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PROFESSOR: OK, starting again. I want to begin with a quick review of what we said last time. This time will be quicker than usual because we're not really continuing from there, we'll be starting a completely new topic today. But I thought I'd remind you, nonetheless, that we had a last lecture-- the universe was not created between then and now.

And what we talked about last time was the spacetime geodesic equation, which is written this way. It's a completely identical to the equation we derived for geodesics in pure spatial situations. The only difference really, is a difference in notation. Instead of using i and j the tradition is to use mu, nu, et cetera for the spacetime indices which are sum from 0 to 3, where x super 0 is identical to t. And in addition, the parameter which is sometimes called s when we're talking about space, is called tau when we're talking about time because what the parameter refers to is the proper time along the trajectory of the object whose geodesic we're calculating.

We then introduced the Schwarzschild metric, which we did not derive, but we claimed it describes the metric for any spherically symmetric mass distribution as long as you're talking about the region outside where the masses are located. And here M is the total mass of the object, which is the only thing the metric really depends on, G is Newton's constant, and C is of course the speed of light. The metric has an interesting feature that the coefficients of both the dt squared term and to dr squared term becomes singular either 0 or infinity, depending on which one you're looking at, a particular value of the radius called the Schwarzschild radius, given by this formula 2 GM divided by c squared. The bigger the mass the bigger the Schwarzschild radius, they're proportional to each other.

The metric is singular at those points, but I mentioned but did not prove, that that particular singularity is in fact what is referred to as a coordinate singularity. It's a singularity that's there only because of the way the coordinates were chosen. So there are other ways of choosing the coordinates where that singularity disappears. There's also singularity at r equals zero, and that singularity is real, there's no way to remove that singularity by a change of coordinates.

However, although r sub s is not a true singularity, it is a horizon. And by that we mean that if any particle or even a light beam gets inside the Schwarzschild radius, it can never get out. There's no geodesic which will take it out of the horizon. And it's not even a matter of geodesics, there are no time-like paths even if you have a rocket which would then not follow geodesic. There's no way to get out from inside a black hole. We didn't show that, but that fact is claimed.

Then we calculated the geodesic for our radially falling object. We solve the problem of an object that is released from rest at some initial value r sub 0, and just let fall straight down towards the center of the sphere. And the the equation describing the geodesic is just a special case of the general equation that we had a few slides ago. And we need only look at the radial component if we want to track how the radius changes with time. So there was a free index mu in the generic form of the equation, we're setting mu equal to the r variable. And then the equation reduces to this form and we know what these G sub t t's and G sub r r's are, they come from the equation for the Schwarzschild metric on the previous slide.

And that equation can be manipulated and eventually it simplifies to something extraordinarily simple. It's just the statement that d squared r d tau squared is equal to minus GM over r squared, which looks exactly like the Newtonian equation for something falling in a spherically symmetric gravitational field. But it's not really the same equation, it just looks like it's the same equation, because the variables both a different meanings. r and tau both have different meanings from the r and t that would have appeared in the Newtonian calculation. The r variable that appears here is not really the distance from the origin. If you wanted to know the distance from the origin you'd have to integrate the metric singular even-- if not even a well defined distance to the origin because the origin singular. And the tau that appears here is a time variable, but it's not the time that would be read on any fixed clock, rather it's the time that would be read on the wristwatch of the person falling into the spherically symmetric object, which we might consider to be a black hole.

And we were able to solve this equation by using essentially conservation of energy techniques, or at least what would be called conservation of energy if we were doing the Newtonian version of the problem, which is the same equation even though the variables have a different interpretation. So we were able to calculate not r as a function of tau, but at least tau a function of r. And we got that equation, which is a little complicated, but the interesting thing about it is that it gives finite answers for every value of r going all the way down to r equals zero. So it means that in a finite amount of time, as seen by the person falling into the black hole, the person would reach r equals zero, at which point he would disappear into the singularity. He'd actually be ripped apart as he approached the singularity because of tidal forces which pull more strongly on the front part of him than on the back part of him, stretching the object out in the radial direction.

However, curiously, if one calculates what this trajectory looks like as a function of the coordinate time, t, we did actually do that calculation but we looked at how it would behave in the limit as you approached-- as the particle approached the Schwarzschild horizon. And we discovered it would take an infinite amount of time, as seen from the outside, for the in falling object to reach the horizon, let go through the horizon and get to r equals zero. So from the outside, it looks like the object never actually falls into the black hole, but just gets closer and closer and closer as t approaches infinity. So it's an example of a very highly distorted spacetime, where you can see very different pictures depending on which observer you're trying to describe the observations of.

And I think that's it. Any questions about any of that? I guess I'll put that back up.

OK. On your homework you'll be applying this geodesic equation to a model universe, to Robertson-Walker universe, and this will serve only as an example for those calculations. I guess there's also a homework problem about the Schwarzschild metric, that orbits in the Schwarzschild metric that you'll be working at. It's all in principal straightforward if you just look at equations and follow what the equations tell you, thinking carefully about what the variables mean.

OK. In that case, let's get started on today's work, which is a change of gear. We're now going to be talking about black body radiation and its effect on the universe. I should say that my original plan was to get into this set of lectures notes-- lecture notes six, which have not been handed out yet-- to get into those last time and to finish them today. I don't think that's going to be possible because I didn't get into them last time, and I don't think I'll be able to finish them today.

But I would like today to be sort of the closing for what's needed for the problem set due Monday and for the quiz next week. So, shortly after today's lecture I will send you an email telling you where the cutoff is as far as the reading and the lecture notes. And I also hope to post the lecture notes by tomorrow.

And I also will be posting a set of review problems as we had for quiz one. And I hope to get that done by tomorrow. You may have noticed that not all of my hopes are filled, but I do my best, and I'll try. OK are there any logistic questions or anything before we go on? Yes.

AUDIENCE: Can you post the solutions to the previous problem set, the one that we turned in on Friday?

PROFESSOR: Oh, um, yes I can, I should. OK I will. OK, that's a third item I should try to get done today. Thanks for reminding me. And the solutions to the problem set that you'll be handing in Monday will be posted very shortly after you hand them in so that people can start talking about them and prepare for the quiz. Yes.

AUDIENCE: Do we have any day at which the videos might be up?

PROFESSOR: Ah, I've request about that and all I was told was that they're doing their best. So, I did look into it, but I don't know the answer. I hope that you'll have all the videos available to study for the quiz, but I don't know if that's going to happen or not.

OK. in that case, the new topic is Blackbody Radiation and the Early History of the

Universe. So far we've dealt with a universe which contains only non-relativistic matter and that, as we said from the beginning, describes our universe for the bulk of its history. But in the early period, the universe was in fact dominated by radiation, as we will now be calculating. And in the more recent period, the universe is dominated by dark energy, which we'll be talking about immediately after we finish talking about radiation.

So the important point here is that even though we don't think of light as having mass, light certainly has energy and relativistically we know that energy and mass are equivalent. The key equation that actually dominates today's lecture is perhaps the most famous equation physics, E equals MC squared, energy and mass are equivalent. And the numbers-- I'll just give you some numbers for this equation. You've probably already aware that the numbers are kind of out of sight. One kilogram, a point to that equation, is equivalent to-- I don't have any figures to give you-- 8.9876, in case you really want to know it accurately, times 10 to the 16th, most important to know the exponent there, joules.

And it's also perhaps interesting to translate this into the kind of energy units that are use when we talk about power consumption in practical situations. It corresponds to 2.497 times 10 to the 10th kilowatt hours, which is a lot of kilowatt hours. An interesting comparison, is the total power consumption of the world, which turns out to be comparable to a kilograms worth of things. I looked things up in the Wikipedia, and it told me that in 2008-- which is the most recent year it had numbers for, which is a little surprising, that's so far in the past-- the total world power consumption for the year equaled about, I'm rounding off here, about 150 petawatt hours.

Now, I always have to look up peta when I see it, I haven't quite learned what peta means yet. But they translated this, this means 150 trillion kilowatt hours. So 150 times 10 to the 12th kilowatt hours. And if we divide this by the number of hours in the year-- to ask how much is per hour, which is I think a natural thing to think about if you're using kilowatt hours to measure the power, especially the energy-- the power that goes with this is about 17 times 10 to the ninth kilowatts. And therefore

17 times 10 to the 9th kilowatt hours per hour.

And if you compare these two numbers, it means that if you could convert one kilogram per hour into pure energy, that would be about equal to 1.5 times the world's power usage in 2008. So if you could convert matter completely to energy, as they do on Star Trek, it would mean that you could fill up your tank of your typical American car and power the world for about two days. But of course we can't undo this, that's the important fact concerning power. Only a small fraction of the mass of the uranium in a nuclear reactor is actually converted as power, a fraction of a percent. So you don't get nearly as much power as this calculation would indicate, but in principle this much energy is contained in the matter that we have around us all the time.

OK. I want to introduce a few formulas which we'll be using sooner or later concerning the relativistic treatment of momentum and energy, which is what we're getting into here. And we're not trying to derive relativity in this course, so I'm just trying to quote the results that we'll be needing. So, it is useful to introduce an energy momentum four-vector, which has a 0-th component and an i-th component, where i refers to the spatial indices 1, 2, and 3. And sometimes I might write this as p zero p with a vector sign over it, where the vector sign indicates the three components 1, 2, 3.

The momentum here is the momentum. It differs in its relationship to velocity from what we have from Newton, and I'll write that in a minute, but this momentum is the conserved physical momentum of an object. And p0 is also conserved. p0 is just an abbreviation for the energy divided by C. And this quantity forms a four dimensional vector in special relativity. And when we say that it's a four-vector, we're actually making a definite statement about how it transforms from one Lorentz frame to another. a four-vector is something which transforms in exactly the same way as x super mu, four spatial and time coordinates transform.

And in particular, we learned that there was an invariant associated with spacetime transformations this s squared, which was x squared plus y squared plus z squared minus C squared t squared. And the same thing will happen here. p squared, which means the Lorentz invariant square of the four-vector is the Lorentz [INAUDIBLE] again here, and it's again equal to the sum of the squares of the 3 spatial components minus the square of the time component. And that can be written out as the square of the momentum-- spatial momentum-- minus E squared C squared.

And the claim is that this is also Lorentz invariant. And we could figure out what Lorentz invariant quantity it's equal to by having knowledge that this is the same in all frames. We can evaluate it in the simplest frame. And the simplest frame would be the rest frame of the object. In the rest frame of the object the p would be zero and this would then just be minus E squared over C squared. E squared would be M squared C to the fourth, so that implies that this is equal to minus M0 C squared squared where M0 is often called the rest mass.

And when I say it's often called the the rest mass, what I mean is that nobody ever mistakes the word "rest mass" to mean anything else, if anybody says "rest mass," and he knows what he's talking about, he means this. But this is often sometimes just called the mass because sometimes people only talk about masses as being rest masses. But I'll try to call this the rest mass because I will use the word mass in other ways.

We could also relate this momentum to the velocity, and in doing that we will again encounter this factor of gamma that we found kinematically earlier.

AUDIENCE: Yes, question. Did you forget to divide by C squared there?

PROFESSOR: Um, yes. There's too many C's here. Absolutely. It should have units of momentum. So it should have units of mass times velocity squared. Thank you. Thank you.

So, the quantity gamma, is the same quantity we encounter at the beginning of the course when we talked about time dilation and Lorentz contraction, it just depends on the velocity and approaches infinity as the velocity approaches the speed of light. And the physical momentum of a particle, relativistically is equal to gamma times M sub 0 times V. Where V is the ordinary velocity-- this is special relativity, we're not concerned with coordinate velocity versus physical velocity yet-- and gamma is that factor. So the momentum is larger than what you would get a la Newton.

The energy can also be written down. And this formula is one expression we can use to find the energy in terms of the momentum. The energy in terms of the momentum is M0 C squared squared plus p squared C squared. And it also can be written in terms of the velocity as just gamma times M0 C squared where the velocity appears in the gamma. And a special case of this is when the particle is at rest. Might as well write this, the energy of a particle at rest, which we might call E0, is just M0 C squared, which gets us back to where we started with E equals MC squared.

Now, I might just say a quick word about where these formulas come from, what idea underlies them. I'm not going to make any attempt to derive them because we just don't have time. It'd be easy to drive them, but we want to do other things.

But logically, where they come from is simply the observation, by Einstein originally, that if one has the Lorentz transformations relating what one inertial observer sees to another inertial observer, if one used those transformations but used the Newtonian definitions of energy and momentum, then you would find immediately that if energy and momentum were conserved in one frame you then know how to calculate what happens in other frames by using the transformations, you'd find it would not be conserved other frames. So if the conservation of energy and momentum are to be a universal principle of physics, which Einstein wanted to maintain, it would be necessary to redefine energy and momentum.

Now they're defined in ways so that they approach the Newtonian values for small velocities, but for velocities of the order of the speed of light they're different. And they have the property-- not completely obvious from what we wrote, well it is actually completely obvious from what we wrote-- they have the property that if it's conserved in on frame, it's conserved in all frames. And what makes it obvious-- maybe the connections between these different formulas are not obvious-- but I did tell you that p super mu transforms as a four-vector, meaning it transforms the same way as x super mu transforms, which are linear transformations. And that's enough to guarantee that if p mu is conserved in one frame, it has to be conserved in all frames because delta p mu, the change in p mu, would also be a four-vector, and if a four-vector vanishes in one frame, it vanishes in all frames.

OK. Well I wanted to give you sort of a quick example of how this works in practice. So I to just talk about the energetics of a hydrogen atom. A hydrogen atom consists of a proton, with a mass that we'll call M sub p, and an electron, with the mass that we'll call M sub E. And, as, if you imagine starting with the electron and proton arbitrarily far apart and bring them together, what you can discover experimentally-- of if you know quantum mechanics you can calculate theoretically-- energy is released, because you're releasing potential energy as you bring the electron into the atom. And the amount of energy released is 13.6 electron volts.

And the important E equals MC squared implication which I want to point out here is that loss of energy. This energy would be extracted from the system as you made the hydrogen atom. The fact that you've extracted energy from the system means that now the system should have less energy than it had to start with.

Initially it had the rest energy of the proton and the rest of electronic, M sub p C squared and M sub E C squared. Now it has less energy by delta E, and that means it also has to have less mass. So the mass of a hydrogen atom is not the sum of the mass of the proton and electron, as it would be in Newtonian mechanics, but is less by an amount proportional to this energy given off, delta E, 13.6 electron volts. And just putting in the C squares in I hope the right places, the mass of a hydrogen atom will be equal to the mass of a proton plus the mass of an electron, but then minus the binding energy expressed in mass units-- delta E over C squared.

OK. So I guess, just probably one more topic I want to talk about in terms of just basic special relativity, and this actually get's into general relativity. I wanted to find the relativistic mass of any system just being its energy divided by C squared. And this means that the relativistic mass of a particle increases with its velocity. The energy of a single moving particle is gamma times M0 C squared. That would say the by this definition the relativistic mass of that particle is gamma times M0.

And I might mention that this concept of relativistic mass is disparaged in many books on special relativity. It's certainly a concept that you can do without, so people who emotionally are bothered by it can get along without it because it is just the energy divided by C squared. And in fact a lot of work in special relativity is done in units where C is equal to 1 and then it is just the energy. Especially if you use C equals 1 you could dispense with this concept of relativistic mass. We're not going to be using C equals 1, so the phrase relativistic mass will allow us to abbreviate E divided by E squared in a convenient way.

But the important thing is not the definitions, the important thing is what properties does this relativistic mass have, whether or not one chooses to call it relativistic mass or E divided by C squared. And it as an important property concerning the gravitational field created by matter. Now the gravitational field of a single moving object is complicated.

If we were talking about, say, a moving star that was moving at a velocity large enough so we care about relativity, the way we calculate that actually as we start with the Schwarzschild metric, which will describe the metric of the star outside of the matter if it were stationary. And then you can just make a transformation to a moving frame. You're allowed to use any coordinates you want in general relativity, so transforming to coordinates that describe the moving frame is no problem. But it distorts the field in a complicated way, nonetheless, a way that you can deal with. And what you find, of course, is that what you get would be asymmetric once you transform to the moving frame, it would show the signs of the velocity that you used to transform from the original spherically symmetric Schwarzschild metric to the new frame. So the bottom line is that the gravitational field of a moving object is not isotropic, it's more complicated than that, just as the electric field would be.

But if we have a gas of particles, which is pretty much what we have in the early universe. If we have a gas of particles in a box moving every which way, then if we thought of this box as being an object that we're only going to look at from the outside-- a black box in the classic use of the phrase black box-- the mass of the black box really would just be the sum of the relativistic masses of the particles. And the isotropy of the metric that any one particle would generate would be canceled by averaging or summing over all the particles going every which direction, because on average the velocity of particles inside the box is 0. So this relativistic mass, when you're talking about a gas, really is the mass per particle. And if you divide that by the volume really you do get the mass density, which is a relevant mass density in terms of talking about how this matter would generate gravitational fields. Yes.

AUDIENCE: So why do some people have emotional problems with it?

PROFESSOR: I think some people have emotional problems with it because when one thinks about pedagogy in a course, for example, one worries about people confusing it with the rest mass. I think that's the reason. And I guess there are other possible sources of confusion, so your question is a good one. Another source of confusion is that this mass does not fit into an F equals ma equation. So in calling this the mass you might suggest to students who aren't paying attention to every word that you say, you might go ahead and put F equals ma for this mass, that does not work. So it has some of the properties of a mass, but not-- by no means all of them.

But in particular for us is important because we're going to be interested in the gravitational field of a gas and then it really is the mass that determines that. In the more formal language in general relativity, it's the mass-- it's the energy density, that appears in the equations that produce gravitational fields, and then this really is the energy density except for a factor of C squared. OK, any questions? Because now I'm going to leave this formalism and get into what role this radiation could play in the early universe.

So now I'd like to talk about radiation in particular, and for now I mean electromagnetic radiation just ordinary photons. And we're not accustomed to thinking of light as having mass, but we know light has energy and energy is related to mass by a factor of C squared. So we can write down the formula that says that rho is equal to u divided by C squared. Where u is the energy density, which we know electromagnetic fields have. And rho will be the mass density of radiation.

Now, photons have zero rest mass, so if we apply for example this formula for a photon we would set M sub 0 equal to 0. And we said that photons have zero rest mass what we mean is that there's no lower limit to the energy a photon can have. In general, the rest mass determines the lowest possible energy a particle could have, which is when it's at rest. Photon can never be rest and there's no lower limit to what its energy can be.

So for photons, M0 is equal to 0 and that implies that the energy of the photon is just C times the magnitude of its momentum. And this formula one can derive just for electromagnetic waves is purely classical EM-- you don't have to be talking about photons, but since it's true for a classical electromagnetic wave it had better also be true for photons because we think of this classical electromagnetic waves as really being made out of photons. So the energy that exists in the universe in the form of electromagnetic radiation will have an energy density, which we know how to calculate. And we know to calculate the momentum of any given photon. Now that will average to zero, but nonetheless if we imagine talking about a box of photons, which are bouncing off the walls, the fact that each photon carries momentum means that there will be a pressure on the walls. And we will be interested in that pressure, we'll be calculating a formula for it in a minute.

So, what we now want to talk about is what happens when we put a photon gas into the universe and allow the universe to expand. What happens to the radiation energy density, or equivalently mass density, as the universe expands? This turns out to be a very easy question to answer if we think of the energy as being made out of photons. We would get an equivalent identical answer if we use classical Maxwell's equations and talked about how energy density is a [INAUDIBLE] of electromagnetic fields behaved. It would be more work actually do it that way, but we would get the same answer.

In terms of photos, we could simply notice that the number density of photons--

photons are not going to disappear as the universe expands, the number density will just keep the same number of photons, but as the universe expands those photons will occupy a larger volume. So it's exactly the same as what we said about non-relativistic matter, the number density of photons will fall like 1 over the cube of the scale factor, which just says that photons are conserved. And the volume of any region grows like a cubed.

I should also mention that I'm using gamma here. Gamma of course also sometimes meant 1 over the square-root of 1 minus V squared over C squared, but besides that use, gamma is also just a label that means photons. It comes from the idea of gamma rays, but it's actually used in this context for any kind of a photon no matter what its frequency is, whether it's a gamma ray or an x-ray or visible light, or infrared. Yes.

AUDIENCE: Is this assumed for time average? Because photons can be absorbed, right? I mean, at least like in a small [? slice ?] can't there be a decided non-relativistic matter?

PROFESSOR: Is this true on the--? Well, the validity of this formula-- you're right this formula's not exact, photons can be absorbed. But in terms of what happens as the universe evolves, that's a very, very, very minor process, especially when we're talking about the early universe when there isn't really anything around to absorb them. So, especially for the early universe and even pretty well today, if we're talking about the cosmic background radiation, which is the bulk of the photons, this formula's a very good approximation. Yes.

AUDIENCE: Even though the photons in the early universe created a lot of massive particles, [INAUDIBLE] didn't that affect the expansion?

PROFESSOR: Are you're asking, will the photons be important if there's a lot of mass of particles? Is that your question?

AUDIENCE: Would the photons, won't they decay into matter antimatter?

PROFESSOR: Will the photon decay into matter antimatter? No, not really. It is, in principle, possible for two photons to collide and produce an electron positron pair. It's actually a rather small cross section for that. And all of these processes in the early universe will rapidly reach an equilibrium, which we'll be talking about more a little later, where there'll be just as many photons converting into E plus E minus pairs as there will be E plus E minus pairs colliding and making photons.

So the early universe is assumed to reach equilibrium very quickly, and all the description we'll be giving will be a description of the universe that's in thermal equilibrium with these processes will tend to iron out. We will learn that they don't always cancel each other because the universe is cooling, and that means it can't be exactly in thermal equilibrium. And there are some cases where the effect of that cooling is significant and we'll be talking about those. But for the most part, if the cooling is slow, which it is for the most part compared to other processes, everything stays in thermal equilibrium. OK. Those are good questions. We've gotten a little ahead of what I wanted to talk about.

So for now I'm just imagining a free photon gas, which is an excellent approximation for the early universe. And those photons just continue to exist as the universe expands, so their number density falls off as 1 over a cubed. But there's another affect that goes on which is that the photons are redshifting. And we already know about that, but now we're going to take it into account in terms of the energy balances.

So we know that the frequency of a photon at some time t2 divide by its frequency at some time t1, and here nu equals frequency, is just diluted by the expansion of the universe. This ratio is 1 over 1 plus z, where z is the redshift between these two times. But written out in more detail which is a formula we'll actually be using, is just a of t1 divided by a of t2. When the scale factor doubles, all the frequency is half. And that means that all the photons are lowering in frequency, and we also know that photons are quantized.

The energy of a photon can't be any old thing, but in fact, the energy of a photon is equal to h times nu. Where little h is what's called Planck's constant. And numerically there are various units you could express it in, but it's 4.136 times 10 to the minus 15th electron volt seconds. So if you measure a frequency an inverse seconds, you get an energy in electron volts from that formula.

The important thing for now though is that this says that the energy of each photon-- being proportional to its frequency, and the frequency being proportional to 1 over the scale factor-- the energy of each photons is proportional to 1 over a of t. And then the total energy density of photons in photon on gas which I'll call u sub gamma, the energy density of the gas, can be thought of as the number density of photons times the energy of each photon. And the number density is falling off like 1 over a cubed, the energy is falling off like 1 over a. And therefore, this is proportional to 1 over a to the fourth. So as the universe expands, the density of non-relativistic matter falls off like 1 over a cubed-- as we've been talking about some time now-- but the energy of radiation falls off faster, like 1 over a to the fourth because of the red shifting of the photons.

OK, now once we know this, we can ask ourselves what happens if we look at our universe going backwards, knowing where we are now where we come from? And if the energy density of photons is falling off faster than the energy density of matter, it would mean that the ratio is getting smaller as we go forward in time. But that of course implies that the ratio gets larger as we go backwards in time. So as we go backwards in time, the radiation becomes more and more important, and there actually is going to be a time when the radiation will equal the matter and at earlier times the radiation will dominate.

Today I'll just give you a number for now, we'll learn later how to calculate it, but for today the total radiation energy density in the universe is equal to 7.01 times 10 to the minus 14 joules per meter cubed. And this actually includes two kinds of radiation, it includes photons and also neutrinos, which at least in the early universe behaved just like radiation.

And we'll be talking more about neutrinos later so don't worry if you don't have any idea what a neutrino is. But for now it's just another contribution to the radiation, and we can measure basically this is all based on measuring the temperature of the cosmic microwave background radiation-- and we'll learn later how to make that conversion. But once you measure the temperature of the cosmic background radiation and have a theory about how many neutrinos there should be, that's actually all theoretical and we'll talk about that later as well.

One can determine what the energy density of that radiation is. And it corresponds to a mass density just dividing by C squared of 7.80 times 10 to the minus 31 kilograms per meter cubed. And when I think of mass densities I always like to think of in centimeters per-- excuse me, grams per centimeter cubed because I'm used to the density of water being one gram per centimeter cubed and I like to be able to make that comparison. So just making that conversion, usually I use SI units, but some things just seem to make more sense in other units. So it's 10 to the minus 34 grams per centimeter cubed, so 10 to the minus 34, or maybe 10 to the minus 33, times the density of water.

And this is incredibly low even compared to the critical density of our universe, and the actual density we know is very near this critical density. Let me remind you that the critical density we derived a formula for, and when we put numbers into that formula we found that was equal to 1.88 times little h0 squared, which is Hubble's units-- Hubble's constant in units of 100 kilometers per second per megaparsec-- I'll write that in a second-- times 10 to the minus 29 grams per centimeter cubed. So just writing down the equation for little h sub 0 is where capital H sub 0, Hubble's expansion rate, is equal to 100 times little h sub 0 kilometers per second per megaparsec. Yep.

AUDIENCE: What's the, I guess, what's the motivation of normalizing the Hubble constant in this way?

PROFESSOR: Well, I think the real motivation is that the astronomers like these peculiar units of kilometers per second per megaparsec, but if your favorite unit is called a kilometer per second per megaparsec you don't want to have to say those units very often. So H0 is dimensionless, so it's a dimensionless way of talking about the Hubble constant. But that's the only importance, there's no real-- no deep significance to it. But it's a standard notation, so it's worth knowing.

And then finally, we can write down how much the radiation contributes to omega-- omega sub r, r is going to indicate radiation. The notation, by the way, will be-- and I realized I've already violated that notation. This really should have been r. Well, I'm not going to change it, it'll get too messy. But I'm going to start using the notation where gamma indicates photons and little r indicates radiation. And the difference is that there are other kinds of radiation besides photons, in particular we've already added in neutrinos in part of what we're calling radiation.

So omega sub r, which now includes photons and neutrinos, is just defined to be the mass density radiation divided by the critical density. And that turns out, when you combine these numbers, to be 4.15 times 10 to the minus 5 little h0 to the minus 2. And then for h0 equals 0.67, which is the Planck satellite value for little h0, we finally get omega sub r is equal to 9.2 times 10 to the minus 5. So roughly 10 to the minus 4. The fraction of the mass density, or energy density today is about 10 to the minus 4 fraction and radiation.

And actually as I write this, this actually calls to mind another reason for defining little h sub 0, which is that if you write formulas in terms of little h sub 0, they remain valid between one year and next year. The observational value of the Hubble parameter is still floating around and differs, for example, every time I teach this course. So the formulas in terms of little h sub 0 stay, and then you plug in the current value of little h sub 0 to get the best value that one can currently write down. Now we know the Hubble constant to within a few percent, which is much better than it used to be, but it's still floating. The Planck value was somewhat lower than the previous except a value, which was about 0.70.

OK. So we know enough information to extrapolate backwards and calculate when this radiation would've equaled the energy density of matter. Because we know how the ratio changes. It changes by a factor of a, the scale factor, because the energy density of non-relativistic matter is falling off like 1 over a cubed. The energy density of radiation is flowing off like 1 over a to the fourth. So the ratio between them just changes by a factor of a, decreasing as a gets larger.

So we can write RHO radiation of t divided by RHO matter of t. The m there means non-relativistic matter. This is just equal to the current value. And the current value is gotten by taking that number and, where I forgot to write down is that omega matter today is about 0.30. So the ratio of the two, omega radiation over omega matter, which is the same as RHO radiation over RHO matter, is about 3.1 times 10 to the minus 4.

And that number is about to appear in this equation. If we want the ratio as a function of time, we start with this value today, 3.1 times 10 to the minus 4, and then we can just multiply that by the scale factor today divided by the scale factor at the time that we want to know it, because we know it falls off as 1 over the scale factor. And by putting an a of t0 here, this just guarantees that if we let t equal t0, we get this number, which is the right ratio for today.

OK. Now it's just a matter of arithmetic. We also know that for a matter-dominated universe. And for now, we're going to estimate when the radiation energy will equal the matter density. This will only be an estimate. We're going to estimate it by assuming that we can approximate the universe as matter dominated between now, all the way back until that time.

There are two errors in that calculation. We're not taking into account here the era of acceleration where dark energy is playing a significant role. We'll learn later how to do that. And also, as we approach this time when they're making equal contributions, we will run into a regime where the contribution of the radiation itself will be relevant. So this is only an estimate.

But what we're going to do is we're going to assume that we can treat this as a matter-dominated universe with a of t proportional to the 2/3. And then we can plug numbers into this formula and ask, when was the ratio 1? And when the ratio was 1, we call that the time of equality, using the subscript EQ for equality to indicate anything having to do with that crossing point.

And what we find is that the z of equality, and z is just the ratio of the a's, is-- according to this calculation, it would be 3.1 times 10 to the minus 4 minus 1. I fibbed when I said that z is the ratio of the a's. It's offset by a little bit. It's 1 plus z that's the ratio of the a's. And that's why there's a minus 1 there.

And numerically, this is about equal to 32,000. So if we look back in the history of the universe, we can define looking back in terms of the redshift.

If you look back to a redshift of 3,200, we get to the time when matter and radiation had the same energy density. And we can know what time that is if we assume t to the 2/3, which again is only a crude approximation. We don't necessarily expect to get the right answer here. But we expect to get the right order of magnitude.

So t-equality, according to this situation, would be about 75,000 years after the Big Bang-- just converting the scale factor that we just calculated to a time using that formula, treating this as t to the 2/3. So this says that about 75,000 years after the Big Bang, the energy densities of matter and radiation were equal. In the earlier times, the universe was radiation dominated. The radiation exceeded the matter in its energy density. Yes?

AUDIENCE: It seems like zEQ is relatively about 31 [INAUDIBLE].

PROFESSOR: You might be right if we would just look to these formulas.

When I calculated this at home, I kept more decimal places all the way and rounded off each answer to one significant figure. And that's not the same as taking the answer to one significant figure and calculating and then rounding off to ones in every figure. So I think there's always an ambiguity of, roughly speaking, 1 in the last decimal place whenever you're rounding numbers off.

AUDIENCE: So we just plug that [INAUDIBLE]

PROFESSOR: 3,220-- starting with this or starting with a more accurate number than the 3.1.

AUDIENCE: Just 1/3.1.

PROFESSOR: 1/3.1. OK. So, OK?

AUDIENCE: Sorry.

AUDIENCE: Wait no, [INAUDIBLE] like we have 1 over 3 times 10 negative 4. That's 3 times 10--

PROFESSOR: It's 3.1.

AUDIENCE: You have 1 over 3.1.

PROFESSOR: 1 over 3.1. So you have to divide.

AUDIENCE: I do that all the time.

AUDIENCE: [INAUDIBLE]

PROFESSOR: So apparently it's even right if you just calculate with that. But there is actually some ambiguity.

The numbers I'm giving you probably have some uncertainty of 1 in the last digit, depending on how you calculate. But the number they give you, I think they're the ones that you get if you start using this and the 0.67 and from then on do everything to large numbers in decimal places and round off at each stage. You'll get the numbers I've given you.

In any case, all these are really, at best, order of magnitude estimates. So worrying about whether or not the last figure is accurate is not a big deal. Yes?

AUDIENCE: I don't really understand how you got t [INAUDIBLE] without telling was a of t dot is [INAUDIBLE].

PROFESSOR: I'm sorry. There is actually a piece of information I used I forgot to write here. You're absolutely right.

I used t0 is equal to 13.8 times 10 to the nine years. And then everything can be related to t0 if you know how things are proportional to t.

You're absolute right. I did not give you all the information necessary for that calculation. Now I think I have. I haven't done the arithmetic for you. But otherwise it's all there.

OK. Now I might mention that in Ryden's book, she does the calculation taking into account everything-- matter, radiation, cosmological constants. And her number for t-equality is 47,000 years, which verifies that we have the right order of magnitude.

And actually, the biggest difference between her number and my number is not that she's taken into account these more sophisticated things, but rather that she used a different value for the Hubble expansion rate than I'm using. She's using a value that was current at the time she wrote her book, which was like '72, I think. h0 equals 0.72 instead of [INAUDIBLE] 0.67. And that does make a significant difference here.

But either of these numbers are, I think, probably within the range of uncertainty of when it really happened. But it's on that scale, on the scale of 50,000 years, 100,000 years, something of that order.

So there was a significantly long period compared to human lifetime when the universe was radiation dominated. But it's a very small fraction of the overall history of the universe, but nonetheless does have important features that happened during that time period. Now, if we want to understand those features, we have to understand how a radiation-dominated universe evolves, which is what we're going to get to next.

OK. The next little chapter than I'm going to be talking about-- the dynamics of a radiation-dominated universe. This is a chapter that you more or less get to work out the equations for yourself on one of the homework problems. That's part of this week's set.

So I will try here to outline the logic. But because all the calculations are in the homework, I will basically skip the calculations themselves, and let you do them for yourselves as part of the homework.

But where we start is we have written down Friedman equations for the matter-dominated case. And I'll start by reminding us what those were. And then in addition to these two equations, which describe our expanding universe, which we've derived sometime ago for-- and to remind us here, this is for a matter-dominated universe. And matter-dominated means non-relativistic matter dominated.

And going along with these equations, we also know that RHO of t for non-relativistic matter falls off like one over a cubed of t. This can be converted into a differential equation for RHO. That is, we can calculate Rho dot from this equation.

And the way to see that is probably most easily to start on a new blackboard and write that equation not as a proportionality, since it's hard to differentiate a proportionality, but we can write it in an arbitrary constant of proportionality. And then it becomes an equality.

So I'm going to write the equation as RHO of t is equal to some constant, b, divided by a cubed of t. And this we know how to differentiate. We can write Rho dot is equal to minus b over a to the fourth of t times a dot. And that is equal to minus 3. I'm sorry--there's a 3 here. Minus 3 times a dot over a times the original RHO.

So we can forget the intermediate steps, and we just arrived at the equation that RHO dot is equal to minus 3 a dot over a times RHO. And we can think of that as going along with equations one and two. Maybe I'll even give that a number. Equation three will be RHO dot is equal to minus 3 a dot over a times RHO.

Now for radiation, there will be a 4 here. The 4 will arrive the same way as the 3 arrives there. It's just the power that appeared in the factor of a.

So for radiation, this last formula we know is going to be modified, which is the key point. Note that these three formulas are not independent of each other. If we know, for example, equation one, which is an equation for a dot, we could differentiate that with respect to time and get an equation for a double dot.

When we do that, everything has to be differentiated. So it involves differentiating a, but that just expresses things in terms of derivatives of a. But the new quantity that gets introduced is RHO. If we wanted to differentiate this equation with respect to time, we have to know what RHO dot is.

But we do. That's what equation three tells us. So we can differentiate equation one, use equation three, and we can derive an equation for a double dot. And if these equations are consistent, it'd better be equation two. And it will be. You can check it.

And actually, I think any two of these equations can be used to derive the third. Those equations just are-- really a set of two independent equations and one dependent equation. You can shuffle it any way you want.

But, now what we want to do is to consider a different kind of matter. Instead of non-relativistic matter, we're considering photon matter. And in particular, we know that it's going to change equation three. So for radiation, 3 gets modified into 3 prime, which is the equation that says that RHO dot is equal to minus 4 a dot over a times RHO.

So how are we going to fix these equations? Now they're inconsistent. If we change three and don't change either one or two, we know that we're inconsistent, because any two of those equations can be used to derive the third. So we're in trouble. Either equations one or two will also have to be modified if we're going to modify equation three.

OK. Before we go on, I'd like to say a little more about why this equation is different from that equation. One might think that it should just be governed by the conservation of energy. After all, we just write down an equation for RHO dot-- how energy density changes with time. Shouldn't conservation of energy determine that? It does.

But there is an extra element to conservation of energy that we need to take into account, and that is the pressure of the gas affects what happens to its energy as it expands. So before we get back to the early universe, I just want to consider a gas in a piston chamber. And I'm going to let the piston have an area a, and inside we're going to have a volume v. Just to define our notation.

If we have a gas inside a piston chamber and let the piston chamber enlarge by pulling out on the piston, the gas has a pressure, in general, and that pressure will exert a force on the piston. So if I allow the piston to move to the right, that gas will be exerting a force on the piston in the direction that it's moving. And that means the gas will be doing work on the piston.

So, by our ordinary notions of Newtonian conservation of energy, we would know that the gas would lose energy, and we can even calculate how much energy it loses. And the formula is easy enough to get in a Newtonian context. It's just du is equal to minus the pressure of the gas times the change in volume. A famous formula.

And this just comes about by saying that the work that's done is the force times the distance. The force is the pressure times the area. And the volume is the area times the distance. And putting those things together, you get this formula immediately.

Now this formula is actually much more general than the quasi derivation that I just showed. It works no matter what the shape of the gas is. If you put a gas in any kind of a container and let that container enlarge, even in an irregular way, the work that the gas will do will always be equal to minus the pressure of the gas times the change in the volume.

We can apply this to the early universe. It actually works. The difference between our two cases is that our non-relativistic matter has no pressure at all. We're just talking about particles sitting at rest.

They're not bouncing off of any walls. They're not creating any pressure, while the photons are moving around all the time. And if you imagine a box of them, they'd be hitting against the walls of that box, exerting a pressure.

And we're now in a position to relate the pressure to the difference between the 3's and the 4's that appear in those two equations for RHO dot in the two cases. To apply this naive idea to a piece of the universe, we can imagine choosing-- we're going to choose some fixed volume in our co-moving coordinate system.

So our box, the volume that we're talking about will actually be expanding with the universe but be fixed in co-moving coordinates. And the physical volume therefore of our box will be a cubed of t times the coordinate volume of the box-- the volume and not just cubed. And this volume will be independent of time.

There's time dependents there. There's time dependents there. The physical volume of our box will be enlarging. The total energy in our box, the total gas energy, which I'll call capital U, will just be the physical volume times the energy density. The energy density is energy per physical volume.

And we can now apply this formula using this U and this v.

And here again is one of these cases where I'm going to be skipping steps because you're going to be doing it in detail on the homework. But by putting these equations together, what you'll find is that d dt of a cubed times RHO times c squared-- this is just d dt of a cubed times the energy density-- basically, the left hand side of that equation divided by dt and divided by v coordinates-- is equal to minus p times d dt of a cubed.

And this is just the PDV term from the right hand side of that equation rewritten in terms of the variables. And you'll be doing this for homework. I'm just getting straight the factors to make sure I have them right.

OK. Reshuffling that equation-- and again, this is a homework problem-- you can turn that into an equation for RHO dot. And what you'll get is minus 3 a dot over a times RHO plus p over c squared.

And now we can see how our two cases emerge. If the pressure is 0, we get minus 3 a dot over 8 times RHO, which is what we had for non-relativistic matter. And the photon gas is going to have a pressure, and we could read off from this formula to know what the pressure has to be to turn the 3 into a 4. The pressure has to be a factor of a third.

So you determine for this that the pressure for light is 1/3 of the energy density, or

1/3 times RHO c squared. And that's what you need to turn the 3 into a 4. So we now have indirectly calculated the pressure of light, and this agrees with any other calculation for the pressure of light that you might do. It's by no means the only way to calculate it.

And now finally, we're in a position-- and we'll just do this quickly to decide how to modify these equations. Now, we're not in a position to determine that rigorously. It can be determined rigorously by doing general relativity, which we're not doing at that level. But we can still motivate the answer.

One of these two equations is going to have to be changed to accommodate a more general expression for RHO dot. The top equation we know is really an equation for conservation of energy. That's how we got it in the Newtonian case where little k ended up being partial to the energy in the Newtonian case.

But this is basically a conservation of energy equation. And that's what you expect, just given your general notion of mechanics as well. If you have a second order equation, a second order differential is with respect to time. That's the force equation.

And if you have a first order differential equation with respect to time, 1/2 mv squared plus v of r equals constant. That's energy conservation. Same thing here. And we know that energy cannot suddenly change. If we imagine-- I guess the first experiment I want to do is imagining somehow there's an explosion throughout all of space.

I imagine putting pieces of TNT throughout space and arranging for, at the same cosmic time, for all of them to be ignited. And that would suddenly change the pressure of the universe, but it would not change the energy density. The energy density would be conserved.

So the bottom line is that pressures can change discontinuously, but energy densities cannot. And since this equation is the conservation of energy equation, we'd expected that nothing can change suddenly here, that the pressure term cannot contribute here, because if it did, the pressure term would changed suddenly. Nothing else in this equation would change suddenly. There would be no way the equation could be satisfied.

But if we added a pressure term to the second equation, that would allow the pressure to change discontinuously as the TNT went off. And that would change a double dot discontinuously. And there's nothing wrong with a double dot changing discontinuously. If you suddenly the apply a new force to a particle, you suddenly change its second derivative of its motion. You suddenly change its acceleration. So that's OK.

So any pressure to this term make sense. Adding pressure to this equation does not make sense. And then we can just ask, what do you have to do to this equation if we're going to add a pressure term to make all three equations now consistent with the new equation for RHO dot? It's your homework problem to answer that question, but the homework tells you the answer, and I'll write the answer on the board right now, and then we'll consider today's lecture over.

The bottom line is that equation number one has to be modified into one prime, which says that equation number two has to be modified. What am I talking about?

And the new equation is a double dot is equal to minus 4 pi over 3 G times RHO plus 3p over c squared times a. And now we have a consistent set of Friedman equations, and these are the Friedman equations that we would have gotten if we had done everything using general relativity from the beginning.

And we'll stop there. And we will meet again next Tuesday. And I'll send you an email about-- there will be at least one homework problem on the problems set that will have to be held over to the following problems set. I'll send you an email about that and post it on the website.