \int \nu^2 d\nu\, d\Omega \, \frac{f_\nu^2 \nu}{2(\nu+\omega)} \, \frac{1-\cos(\nu-\omega)t}{(\nu-\omega)^2}$$

replace $\dfrac{1-\cos(\nu-\omega)t}{(\nu-\omega)^2}$ by $\pi t\, \delta(\nu - \omega)$:



$$\int \frac{1-\cos xt}{x^2}\, dx = \pi t$$

$$E_{rad}(t) = \frac{Vt\pi}{2(2\pi c)^3} \int d\Omega\, \nu^2 d\nu \, \frac{f_\nu^2}{4}\, \delta(\nu-\omega)$$

substitute f_ν

$$P = \frac{dE_{rad}}{dt} = \int d\Omega \, \frac{\pi V \nu^2}{2(2\pi c)^3} \, \frac{e^2 a^2 \omega^2 \frac{8\pi}{V} \sin^2\theta}{4} \Bigg|_{\nu=\omega}$$

$$P = \int d\Omega \, \frac{e^2 \omega^4 a^2}{8\pi c^3} \sin^2\theta$$

$$\frac{dP}{d\Omega} = \frac{c}{8\pi} k^4 a^2 e^2 \sin^2\theta \quad \text{(agrees w. Jackson)}$$