Lecture 6 - Topics

- The relativistic point particle: Action, reparametrizations, and equations of motion

Reading: Zwiebach, Chapter 5

Continued from last time.

$$\frac{\partial \mathcal{P}^t}{\partial t} + \frac{\partial \mathcal{P}^x}{\partial x} = 0$$

$\mathcal{P}^t = \mu_0 \partial y/\partial t$
$\mathcal{P}^x = -T_0 \partial y/\partial x$

Similar to $\partial_\mu J^\mu = 0$, $\partial\rho/\partial t + \nabla \cdot \vec{J} = 0$, $Q = \int dx\rho$

Free BC (Neumann BC):

$$\mathcal{P}^x(t, x_*) = 0$$

$$P_y = \int_0^a \mu_0 dx(\partial y/\partial t) = \int_{0^a} dx \mathcal{P}^t$$

$$\partial P_y/\partial t = \int_0^a dx \partial \mathcal{P}^t/\partial t = -\int_0^a dx \partial \mathcal{P}^x/\partial x = -[\mathcal{P}^x(t, x=a) - \mathcal{P}^x(t, x=0)]$$

Conservation of momentum?

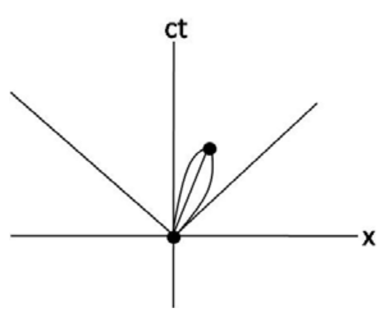
Free Relativistic Particle

Non-relativistic Action:

$$S = \int dt \left(\frac{1}{2} m v^2\right)$$

Calculation: $dv/dt = 0$

Relativistic Particles:



Everyone should agree on action. It's a Lorentz invar.

$$-ds^2 = -\eta_{\mu\nu}dx^\mu dx^\nu$$
$$ds = cdt\sqrt{1 - v^2/c^2} = cd\tau$$

$$s = -mc^2\int_{\mathcal{P}}\frac{ds}{c} = -mc\int_{\mathcal{P}}ds$$

So: $s = -mc^2\int_{t_i}^{t_f} dt\sqrt{1-v^2/c^2}$

Check:

Lagrangian:

$$\begin{aligned} L &= -mc^2\sqrt{1-\frac{v^2}{c^2}} \\ &= -mc^2(1 - \frac{1}{2}\frac{v^2}{c^2} - \ldots) \qquad \text{Taylor Expansion} \\ &= \underbrace{-mc^2}_{\text{rest energy}} + \underbrace{\frac{1}{2}mv^2}_{\text{kinetic energy}} \end{aligned}$$

Momentum:

$$\begin{aligned} \vec{P} &= \frac{\partial L}{\partial \vec{v}} \\ &= -mc^2 \cdot \frac{\frac{1}{2}\frac{-2\vec{v}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \\ &= \frac{mv^2}{\sqrt{1-\frac{v^2}{c^2}}} \end{aligned}$$

Hamiltonian:

$$H = \vec{p}\cdot\vec{v} - L = \ldots = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

Parameterization

Have parameterization $x^\mu(\tau)$ (the x^μ's are functions of τ)

$$ds^2 = -\eta_{\mu\nu}dx^\mu dx^\nu$$

$$ds = \sqrt{-\eta_{\mu\nu}\left(\frac{dx^\mu}{d\tau}\right)\left(\frac{dx^\nu}{d\tau}\right)}d\tau$$

$$s = -mc\int_{t_i}^{t_f}\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}d\tau$$

$\tau'(\tau)$:

$$\frac{dx^\mu}{d\tau} = \frac{dx^\mu}{d\tau'}\frac{d\tau'}{d\tau}$$

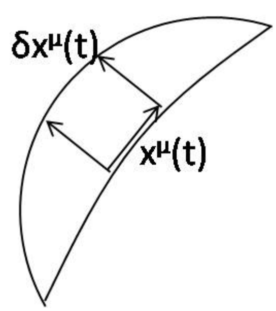
$$\begin{aligned} s &= -mc\int_{t_i}^{t_f}\sqrt{-\eta_{\mu\nu}\left(\frac{dx^\mu}{d\tau'}\frac{dx^\nu}{d\tau'}\right)}\frac{d\tau'}{d\tau}d\tau \\ &= -mc\int_{t_i}^{t_f}\sqrt{-\eta_{\mu\nu}\left(\frac{dx^\mu}{d\tau'}\frac{dx^\nu}{d\tau'}\right)}d\tau' \end{aligned}$$

So using a different parameter, τ' (instead of τ) gets same action s. s is reparameterization-invariant.

Quick calculation to find equation of motion from $s = -mc\int\sqrt{1-\frac{v^2}{c^2}}dt$. Should get derivative of rel. momentum with respect to time = 0.

$$S = -mc\int dS$$

$$\delta S = -mc\int\delta(dS)$$



$$dS^2 = -\eta_{\mu\nu}dx^\mu dx^\nu$$

$$(dS)^2 = -\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}(d\tau)^2$$

$$2(dS)\cdot\delta(dS) = -2\eta_{\mu\nu}\delta\left(\frac{dx^\mu}{d\tau}\right)\frac{dx^\nu}{d\tau}(d\tau)^2$$

$$\boxed{\delta(dS) = -\eta_{\mu\nu}\frac{d}{d\tau}(\delta x^\mu)\frac{dx^\nu}{ds}d\tau}$$

Must vary with $dx^\mu/d\tau$ and $dx^\nu/d\tau$, but since $\eta_{\mu\nu}$ is symmetric sufficient to vary just $dx^\mu/d\tau$ and multiply by 2.

$$\boxed{\delta(dS) = -\frac{d}{d\tau}(\delta x^\mu)\frac{dx_\mu}{dS}d\tau}$$

$$\begin{aligned}\delta S &= \int_{\tau_i}^{\tau_f} \frac{d(\delta x^\mu)}{d\tau}\left(mc\frac{dx^\mu}{ds}\right)d\tau \\ &= \int_{\tau_i}^{\tau_f}\left[\frac{d}{d\tau}(\delta x^\mu P_\mu) - \delta x^\mu \frac{dP_\mu}{d\tau}\right]d\tau\end{aligned}$$

$$\delta x^\mu(\tau_i) = \delta x^\mu(\tau_f) = 0$$

$$\boxed{dS = -\int_{\tau_i}^{\tau_f}\left(\delta x^\mu(\tau)\frac{dP_\mu}{d\tau}\right)d\tau}$$

Equation of Motion:

$$\boxed{\frac{dP_\mu}{d\tau} = 0}$$

This means that P_μ constant on world-line. Constant as a function of any parameter!

$$\underbrace{\frac{dP_\mu}{dt}}_{0} = \underbrace{\frac{dP_\mu}{d\tau}}_{0}\cdot\underbrace{\frac{d\tau}{dt}}_{\neq 0}$$

Therefore: $\frac{d}{d\tau}\left(\frac{dx^\mu}{ds}\right) = 0$, $\frac{d^2}{ds^2}(dx^\mu) = 0$ (if $\tau = s$. Okay because τ is arbitrary.)

But can't assign $s = \tau$: $d^2x^\mu/d\tau^2 \neq= 0$.

$$\frac{d}{ds}\left(\frac{dx^\mu}{d\tau}\right) \neq 0$$

Coupling to Electromagnetism

Lorentz Force Equation:

$$\frac{dP_\mu}{dS} = \frac{q}{c} \cdot F_{\mu\nu} \frac{dx^\nu}{ds}$$
$$\frac{dP_\mu}{d\tau} = \frac{q}{c} \cdot F_{\mu\nu} \frac{dx^\nu}{d\tau}$$

$$S = -mc \int_P dS + \frac{q}{c} \int_{\mathcal{P}} A_\mu(x(\tau)) \frac{dx^\mu}{d\tau} d\tau$$

A: Nevitz-Schwartz Tensor