

## 8.251 Test

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Only personal 2-page notes allowed. Test duration: 90 minutes.

**Problem 1.** (10 points) Boundary conditions for open strings.

Consider two static D2-branes in four dimensional spacetime  $(ct, x^1, x^2, x^3)$ . The first one is located at  $x^3 = 0$  and the second is located at  $x^2 = 0$ . Sketch the branes. Consider open strings with  $\sigma \in [0, \sigma_1]$  that stretch from the first brane ( $\sigma = 0$ ) to the second brane ( $\sigma = \sigma_1$ ).

State the boundary conditions (Free or Dirichlet, with value) for the string coordinates  $X^\mu(t, \sigma_*)$  (list the eight conditions –  $\mu = 0, 1, 2, 3$  and  $\sigma_* = 0, \sigma_1$ ).

**Problem 2.** (20 points) Conical spaces

Consider the complex  $z$  plane, with the identification

$$z \sim e^{2\pi i/3} z .$$

- On the diagram on the last page of the test, you are to draw a fundamental domain for this space. One boundary of the fundamental domain is already drawn: the real  $z$  axis. The fundamental domain is to lie above this line. You are to draw the second boundary. For any line that you draw, you should indicate the numerical value of the angle that the line makes with respect to one of the axes of the diagram. [BE SURE TO RIP OFF THE DIAGRAM AND TURN IT IN WITH YOUR SOLUTIONS!]
- How many geodesics are there that connect the points  $A$  and  $B$ ? Using a ruler if available, draw these geodesics on the diagram. You may find it useful to draw points or lines outside the fundamental domain, but you should also show each of these geodesics within the fundamental domain.

**Problem 3.** (10 points)

Observations indicate that the expansion of the universe is currently accelerating, an effect which is possibly caused by a cosmological constant. Although Einstein did not look at it this way, the cosmological constant that he defined is equivalent to assigning a nonzero mass density to the vacuum. The observed acceleration implies a vacuum mass density equal to  $0.76 \pm 0.05$  times the total mass density of the universe, which gives

$$\rho_{\text{vacuum}} = 7.6 \times 10^{-27} \text{ kg/m}^3 .$$

(This is equal, by the way, to about 4.5 proton masses per cubic meter.) Some physicists are exploring ways to explain this effect by introducing modifications to gravity at short distances, so it is useful for them to know what length scales might be relevant to such an explanation. If one assumes that the only relevant parameters are  $\rho_{\text{vacuum}}$ ,  $\hbar$ , and  $c$ , one can construct a length parameter  $\ell_{\text{vacuum}}$  by multiplying powers:

$$\ell_{\text{vacuum}} = \rho_{\text{vacuum}}^\alpha \hbar^\beta c^\gamma .$$

- What must be the values of  $\alpha$ ,  $\beta$ , and  $\gamma$  in the above equation?

- (b) Given that  $\hbar = 1.055 \times 10^{-34}$  J·s and  $c = 2.998 \times 10^8$  m/s, what is the numerical value of  $\ell_{\text{vacuum}}$ ? Express your answer in  $\mu\text{m}$ , where  $1 \mu\text{m} = 10^{-6}$  m. (Note that m = meter, s = second, J = joule = newton·meter.)

**Problem 4.** (20 points) The induced metric on a two-dimensional surface.

A two-dimensional surface in flat three-dimensional space is described by the height function

$$z = h(x, y).$$

This means that points on the surface take the form  $(x, y, h(x, y))$ . This surface is parameterized by  $x$  and  $y$ . Recall also the formula for the metric  $g_{ij}(\xi)$  induced on a surface parameterized by  $(\xi^1, \xi^2)$  and described by embedding functions  $\vec{x}(\xi^1, \xi^2)$ :

$$g_{ij}(\xi) = \frac{\partial \vec{x}}{\partial \xi^i} \cdot \frac{\partial \vec{x}}{\partial \xi^j}.$$

- (a) Calculate the components of the metric  $g_{ij}$  in terms of  $h$  and its derivatives. Write a formula for  $g_{ij}$  of the form  $g_{ij} = \delta_{ij} + \dots$
- (b) Write the fully simplified appropriate form of the area integral ( $A = \int d\xi^1 d\xi^2 \sqrt{g}$ ).

**Problem 5.** (40 points) Motion of an open string stretched between two parallel D2-branes.

Consider two D2-branes located at  $z = 0$  and  $z = a > 0$  and an open string suspended between them, with the point  $\sigma = 0$  on the  $z = 0$  brane and the point  $\sigma = \sigma_1$  on the  $z = a$  brane.

It is convenient to use a two-component vector  $\vec{x}(t, \sigma)$  to describe motion in the  $x$  and  $y$  directions and a function  $z(t, \sigma)$  to describe the displacement along  $z$ . We then write general solutions of the wave equations as follows:

$$\begin{aligned} \vec{x}(t, \sigma) &= \frac{1}{2} \left( \vec{F}(ct + \sigma) + \vec{G}(ct - \sigma) \right), \\ z(t, \sigma) &= \frac{1}{2} \left( f(ct + \sigma) + g(ct - \sigma) \right). \end{aligned}$$

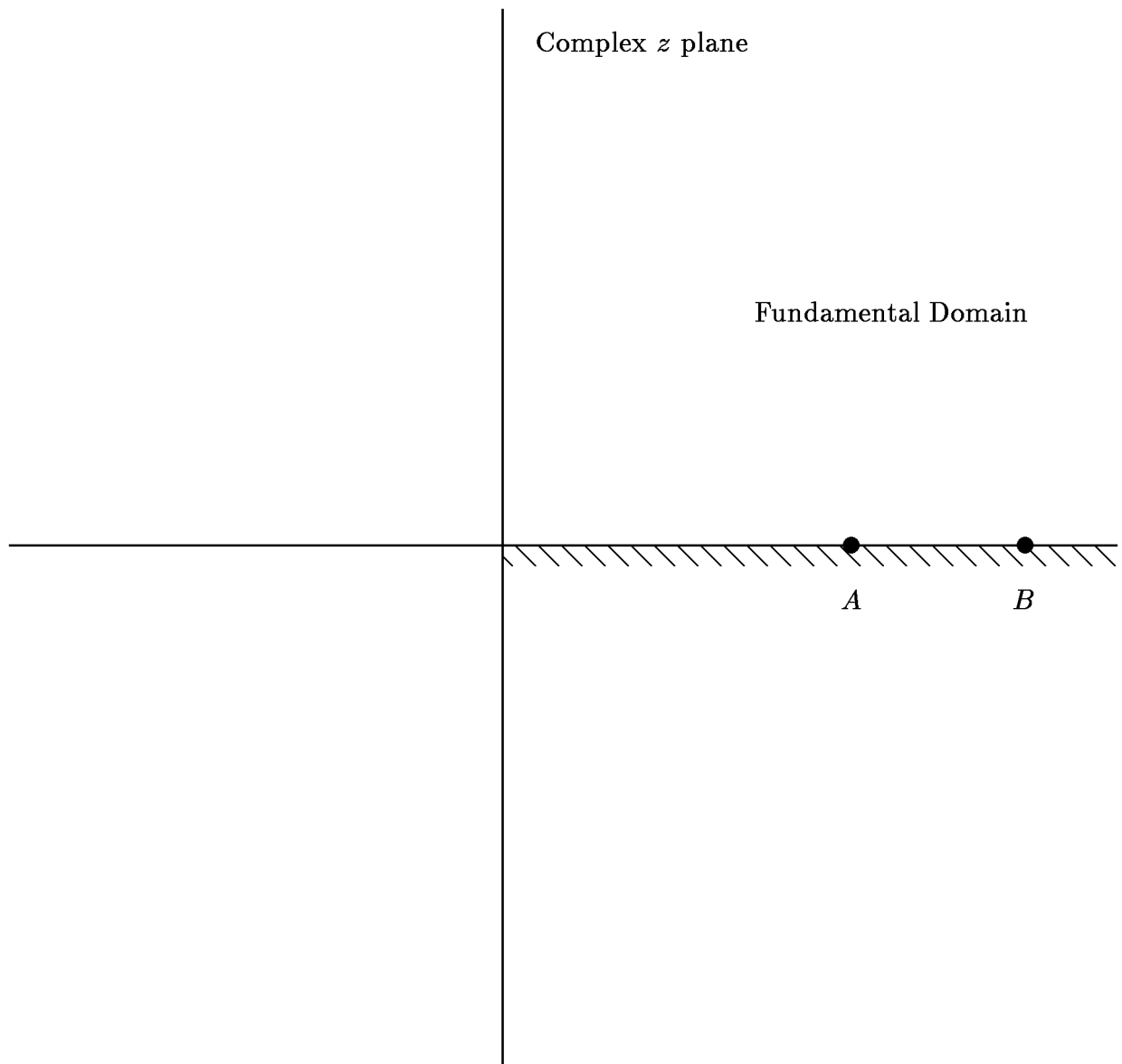
- (a) Use the boundary conditions to express  $\vec{G}$  in terms of  $\vec{F}$ ,  $g$  in terms of  $f$ , and to obtain the relevant periodicity conditions. Write your simplified formulae for  $\vec{x}(t, \sigma)$  and  $z(t, \sigma)$ .
- (b) Find the conditions implied by the parameterization conditions

$$\left( \frac{\partial \vec{X}}{\partial \sigma} \pm \frac{1}{c} \frac{\partial \vec{X}}{\partial t} \right)^2 = 1.$$

We observe that the  $\sigma = 0$  endpoint of the open string traces a circle of radius  $R$  on the D2-brane as it moves with constant angular velocity  $\omega$ . The circle is centered around  $(x, y) = (0, 0)$ . Assume that the string has the *lowest* possible energy consistent with given values of  $\omega$  and the separation  $a$  between the branes. As we will see, both this energy  $E$  and the radius  $R$  of the circle at the D-brane are determined by  $\omega$ ,  $a$ , and constants such as  $c$  and the string tension  $T_0$ .

- (c) Write an explicit formula for  $\vec{F}(u)$ . Calculate the value of  $\sigma_1$ .
- (d) Determine the function  $f$ . What is the radius  $R$  of the circle traced at the D-brane? Write your answer in the form  $\frac{\omega R}{c} = \dots$

Diagram for Problem 2, Conical Spaces:



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