Chapter 6

Fluid Mechanics

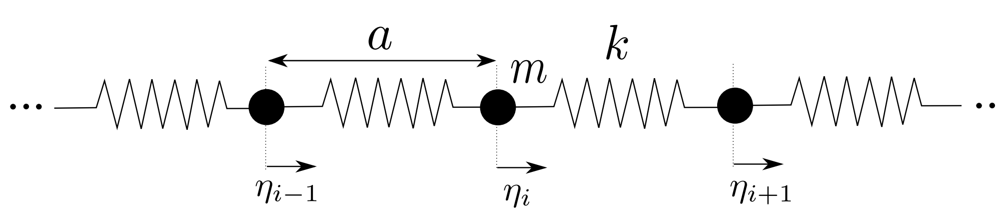
So far, our examples of mechanical systems have all been discrete, with some number of masses acted upon by forces. Even our study of rigid bodies (with a continuous mass distribution) treated those bodies as single objects. In this chapter, we will treat an important continuous system, which is that of fluids.

Fluids include both liquids and gases. (It also includes plasmas, but for them a proper treatment requires the inclusion of electromagnetic effects, which will not be discussed here.) For our purposes, a fluid is a material which can be treated as *continuous*, which has the ability to *flow*, and which has very little resistance to deformation (that is, it has only a small support for shear stress, which refers to forces parallel to an applied area). Applications include meteorology, oceanography, astrophysics, biophysics, condensed matter physics, geophysics, medicine, aerodynamics, plumbing, cosmology, heavy-ion collisions, and so on.

The treatment of fluids is an example of classical field theory, with continuous field variables as the generalized coordinates, as opposed to the discrete set of variables q_i that we have treated so far. Therefore the first step we have to take is understanding how to make the transition from discrete to continuum.

6.1 Transitioning from Discrete Particles to the Continuum

Rather than starting with fluids, lets instead consider a somewhat simpler system, that of an infinite one dimensional chain of masses m connected by springs with spring constant k, which we take as an approximation for an infinite continuous elastic one-dimensional rod.



If the equilibrium separation of masses is a and the distance the i'th mass is translated from equilibrium is η_i, then

$$V = \frac{k}{2}\sum_i (\eta_{i+1} - \eta_i)^2 \qquad T = \frac{m}{2}\sum_i \dot{\eta}_i^2, \tag{6.1}$$

where V is the potential energy from the springs, and T is the kinetic energy. It is convenient to write the Lagrangian as

$$L = T - V = \frac{1}{2}\sum_i a\left(\frac{m}{a}\dot{\eta}_i^2 - ka\left(\frac{\eta_{i+1} - \eta_i}{a}\right)^2\right), \tag{6.2}$$

and the corresponding equations of motion obtained from the Euler-Lagrange equations as

$$\frac{m}{a}\ddot{\eta}_i - ka\Big(\frac{\eta_{i+1} - \eta_i}{a^2}\Big) + ka\Big(\frac{\eta_i - \eta_{i-1}}{a^2}\Big) = 0. \tag{6.3}$$

Technically both the Lagrangian and the equations of motion are independent of a, but we have introduced factors of a to facilitate taking the continuous limit $a \to 0$. In this limit the masses become progressively more densely distributed along the line. The important question when taking this limit is which quantities do we hold fixed.

Lets define $\mu \equiv \frac{m}{a}$ as the mass density and $Y = ka$ as the Young's modulus. Here Y is equivalent to the spring constant for a continuous rod. (For a rod, the force $F = Y\xi$ where ξ is the longitudinal extension per unit length, or in other words, the strain.) We intend to hold μ and Y fixed when taking the continuous limit.

The key change in the continuous limit is that the discrete position index i becomes a continuous position label x, so instead of $\eta_i = \eta_i(t)$, now $\eta_x = \eta_x(t)$, or with more conventional notation, $\eta = \eta(x,t)$. This also means that

$$\frac{\eta(x+a,t) - \eta(x,t)}{a} \to \frac{\partial \eta}{\partial x}, \tag{6.4}$$

$$\frac{1}{a}\left(\frac{\eta(x+a,t) - \eta(x,t)}{a} - \frac{\eta(x,t) - \eta(x-a,t)}{a}\right) \to \frac{\partial^2 \eta}{\partial x^2}, \tag{6.5}$$

$$\sum_i a \to \int dx\,. \tag{6.6}$$

Using these results in Eq. (6.2) gives $L = \int \mathcal{L}\, dx$ where

$$\mathcal{L} = \frac{1}{2}\left(\mu\left(\frac{\partial \eta(x,t)}{\partial t}\right)^2 - Y\left(\frac{\partial \eta(x,t)}{\partial x}\right)^2\right) \tag{6.7}$$

is the *Lagrangian density.* Likewise, using them in Eq. (6.3) gives the equations of motion

$$\mu\frac{\partial^2 \eta}{\partial t^2} - Y\frac{\partial^2 \eta}{\partial x^2} = 0\,, \tag{6.8}$$

which we recognize as the wave equation.

The results for the Lagrange density and Euler Lagrange equations are of course not indepedent. We can also use

$$\mathcal{L} = \mathcal{L}\Big(\eta, \frac{\partial \eta}{\partial x}, \frac{\partial \eta}{\partial t}, x, t\Big) \tag{6.9}$$

with Hamilton's principle,

$$\delta S = \delta \int_{t_1}^{t_2} \int_{-\infty}^{\infty} \mathcal{L}\, dx\, dt = 0. \tag{6.10}$$

to formulate the dynamics, and thus derive the Euler-Lagrange equations. Because $\eta = \eta(x,t)$ has two parameters, if we follow the standard procedure of varying the path η takes between the two endpoints, we get variations from the dependence of the Lagrange density on its first three arguments. Integrating by parts in each of t and x, and setting the surface terms to zero, then yields

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \eta}{\partial t}\right)}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \eta}{\partial x}\right)}\right) - \frac{\partial \mathcal{L}}{\partial \eta} = 0 \tag{6.11}$$

as the continuum Euler-Lagrange equation. Recall that for N particles we expect N E-L equations for the time dependence, but here we have just one equation. However actually by that counting, this result corresponds to an infinite number of equations, one for each value of x. From this point of view, the derivatives with respect to x are what couples these equations together.

Example For the Lagrangian density of the elastic rod,

$$\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \eta}{\partial t}\right)} = \mu \frac{\partial \eta}{\partial t}, \qquad \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \eta}{\partial x}\right)} = -Y \frac{\partial \eta}{\partial x}, \qquad \frac{\partial \mathcal{L}}{\partial \eta} = 0\,. \tag{6.12}$$

Putting these results together in Eq. (6.11) gives the wave equation $\mu \frac{\partial^2 \eta}{\partial t^2} - Y \frac{\partial^2 \eta}{\partial x^2} = 0$ as anticipated.

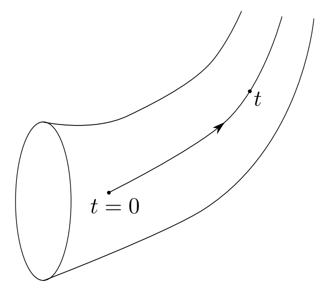
In our above analysis, $\eta = \eta(x,t)$ is a continuum generalized coordinate called a *classical field*. Here t is a parameter and x is a continuous label as well.

Although we have been talking about one dimension so far, it is easy to generalize the above discussion to a higher number of dimensions. For example, in three dimensions we simply have dependence on three continuous label parameters, $\eta = \eta(x,y,z,t)$ or $\eta = \eta(\mathbf{r},t)$. The field $\eta(\mathbf{r},t)$ is called a scalar field because the output is a single number. With multiple dimensions we also have vector fields $\eta(\mathbf{r},t)$, where the output is a vectors. An example of vector fields that you are familiar with are the electromagnetic fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$. In fact, classical fields of exactly this sort are also the starting point for formulating quantum field theory. One formulates a classical Lagrangian density $\mathcal{L}$ (that is most often Lorentz invariant) which depends on fields like the electromagnetic scalar and vector potentials $\phi(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t)$. Then one quantizes these fields.

Our description of fluids will make use of classical field variables in 3-dimensions without considering quantization.

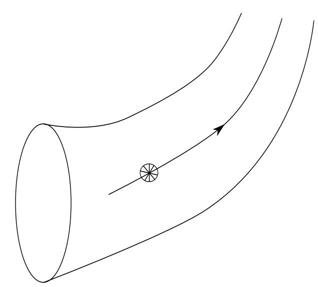
Eulerian & Lagrangian Variables

Let us consider fluid flowing in a tube. A natural set of variables would be to take some initial time $t = 0$ and label all fluid elements by their coordinates at this time: $\mathbf{r}_0$. We will often refer to fluid element, which is a point in the fluid, or sometimes an infinitesimally small region of the fluid. The motion of the fluid could then be described by $\mathbf{r} = \mathbf{r}(\mathbf{r}_0, t)$, which determines the subsequent coordinates of the fluid element labeled by $\mathbf{r}_0$ at the later time t.



If we continue in this direction it leads to the Lagrangian formulation of fluid dynamics. The advantage is that the usual laws of classical mechanics apply to fluid particles. The disadvantage is that it is more work to make the connection to measurable quantities that characterize the fluid.

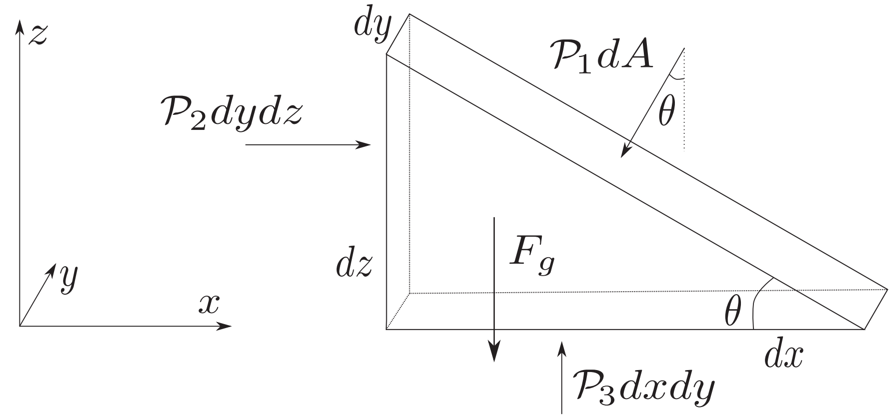
Instead, we will explore an alternate formulation. Pretend we sit at a fixed point $\mathbf{r}$ in the fluid and ask what happens there as a function of the time t. We can think of it like placing a measuring device at $\mathbf{r}$ and measuring, at any time t, quantities like the fluid velocity $\mathbf{v}$, density ρ, or pressure $\mathcal{P}$. In fact, these 5 variables (as density and pressure are scalars, while velocity is a vector) are enough to describe the state of the moving fluid. This is the Eulerian formulation of fluid dynamics. Additionally, ρ and $\mathcal{P}$ are enough to determine all thermodynamic quantities (assuming we know the equation of state).



Example for an ideal gas at temperature T, we have $\mathcal{P}V = nRT$, where V is volume, n is the number of moles, R is contant, and T is temperature. Dividing by V we have that pressure is simply proportional to density, $\mathcal{P} = \rho R'T$ for a rescaled gas constant R'.

Pressure is *isotropic* in a fluid. This means it is the same from all directions, so there is only 1 number for each $(\mathbf{r}, t)$; thus, pressure is a scalar field, $\mathcal{P}(\mathbf{r}, t)$.

To prove this consider the infinitesimal wedge-shaped fluid element below, which we take to be at rest in a gravitational field. Recall that pressure is force per unit area, $\mathcal{P} = \hat{n} \cdot \mathbf{F}/A$, where $\hat{n}$ is a unit vector perpendicular to the area A that the force $\mathbf{F}$ is acting on. In the figure we label three pressures $\mathcal{P}_1$, $\mathcal{P}_1$, $\mathcal{P}_1$ acting on three of the sides, that have corresponding force components in the x and z directions. (There is also pressure on the other two faces with forces in the y direction, but the above will suffice to argue why the pressure is isotropic.)



Next we balance forces for the wedge at rest. Gravity pulls down, and the volume of the wedge is $dxdydz/2$ so the force of gravity is

$$F_g = \rho g \, \frac{dx \, dy \, dz}{2} \, . \tag{6.13}$$

Also by simple trigonometry the area dA of the slanted face can be written in two different ways

$$dA = dy \, \frac{dz}{\sin(\theta)} = dy \, \frac{dx}{\cos(\theta)}.$$

Balancing forces in the x and z directions then means

$$\begin{aligned} 0 = dF_x &= \mathcal{P}_2 \, dy \, dz - \mathcal{P}_1 \sin(\theta) \, dA = (\mathcal{P}_2 - \mathcal{P}_1) \, dy \, dz, \\ 0 = dF_z &= \mathcal{P}_3 \, dx \, dy - \mathcal{P}_1 \cos(\theta) \, dA - \frac{1}{2} \rho g \, dx \, dy \, dz. \end{aligned} \tag{6.14}$$

The first equation implies $\mathcal{P}_1 = \mathcal{P}_2$. In the second equation we can pull out a common $dxdy$ to give

$$\mathcal{P}_3 = \mathcal{P}_1 + \frac{1}{2} \rho g \, dz, \tag{6.15}$$

then as the infinitesimal distance $dz \to 0$ we have

$$\mathcal{P}_1 = \mathcal{P}_3. \tag{6.16}$$

Thus, pressure is the same in all directions. Even if the fluid is moving or even accelerating we would come to the same conclusion. For example, if we had to balance the force against acceleration this would lead to adding term

$$\rho \mathbf{a} \, dx \, dy \, dz \tag{6.17}$$

for acceleration **a**, which again drops out for an infinitesimal fluid element just like the gravitational force did.

Time Derivatives:

The total time derivative $\frac{d}{dt}$ tells us the rate at which a quantity changes as we move with a fluid element. The partial time derivative $\frac{\partial}{\partial t}$ tells us the rate of change of a quantity at a fixed position **r**. We can work out a relation between them.

Example For $\mathcal{P}$

$$\begin{aligned}\frac{d\mathcal{P}}{dt} &= \frac{\partial \mathcal{P}}{\partial t} + \frac{\partial \mathcal{P}}{\partial x}\dot{x} + \frac{\partial \mathcal{P}}{\partial y}\dot{y} + \frac{\partial \mathcal{P}}{\partial z}\dot{z} \qquad (6.18)\\ &= \frac{\partial \mathcal{P}}{\partial t} + \mathbf{v}\cdot\nabla\mathcal{P}.\end{aligned}$$

In general, the time derivative acts as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla \qquad (6.19)$$

on any fluid field (scalar, vector, or tensor) that is a function of (x, y, z, t).

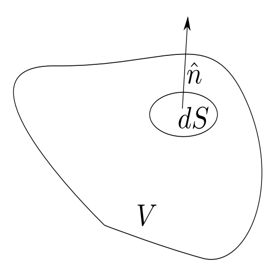
6.2 Fluid Equations of Motion

6.2.1 Continuity Equations

Let us consider a volume V of fluid. Then from Gauss' divergence theorem (which you may recall from electomagnetism, but which is a general result of vector calculus for any vector field):

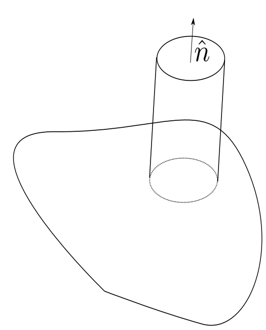
$$\int_V dV\, \nabla\cdot\mathbf{v} = \int_{\partial V} d\mathcal{S}\, \hat{n}\cdot\mathbf{v} = \int_{\partial V} d\boldsymbol{\mathcal{S}}\cdot\mathbf{v}, \qquad (6.20)$$

where ∂V is the closed area that bounds the volume V, $\hat{n}$ is unit vector orthogonal to the surface element dS, and $d\boldsymbol{\mathcal{S}} \equiv dS\hat{n}$.



Lets ask: As the fluid moves, how does V change?

The quantity $\hat{n}\cdot\mathbf{v}$ is the outward velocity of the surface $d\mathcal{S}$, so $\hat{n}\cdot\mathbf{v}\, d\mathcal{S}\, dt$ is the volume added in a time interval dt, as illustrated on the right.



This means that the change in volume can be determined by adding up all the changes from integrating over the entire surface

$$\frac{dV}{dt} = \int_{\partial V} \mathbf{v} \cdot d\boldsymbol{\mathcal{S}} = \int_V \nabla \cdot \mathbf{v} \, dV \,. \tag{6.21}$$

This result becomes even simpler if we consider an infinitesimal volume δV over which $\nabla \cdot \mathbf{v}$ does not vary, then we can pull $\nabla \cdot \mathbf{v}$ outside the integral to obtain simply

$$\frac{d\delta V}{dt} = \delta V \nabla \cdot \mathbf{v}. \tag{6.22}$$

Thus the divergence of the velocity, $\nabla \cdot \mathbf{v}$, controls how the fluid volume expands with time. If $\nabla \cdot \mathbf{v} = 0$ everywhere then we say the fluid is *incompressible* because for every volume element $\frac{dV}{dt} = 0$.

Even if the volume changes, the mass of the fluid element will not,

$$\frac{d\delta m}{dt} = \frac{d}{dt} (\rho \delta V) = 0. \tag{6.23}$$

This implies that

$$0 = \delta V \frac{d\rho}{dt} + \rho \frac{d\delta V}{dt} = \delta V \left(\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} \right), \tag{6.24}$$

so

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0. \tag{6.25}$$

Expanding out the time derivative into partial derivatives this yields

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v} = 0, \tag{6.26}$$

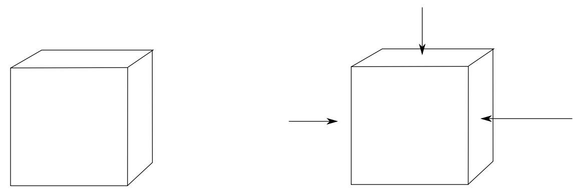
and simplifying the result gives

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \,. \tag{6.27}$$

This is an important result the *continuity equation* for mass of the fluid, which is a partial differential equation in the fluid variables.

Here $\rho \mathbf{v}$ is the mass density flux (the flow of mass density), and this equation says that nowhere is the matter making up the fluid created or destroyed. If the density changes, then there must be an incoming or outgoing flux. This is easier to see if we use the divergence theorem for the vector $(\rho \mathbf{v})$ to write this result in integral form,

$$\frac{\partial}{\partial t}\int_V dV\,\rho = -\int_{\partial V} dS\,\hat{n}\cdot(\rho\mathbf{v})\,. \tag{6.28}$$



Mass increase equals the inflowing mass

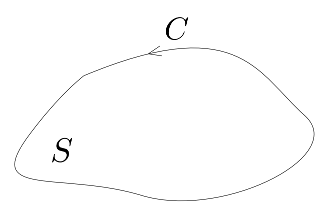
Here $\frac{\partial}{\partial t}\int_V dV\rho$ is the increase of mass in the volume V, while $\int_{\partial V} dS\hat{n}\cdot(\rho\mathbf{v})$ is the outflow of mass through the surface surrounding this volume (which becomes an inflow with the minus sign).

We've talked about $\nabla\cdot\mathbf{v}$ thus far, so it's natural to ask: is there a physical interpretation to $\nabla\times\mathbf{v}$? There is indeed.

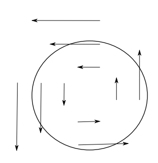
The quantity $\mathbf{\Omega} = \nabla\times\mathbf{v}$ is the *vorticity*. If the velocity is constant or uniform in a single direction $\vec{v} = v_x(x)\hat{x}$, then its obvious that $\mathbf{\Omega} = 0$. To consider what it measures we can use Stoke's theorem for the velocity vector field,

$$\int_{\mathcal{S}}(\nabla\times\mathbf{v})\cdot\hat{n}\,d\mathcal{S} = \oint_{\partial\mathcal{S}}\mathbf{v}\cdot d\mathbf{l}, \tag{6.29}$$

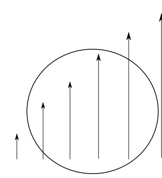
where $\mathcal{S}$ is now an open surface and $\partial\mathcal{S}$ is its closed boundary curve. We can use this result to determine when $\mathbf{\Omega} = \nabla\times\mathbf{v}$ is non-zero.



Consider a rotating fluid and the circular curve shown. When we integrate the velocity along this circular curve the terms always add up, so $\mathbf{\Omega}\neq 0$. It thus provides a measures of the rate of rotation of the fluid.



If we consider flow in one direction with a velocity gradient, again $\mathbf{\Omega}\neq 0$. Here a larger positive contribution to the integral is obtained from the right side of the circle, relative to the negative contribution from the left side.



If $\nabla\times\mathbf{v} = 0$ everywhere in a moving fluid, we say the flow is *irrotational*.

We can determine more precisely what Ω is as follows. Consider a rotating coordinate system with constant angular velocity $\boldsymbol{\omega}$, so $\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}$. Then

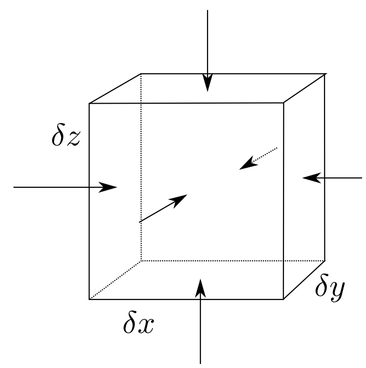
$$\begin{aligned}\nabla \times \mathbf{v} &= \nabla \times \mathbf{v}' + \nabla \times (\boldsymbol{\omega} \times \mathbf{r}) \\ &= \nabla \times \mathbf{v}' + \boldsymbol{\omega} \nabla \cdot \mathbf{r} - \boldsymbol{\omega} \cdot \nabla \mathbf{r} \\ &= \nabla \times \mathbf{v}' + 2\boldsymbol{\omega} \,.\end{aligned}$$

Now $\boldsymbol{\omega}$ was constant, but we have not yet specified its value. If we pick $\boldsymbol{\omega} = \frac{1}{2} \nabla \times \mathbf{v}$ at a point $\mathbf{r}$, then $\nabla \times \mathbf{v}' = 0$ and the fluid is irrotational at $\mathbf{r}$ in the rotating frame. This argument can be repeated for other points $\mathbf{r}$. Considering this from the point of view of the origianl frame we thus see that $\boldsymbol{\Omega}/2$ is the angular velocity of the fluid at position $\mathbf{r}$.

6.2.2 Ideal Fluid: Euler's Equation and Entropy Conservation

Let us consider an *ideal fluid* which has no energy dissipation due to internal friction (meaning no viscosity) and no heat exchange between different parts of the fluid (meaning no thermal conductivity).

The force on a fluid element δV from pressure in the $\hat{x}$ direction is



$$\begin{aligned}\delta F_x &= F_x(x) - F_x(x + \delta x) = \\ &= \frac{\mathcal{P}(x) - \mathcal{P}(x + \delta x)}{\delta x} \, \delta y \, \delta z \, \delta x = -\frac{\partial \mathcal{P}}{\partial x} \delta V \,.\end{aligned} \tag{6.30}$$

More generally accounting for all directions we have

$$\delta \mathbf{F} = -\nabla \mathcal{P} \, \delta V. \tag{6.31}$$

From external forces $\delta \mathbf{F}$ it is useful to define the force density $\mathbf{f}$ by $\delta \mathbf{F} = \mathbf{f} \delta V$. This means that Newton's law, $m\mathbf{a} = \mathbf{F}$ becomes

$$\rho \delta V \frac{d\mathbf{v}}{dt} = (-\nabla \mathcal{P} + \mathbf{f}) \delta V. \tag{6.32}$$

Writing out the total time derivatives this becomes

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{\nabla \mathcal{P}}{\rho} = \frac{\mathbf{f}}{\rho} \tag{6.33}$$

which is the Euler equation for fluid dynamics. There are two special cases which are useful to consider.

1. Often $\mathbf{f}$ can be derived from a potential: $\mathbf{f} = -\rho \nabla \Phi$. Note that Φ here is a potential energy per unit mass. For example, with gravity, $\Phi = gz$, so $\mathbf{f} = -\rho g \hat{z}$. In general, then
$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{\nabla \mathcal{P}}{\rho} + \nabla \Phi = 0 \tag{6.34}$$
is a rewriting of the Euler equation.

2. We can use $\mathbf{v}\times\boldsymbol{\Omega} = \mathbf{v}\times(\nabla\times\mathbf{v}) = \nabla\left(\frac{\mathbf{v}^2}{2}\right) - \mathbf{v}\cdot\nabla\mathbf{v}$. This gives

$$\frac{\partial\mathbf{v}}{\partial t} + \nabla\Big(\frac{\mathbf{v}^2}{2} + \Phi\Big) + \frac{\nabla\mathcal{P}}{\rho} - \mathbf{v}\times\boldsymbol{\Omega} = 0 \tag{6.35}$$

as another rewriting of the Euler equation. For constant ρ, we can take the curl and use the fact that $\nabla\times(\nabla h) = 0$ for any h to obtain

$$\frac{\partial\boldsymbol{\Omega}}{\partial t} = \nabla\times\big(\mathbf{v}\times\boldsymbol{\Omega}\big). \tag{6.36}$$

We will have occasion to use this result later on.

To solve the Euler and continuity partial differential equations we need boundary conditions. The boundary conditions for ideal fluids are simply that they cannot penetrate solid surfaces, so if a surface with normal vector $\hat{n}$ is stationary then

$$\mathbf{v}\cdot\hat{n}\,\Big|_{\text{surface}} = 0\,. \tag{6.37}$$

If the surface does move then

$$\mathbf{v}\cdot\hat{n}\,\Big|_{\text{surface}} = v_{\text{surface}}. \tag{6.38}$$

So far we have four equations (continuity for the scalar density ρ, and the Euler equation for the vector velocity $\mathbf{v}$) for five unknowns. For an ideal fluid the 5^{th} equation,

$$\frac{dS}{dt} = 0, \tag{6.39}$$

is the statement that the entropy S is conserved, so there is no heat exchange. Effectively, this provides a relationship between pressure and density through $\mathcal{P} = \mathcal{P}(\rho, S)$. A simple example is an ideal gas at constant temperature, where $\mathcal{P} = \rho\, R'T$.

6.2.3 Conservation of Momentum and Energy

Due to the term $(\mathbf{v}\cdot\nabla)\mathbf{v}$ which has a $v_i v_j$, the Euler equation is nonlinear. For simpler situations it is therefore very useful to consider conservation laws.

Let us start by considering *Momentum Conservation*. The quantity $\rho\mathbf{v}$ is the flux of mass density, which is also the density of momentum (in direct analogy to $\mathbf{p} = m\mathbf{v}$). Consider $\frac{\partial}{\partial t}(\rho v_i) = \frac{\partial\rho}{\partial t}v_i + \rho\frac{\partial v_i}{\partial t}$. Using the continuity and Euler equations to replace these two partial derivatives, and once again implicitly summing over repeated indices until further notice, this becomes

$$\frac{\partial}{\partial t}(\rho v_i) = -v_i\frac{\partial}{\partial x_j}(\rho v_j) + \rho\left(-v_j\frac{\partial v_i}{\partial x_j} - \frac{1}{\rho}\frac{\partial\mathcal{P}}{\partial x_i}\right) + f_i. \tag{6.40}$$

This is rearranged to give

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial\mathcal{P}}{\partial x_i} + \frac{\partial}{\partial x_j}(\rho v_i v_j) = f_i. \tag{6.41}$$

We define the *stress tensor* for an ideal fluid as

$$T_{ij} = \mathcal{P}\delta_{ij} + \rho v_i v_j \tag{6.42}$$

which gives the momentum flux density (which is to say, the density of momentum in the direction $\mathbf{e}_i$ flowing in the direction of $\mathbf{e}_j$). Note that T_{ij} is symmetric. Then, in vector form, the equation above becomes

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot \mathbb{T} = \mathbf{f}\,. \tag{6.43}$$

This describes the conservation of linear momentum density with sources given by the external force densities $\mathbf{f}$. Comparing to the continuity equation where the density ρ is a scalar so its flux $\rho\mathbf{v}$ is a vector, here the momentum density $\rho\mathbf{v}$ is a vector so its flux $\mathbb{T}$ is a tensor.

Next consider *Conservation of Energy.* Conservation of energy can be said to arise from the Euler equation $d\mathbf{v}/dt + \nabla\mathcal{P}/\rho + \nabla\Phi = 0$. Note that here we are switching back to the total time derivative since this is more useful for our discussion of energy. For a volume element δV, we take the inner product of the Euler equation with $\rho\mathbf{v}\delta V$ to obtain

$$\delta V \rho\, \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} + \delta V\, \mathbf{v} \cdot \nabla\mathcal{P} + \delta V \rho\, \mathbf{v} \cdot \nabla\Phi = 0\,. \tag{6.44}$$

Given that $\frac{d}{dt}(\rho\delta V) = 0$ by the conservation of mass we can move this combination inside of total time derivatives. Also recall that $\mathbf{v}\cdot\nabla = d/dt - \partial/\partial t$. Using these two facts we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\rho\mathbf{v}^2\delta V\right) + (\mathbf{v}\cdot\nabla\mathcal{P})\delta V + \frac{d}{dt}(\rho\Phi\delta V) - \rho\frac{\partial\Phi}{\partial t}\delta V = 0. \tag{6.45}$$

Next we try to convert the second term to time derivatives. Consider using

$$\frac{d}{dt}(\mathcal{P}\delta V) = \frac{\partial\mathcal{P}}{\partial t}\delta V + (\mathbf{v}\cdot\nabla\mathcal{P})\delta V + \mathcal{P}(\nabla\cdot\mathbf{v})\delta V, \tag{6.46}$$

where we recalled that $d\delta V/dt = (\nabla\cdot\mathbf{v})\delta V$. Using this to eliminate $(\mathbf{v}\cdot\nabla\mathcal{P})\delta V$ gives

$$\frac{d}{dt}\left(\frac{1}{2}\rho\mathbf{v}^2\delta V + \rho\Phi\delta V + \mathcal{P}\delta V\right) = \left(\frac{\partial\mathcal{P}}{\partial t} + \rho\frac{\partial\Phi}{\partial t}\right)\delta V + \mathcal{P}\nabla\cdot\mathbf{v}\delta V, \tag{6.47}$$

where $\frac{1}{2}\rho\mathbf{v}^2\delta V$ is the kinetic energy, $\rho\Phi\delta V$ is the external potential energy, and $\mathcal{P}\delta V$ is the internal potential energy due to pressure. The terms with partial time derivatives act like sources. Unfortunately there is still a term without a total or partial time derivative, however this term is easier to interpret. It is related to the work W_u done by δV when it expands and exerts pressure on the surrounding fluid. In particular

$$\frac{dW_u}{dt} = \mathcal{P}\frac{d(\delta V)}{dt} = \mathcal{P}\nabla\cdot\mathbf{v}\,\delta V \equiv -\frac{d}{dt}(U\delta m) \tag{6.48}$$

where in the last step we have defined the work as a negative potential energy U (per unit mass). If the equation of state is given, this U can be calculated either as an integral in density or pressure,

$$U = \int d(\delta V)\,\frac{\mathcal{P}}{\delta m} = \int_{\rho_0}^{\rho} d\rho'\,\frac{\mathcal{P}(\rho')}{\rho'^{\,2}} = \int_{\mathcal{P}_0}^{\mathcal{P}} \mathcal{P}'\Big(\frac{1}{\rho^2}\frac{d\rho}{d\mathcal{P}'}\Big)(\mathcal{P}')\;d\mathcal{P}' \tag{6.49}$$

where we used the fixed $\delta m = \delta V \rho$ to switch variables between δV and ρ. Using Eq. (6.48) means we can now write everything in terms of time derivatives,

$$\frac{d}{dt}\left(\left(\frac{1}{2}\rho\mathbf{v}^2 + \rho\Phi + \mathcal{P} + \rho U\right)\delta V\right) = \left(\frac{\partial\mathcal{P}}{\partial t} + \rho\frac{\partial\Phi}{\partial t}\right)\delta V \tag{6.50}$$

This is the equation for *energy conservation* in an ideal fluid. If $\mathcal{P}$ and Φ are not explicitly dependent on time at any point in space (which is often the case), then any fluid element has a constant total energy as it moves along (recall that this is the meaning of d/dt). For applications to fluids it is more convenient to divide this result by $\delta m = \rho\delta V$ to give

$$\frac{d}{dt}\left(\frac{\mathbf{v}^2}{2} + \Phi + \frac{\mathcal{P}}{\rho} + U\right) = \frac{1}{\rho}\frac{\partial\rho}{\partial t} + \frac{\partial\Phi}{\partial t}\,. \tag{6.51}$$

This is *Bernoulli's equation*.

6.3 Static Fluids & Steady Flows

Having derived the equations of motion and conservation laws for ideal fluids, let us now consider some important special cases.

<u>Static Fluids</u>

Static fluids have $\mathbf{v} = 0$ everywhere, so the fluid is at rest (implying mechanical equilibrium). Continuity then says $\frac{\partial\rho}{\partial t} = 0$, so ρ and $\mathcal{P}$ are independent of time. If $\Phi = gz$ for gravity, the Euler equation says

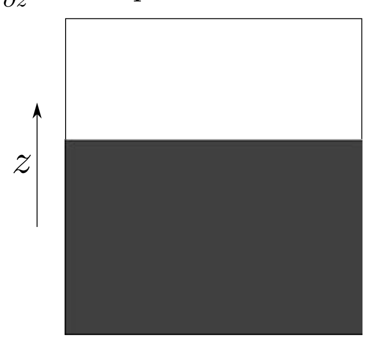
$$\frac{\nabla\mathcal{P}}{\rho} = -g\hat{z}, \tag{6.52}$$

so $\mathcal{P}$ and ρ can each only be functions of z, while in fact $\frac{1}{\rho}\frac{\partial\mathcal{P}}{\partial z}$ is independent of z.

Example if the density ρ is constant, then if $\Phi = gz$, then

$$\mathcal{P}(z) = \mathcal{P}_0 - \rho g z, \tag{6.53}$$

so the pressure decreases with height.



Example let us pretend the atmosphere is an ideal gas at a uniform constant temperature T. As $\mathcal{P} = \rho R'T$, then

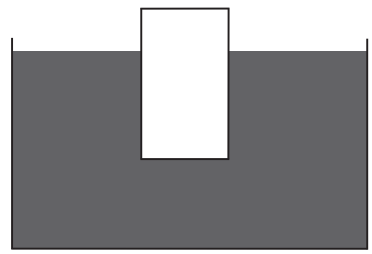
$$\frac{\partial \mathcal{P}}{\partial z} = -\frac{g}{R'T}\mathcal{P}, \tag{6.54}$$

so

$$\mathcal{P}(z) = \mathcal{P}_0 e^{-\frac{gz}{R'T}}\,, \tag{6.55}$$

and the pressure falls exponentially.

Example the Archimedes principle says that the pressure balances the weight of any displaced fluid. This follows from our first example above. The pressure on the bottom of the object displacing the fluid is $\rho g z$, where z is the distance from the surface. The force is pressure times area, so that is the volume displaced times ρg, or the mass displaced times g.



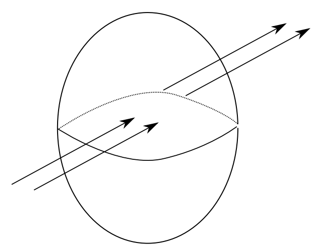
Note that if the temperature is not uniform, then the mechanical equilibrium is not stable, as the temperature gradients result in *convection* currents which mix the fluid. Therefore we have used the fact that we are discussing an ideal fluid.

Steady Flows

Steady flows are ones in which $\frac{\partial \rho}{\partial t} = 0$, $\frac{\partial \mathcal{P}}{\partial t} = 0$, and $\frac{\partial \mathbf{v}}{\partial t} = 0$ at every position $\mathbf{r}$.

In this case, the continuity equation becomes $\nabla \cdot (\rho \mathbf{v}) = 0$, so what flows in must flow out. This is most easily implemented by using the integral form, where for any closed surface S we have

$$\int_S d\mathbf{S} \cdot (\rho \mathbf{v}) = 0\,. \tag{6.56}$$



For a steady flow the Bernoulli equation becomes the statement that

$$\frac{\mathbf{v}^2}{2} + \Phi + \frac{\mathcal{P}}{\rho} + U = B \tag{6.57}$$

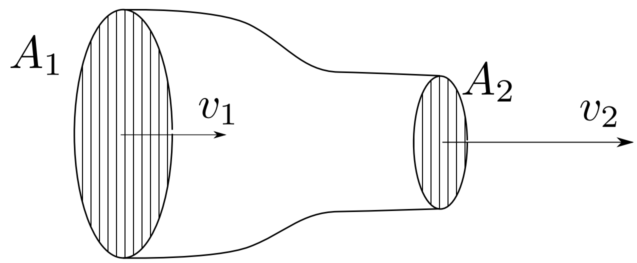
where B is a constant along the paths of fluid elements. Most often we will consider gravity where $\Phi = gz$.

A *steady incompressible fluid* is one where $\nabla \cdot \mathbf{v} = 0$ and U is constant. Continuity now says $\nabla \rho = 0$ so ρ is constant as well. Moreover, the Bernoulli equation now says

$$\frac{\mathbf{v}^2}{2} + \Phi + \frac{\mathcal{P}}{\rho} = B' \tag{6.58}$$

is also constant.

Example for a horizontal pipe filled with an incompressible fluid (which is approximately true of water at room temperature) of constant density ρ.



Lets consider the two ends to be at approximately at the same height so we can drop the term $\Phi = gz$. At the hatched areas shown we know that the flow must be tangential to the edge of the pipe, so a valid solution is to simply consider the velocities to be uniform and tangential to the enclosing pipe across each of these areas. Using Bernoulli this implies that

$$\frac{\rho v_1^2}{2} + \mathcal{P}_1 = \frac{\rho v_2^2}{2} + \mathcal{P}_2. \tag{6.59}$$

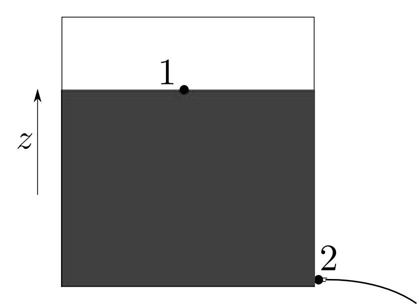
Furthermore, the continuity equation for the enclosed areas shown (some sides being those of the pipe) implies that the flux in at one end must equal the flux out at the other $A_1 v_1 = A_2 v_2$. Together this gives

$$\mathcal{P}_1 = \mathcal{P}_2 + \frac{1}{2}\rho v_2^2 \left(1 - \left(\frac{A_2}{A_1}\right)^2\right). \tag{6.60}$$

Since $A_1 > A_2$ we have $v_2 > v_1$, and this implies $\mathcal{P}_1 > \mathcal{P}_2$.

Example let us consider a water tank filled to a height z_1 with a hole at height $z_2 < z_1$ that produces a jet of water.
Let us assume $v_1 \approx 0$ (so the tank is much larger than the hole). Then

$$\frac{1}{2}\rho v_2^2 + \mathcal{P}_2 + \rho g z_2 = \mathcal{P}_1 + \rho g z_1. \tag{6.61}$$



Additionally, $\mathcal{P}_2 = \mathcal{P}_1 = \mathcal{P}_{\text{atmosphere}}$, so the pressure terms cancel out, and we can solve for the v_2 velocity to give

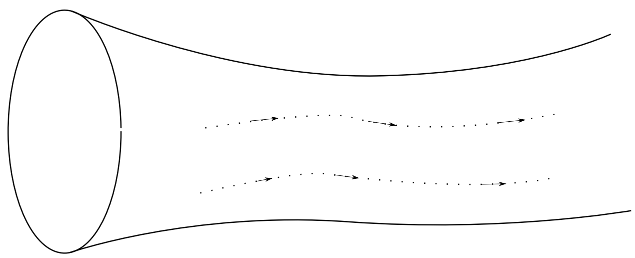
$$v_2 = \sqrt{2g(z_1 - z_2)}. \tag{6.62}$$

This is the same velocity as that for any mass falling from rest through a height $z_1 - z_2$. Of course a key difference for the jet of water is that this velocity is horizontal rather than vertical.

Lets discuss two common ways to picture flows. One is through *stream lines*, which are lines that are everywhere tangent to the instantaneous velocity, meaning

$$\frac{d\mathbf{x}(s)}{ds} \times \mathbf{v} = 0 \tag{6.63}$$

for some parameter s that determines the distance along the streamline. These lines are drawn at some fixed time and never cross since there is a unique velocity at every point.



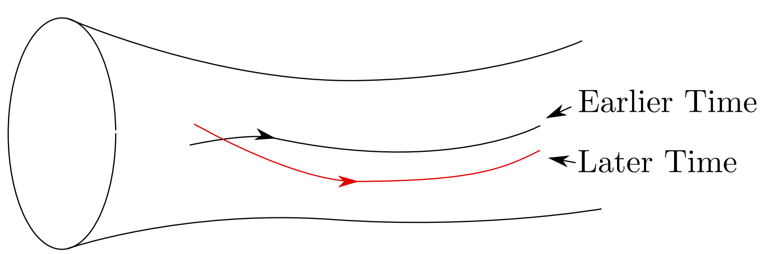
Writing out the cross product we find $0 = d\mathbf{x}/ds \times \mathbf{v} = \hat{z}\big(v_y \frac{dx}{ds} - v_x \frac{dy}{ds}\big) + \ldots$ implying that we have

$$\frac{dy}{dx} = \frac{v_y}{v_x}\,, \qquad \frac{dz}{dx} = \frac{v_z}{v_x}\,, \qquad \left(\text{or } \frac{dz}{dy} = \frac{v_z}{v_y}\right). \tag{6.64}$$

When we use the equations in this form we would need to be able to switch from s to the variable x to uniquely parameterize the curve.

Another method of picturing the flow is through *flow lines*, which are paths that are followed by fluid elements, meaning

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}, t). \tag{6.65}$$



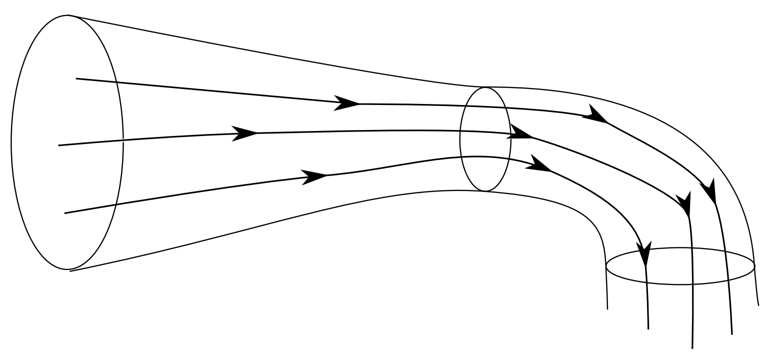
Since flow lines are time dependent, they can in general cross since the path a fluid element takes through a point may differ at a later time.

For a steady flow, the lines are time independent, and stream lines and flow lines are identical. To prove this we consider the flow line equations $dx/dt = v_x$, $dy/dt = v_y$, and note that since the velocities are time independent that we can eliminate time through the ratio

$dy/dx = v_y/v_x$ (and similar for other directions), which is the equation for the stream lines. To go the opposite direction we simply pick $s = t$ to parameterize the streamline, and note that $\mathbf{v} \times \mathbf{v} = 0$. Furthermore, for a steady fluid we have Bernoulli's law

$$\frac{\mathbf{v}^2}{2} + \Phi + \frac{\mathcal{P}}{\rho} + U = B \tag{6.66}$$

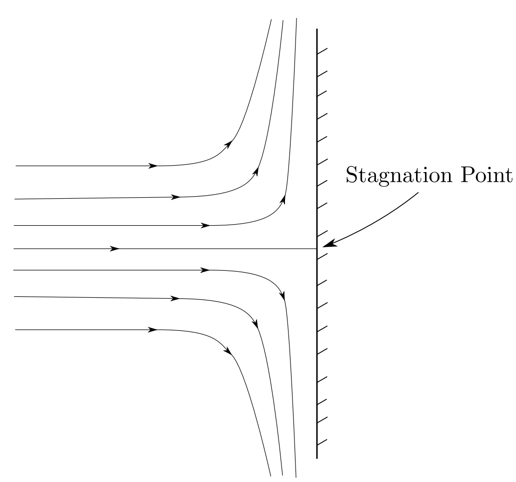
which we now understand is constant along stream lines.



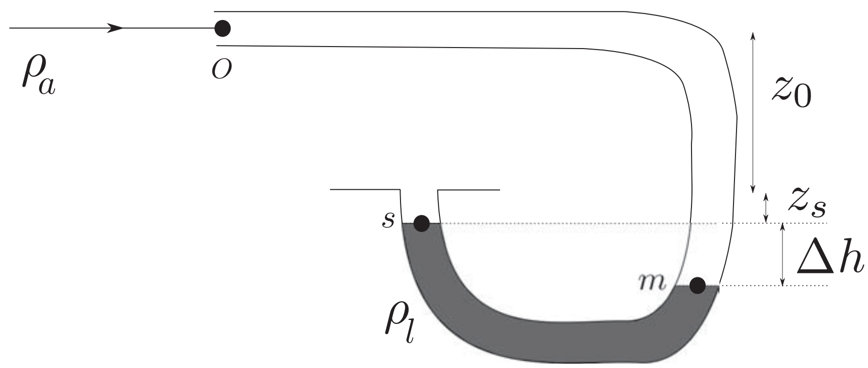
Stagnation Points and a Pitot Tube

Consider an ideal incompressible fluid in a steady flow which flows in an almost uniform manner, and then hits a wall.

Since the velocity must be tangential at the surface, the flow can either curve left or right, and it is often the case that parts of the flow will go in both directions. In this case there is a stagnation point o, where $v_o = 0$. If we consider the streamline that hits the stagnation point then $\mathcal{P}_o = \mathcal{P}_\infty + \frac{1}{2}\rho v_\infty^2$, where v_∞ and $\mathcal{P}_\infty$ are, respectively, the velocity and pressure infinitely far away. Thus the largest pressure in the entire flow occurs at the stagnation point where there is only pressure and no kinetic energy.



Now let us consider a Pitot tube, which is a device used to measure velocity (for example on airplanes).



If the density is ρ_a in the air and ρ_l in the liquid, we can write down the Bernoulli equations obtained by comparing the air and liquid flows at ∞, the stagnation point o, at the point s near the surface (where the air velocity is the same as at ∞ and liquid is at rest), and at the point m at the top of the liquid inside the column. This gives:

$$\begin{aligned}\mathcal{P}_o &= \mathcal{P}_\infty + \frac{1}{2}\rho_a v_\infty^2 \\ \mathcal{P}_s &= \mathcal{P}_\infty + \frac{1}{2}\rho_a g(z_o + z_s) \\ \mathcal{P}_m &= \mathcal{P}_o + \rho_a g(z_0 + z_s + \Delta h) \\ &= \mathcal{P}_s + \rho_l g \Delta h\end{aligned}$$

Subtracting the 2$^{\text{nd}}$ equation from the 1$^{\text{st}}$, subtracting the 4$^{\text{th}}$ equation from the 3$^{\text{rd}}$, and then adding these two results cancels all the pressure terms, and leaves

$$\frac{1}{2}\rho_a v_\infty^2 = g(\rho_l - \rho_a)\Delta h. \tag{6.67}$$

This can be rearranged to write

$$v_\infty^2 = 2g\left(\frac{\rho_l}{\rho_a} - 1\right)\Delta h \approx 2g\frac{\rho_l}{\rho_a}\Delta h \tag{6.68}$$

allowing us to determine the velocity of the air v_∞ in terms of the known ratio of densities $\frac{\rho_l}{\rho_a} \gg 1$ and simply the measured height between the liquid on each side, Δh.

6.4 Potential Flow

When a flow is everywhere both irrotational and incompressible it is known as *potential flow*. Such flows may be steady or not steady. Since $\mathbf{\Omega} = \nabla \times \mathbf{v} = 0$ the velocity field is conservative. This means there exists a *velocity potential*

$$\phi(\mathbf{r}) = \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{v}(\mathbf{r}', t) \cdot d\mathbf{r}' \tag{6.69}$$

which depends only on the endpoints of the integration, and not the path taken between them, such that

$$\mathbf{v} = \nabla\phi \tag{6.70}$$

Since $\nabla \cdot \mathbf{v} = 0$ for an incompressible fluid, then ϕ must solve Laplace's equation

$$\nabla^2\phi = 0\,. \tag{6.71}$$

Solving for the scalar ϕ (with suitable boundary conditions), then immediately gives $\mathbf{v}$.

We can then use the Euler equation to immediately get the pressure. For $\mathbf{\Omega} = 0$ one form of the Euler equation was

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\frac{\mathbf{v}^2}{2} + \Phi\right) + \frac{\nabla\mathcal{P}}{\rho} = 0\,. \tag{6.72}$$

If ρ is constant, then

$$\nabla\left(\frac{\partial\phi}{\partial t} + \frac{\mathbf{v}^2}{2} + \Phi + \frac{\mathcal{P}}{\rho}\right) = 0, \tag{6.73}$$

so

$$\frac{\partial\phi}{\partial t} + \frac{\mathbf{v}^2}{2} + \Phi + \frac{\mathcal{P}}{\rho} = b(t) \tag{6.74}$$

for some function b. For each t we can pick the zero of ϕ so that $b(t)$ is constant. (This is equivalent to shifting $\phi \to \phi + \int^t b(t')\,dt'$, where adding this constant that is independent of $\mathbf{x}$ gives a solution that is equally valid.) The remaining constant b be fixed by a boundary condition on the pressure. Thus the full pressure as a function of $\mathbf{x}$ and t is determined by

$$\frac{\mathcal{P}}{\rho} = -\frac{\partial\phi}{\partial t} - \frac{\mathbf{v}^2}{2} - \Phi + b\,, \tag{6.75}$$

where in principal the first three terms on the right hand side carry both spatial and time dependence. Often we are interested in a steady flow, in which case the term $\partial\phi/\partial t = 0$.

Example if $\phi = v_0 x$, then $\mathbf{v} = v_0\hat{x}$, which is a specific case of a uniform flow.

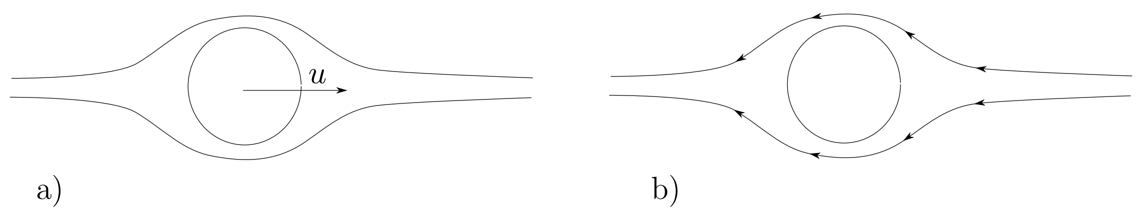
Example if $\phi = A\ln(r)$ in 2 dimensions, then for all $r > 0$

$$\mathbf{v} = \frac{\partial\phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta} = \frac{A}{r}\hat{r}, \tag{6.76}$$

which is a point source. An analogous point source in 3 dimensions would arise from $\phi = -\frac{A}{r}$, which gives $\mathbf{v} = \frac{A}{r^2}\hat{r}$ for all $r > 0$. (At $r = 0$ there would be a delta function source, so we do not satisfy Laplace's equation at this point.)

Example Consider $\phi = \frac{\Gamma}{2\pi}\theta$ in polar coordinates in the 2 dimensional plane. Then $v_r = \frac{\partial\phi}{\partial r} = 0$ and $v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{\Gamma}{2\pi r}$ for all $r > 0$. This corresponds to a potential vortex about the point $r = 0$.

Example Consider a sphere of radius R moving with constant velocity $\mathbf{u}$ through an incompressible ideal fluid and find its velocity by solving for its velocity potential. Equivalently we can consider the problem of finding the velocity when a sphere is held at rest and the fluid flows in from far away with uniform velocity $-\mathbf{u}$ at infinity. The first situation is shown in figure a) and the second as figure b).



We use coordinates centered on the sphere, and define the axes so that $\mathbf{u} = u\hat{x}$. The problem is spherically symmetric other than the directionality from $\mathbf{u}$, so its natural to expect $\mathbf{v} \propto \mathbf{u}$. Since $\nabla\phi = \mathbf{v}$ this means that we expect $\phi \propto \mathbf{u}$. (Effectively the boundary condition is linear in $\mathbf{u}$ and the equation $\nabla^2\phi = 0$ is linear. We could also explicitly demonstrate the proportionality $\mathbf{v} \propto \mathbf{u}$ using dimensional analysis, as we will discuss in more detail later in this chapter.)

It is actually easier to consider the sphere being at rest with the fluid moving past it as in b), so lets start with this case. Since $\nabla^2\phi = 0$ is linear, we can solve using superposition. The velocity potential

$$\phi(\mathbf{x}) = -ux + \phi'(\mathbf{x}) \tag{6.77}$$

has a term $-ux$ giving the correct uniform flow far away from the sphere. Therefore, with r the distance from the center of the sphere, we have

$$\lim_{r\to\infty} \phi'(\mathbf{x}) = 0. \tag{6.78}$$

Another way to see this is that taking the gradient of Eq. (6.77) gives

$$\mathbf{v} = -u\hat{x} + \mathbf{v}', \tag{6.79}$$

which is simply the translation between the velocity field $\mathbf{v}$ for b) and the velocity field $\mathbf{v}'$ for a). For the situation a) we would anticipate Eq. (6.78) as the correct boundary condition, since the fluid is at rest at infinity when it is infinitely far away from the disturbance caused by dragging the sphere. We can look for a solution for ϕ'.

As $\nabla^2\phi' = 0$ with $\lim_{r\to\infty}\phi' = 0$, one option could be $\phi' = \frac{1}{r}$ in 3 dimensions, but this would give a point source solution with velocity moving radially outward from our sphere and hence make it impossible to satisfy the appropriate boundary condition on the sphere (its also not $\propto \mathbf{u}$). Instead, let us consider a dipole source

$$\phi' = A\mathbf{u}\cdot\nabla\left(\frac{1}{r}\right) \tag{6.80}$$

for some constant A. This proposal is linear in $\mathbf{u}$, and satisfies $\nabla^2\phi' = 0$ for $r > 0$ since the derivatives commute: $\nabla^2\phi' = A(\mathbf{u}\cdot\nabla)\nabla^2(1/r) = 0$. It remains to compute ϕ' and $\mathbf{v}$ explicitly and check that we can satisfy the boundary conditions (and proper dimensions) with this solution. Since

$$\nabla\left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3}, \tag{6.81}$$

we have

$$\phi' = -\frac{A\mathbf{u}\cdot\mathbf{r}}{r^3}. \tag{6.82}$$

Moreover,

$$\nabla(\mathbf{u}\cdot\mathbf{r}) = \mathbf{u}, \qquad \nabla r^{-n} = -nr^{-(n+2)}\mathbf{r}, \tag{6.83}$$

so the solution for $\mathbf{v}'$ is

$$\mathbf{v}' = \nabla\phi' = \frac{A}{r^3}\left(-\mathbf{u} + \frac{3(\mathbf{u}\cdot\mathbf{r})\mathbf{r}}{r^2}\right) \tag{6.84}$$

and we then also have obtained $\mathbf{v} = -\mathbf{u} + \mathbf{v}'$. The boundary condition on the surface of the sphere (which is $r = R$ for case b) where the sphere is at rest) is $\mathbf{v}\cdot\hat{r} = 0$. This means

$$0 = -\mathbf{u}\cdot\hat{r} - \frac{A\mathbf{u}\cdot\hat{r}}{R^3} + \frac{3A\mathbf{u}\cdot\hat{r}}{R^3}, \tag{6.85}$$

which has the solution $A = \frac{R^3}{2}$. Thus,

$$\mathbf{v} = -\mathbf{u} + \frac{R^3}{2r^3}\left(\frac{3(\mathbf{u}\cdot\mathbf{r})\mathbf{r}}{r^2} - \mathbf{u}\right). \tag{6.86}$$

This solution has the right dimensions and satisfies the boundary conditions on the sphere and at infinity.

For steady flow, we can then use Bernoulli's equation to get the pressure on the sphere, constant $= (\mathcal{P} + \frac{1}{2}\rho v^2)_{r=R} = (\mathcal{P} + \frac{1}{2}\rho v^2)_{r=\infty}$. Squaring our result for the velocity on the sphere, setting $\mathbf{u}\cdot\mathbf{r} = ur\cos\theta$, and simplifying we find

$$\mathcal{P} = \mathcal{P}_\infty + \frac{\rho\mathbf{u}^2}{8}\left(9\cos^2\theta - 5\right). \tag{6.87}$$

This result for the pressure says that it is the same on the front and back of the sphere, since its unchanged by taking $\theta \to \pi/2 - \theta$. This is quite counterintuitive, since we expect a force on the sphere in b) that would try to push it downstream. This actually results from our approximation that the fluid is ideal (viscosity can not be neglected when trying to answer questions near surfaces).

Another possibility is that our approximation of potential flow is suspect. To explore this, lets ask how common is potential flow? Consider

$$\frac{d}{dt}(\nabla\times\mathbf{v}) = \nabla\times\frac{d\mathbf{v}}{dt} = \nabla\times\left(\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right), \tag{6.88}$$

which expands into

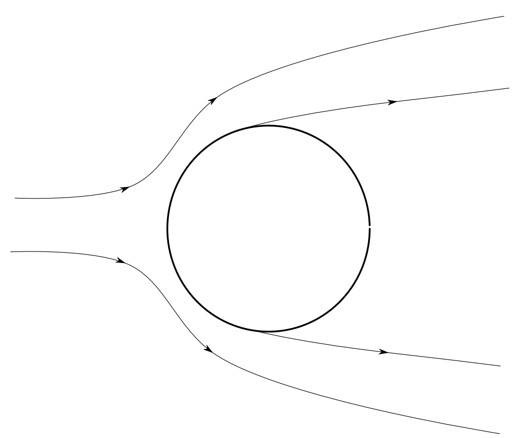
$$\frac{\partial}{\partial t}(\nabla \times \mathbf{v}) + \nabla \times \left(\nabla \left(\frac{\mathbf{v}^2}{2}\right) - \mathbf{v} \times (\nabla \times \mathbf{v})\right) = \frac{\partial \boldsymbol{\Omega}}{\partial t} - \nabla \times (\mathbf{v} \times \boldsymbol{\Omega}) = 0 \tag{6.89}$$

where in the last step we used our result in Eq. (6.36) that we derived from the Euler equation. Thus, $\nabla \times \mathbf{v}$ is conserved along flow lines. If we consider a steady flow which starts out as uniform infinitely far away, then there is no vorticity at infinity, and

$$\nabla \times \mathbf{v} = 0 \tag{6.90}$$

on every stream line, and remains that way for the entire flow. Thus we have a potential flow with no vorticity.

If we wanted to get around the counterintuitive behavior we have found in our ideal fluid solution, but stick with the ideal fluid framework, then we would have to allow for the existence of discontinuous solutions. For an ideal fluid flowing past a sphere, we could propose that stream lines exist that start tangential to the spherical surface, and hence satisfy the boundary conditions, and can have $\nabla \times \mathbf{v} \neq 0$ since they are not connected to infinity. Behind the sphere we could then imagine having fluid at rest.



The correct treatment of boundary layers near objects and of wakes, does not require discontinuous solutions of this sort, but instead simply requires the inclusion of viscosity, which we will turn to shortly, after treating one final important example from ideal fluids.

6.5 Sound Waves

In this section we will explore an example where $\nabla \cdot \mathbf{v} \neq 0$ plays an important role. To set things up, consider a compressible fluid at rest with pressure $\mathcal{P}_0$ and density ρ_0 in equilibrium with an external force density $\mathbf{f}_0$. If $\mathcal{P}_0$ and ρ_0 are constant and uniform, then

$$\frac{\nabla \mathcal{P}_0}{\rho_0} = \frac{\mathbf{f}_0}{\rho_0} \tag{6.91}$$

from the Euler equation. Now lets add disturbances $\mathcal{P}'$ and ρ' to this system

$$\mathcal{P} = \mathcal{P}_0 + \mathcal{P}', \tag{6.92}$$

$$\rho = \rho_0 + \rho', \tag{6.93}$$

with $\mathcal{P}' \ll \mathcal{P}_0$ and $\rho' \ll \rho_0$. These disturbances will induce a velocity field as well, $\mathbf{v}(\mathbf{r}, t)$, which we will also assume is small, so that perturbation theory applies. We will therefore drop terms that are second order or higher in any of $\{\mathcal{P}', \rho', \mathbf{v}\}$. Using perturbation theory on the Euler equation, that is

$$0 = \rho\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p - \mathbf{f}_0 = \nabla\mathcal{P}_0 - \mathbf{f}_0 + \rho_0\frac{\partial \mathbf{v}}{\partial t} + \nabla\mathcal{P}' + \ldots \tag{6.94}$$

where we have dropped terms $\rho\mathbf{v}\cdot\nabla\mathbf{v} = \mathcal{O}(\rho_0\mathbf{v}^2)$ and $\mathcal{O}(\rho'\mathbf{v})$. Using $\mathbf{f}_0 = \nabla\mathcal{P}_0$ then gives

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{\nabla\mathcal{P}'}{\rho_0} \tag{6.95}$$

as the Euler equation to 1^{st} order in perturbations. To the same order, continuity says

$$\frac{\partial \rho'}{\partial t} = -\nabla\cdot(\rho\mathbf{v}) = -\rho_0\nabla\cdot\mathbf{v}. \tag{6.96}$$

where we have again dropped second order terms. Finally the appropriate thermodynamic equation of state is

$$\rho' = \frac{\rho_0}{B}\mathcal{P}' \tag{6.97}$$

where B is a constant known as the bulk modulus. The bulk modulus describes a substance's resistance to compression, and this formula arises from $B = \rho\frac{\partial\mathcal{P}}{\partial\rho} \approx \rho_0\frac{\mathcal{P}'}{\rho'}$. Using this result we can eliminate density ρ' to get a second equation involving only the pressure and velocity disturbances,

$$\frac{\partial\mathcal{P}'}{\partial t} = \frac{B}{\rho_0}\frac{\partial\rho'}{\partial t} = -B\nabla\cdot\mathbf{v} \tag{6.98}$$

Combining Eq. (6.95) and Eq. (6.98) we can derive a differential equation for the pressure disturbance

$$\frac{\partial^2\mathcal{P}'}{\partial t^2} = -B\nabla\cdot\frac{\partial\mathbf{v}}{\partial t} = \frac{B}{\rho_0}\nabla^2\mathcal{P}', \tag{6.99}$$

which can be written more simply as

$$\frac{\partial^2\mathcal{P}'}{\partial t^2} - c_{\mathrm{S}}^2\,\nabla^2\mathcal{P}' = 0 \tag{6.100}$$

which is a wave equation for $\mathcal{P}'$, whose solutions move at a velocity $c_{\mathrm{S}} = \sqrt{\frac{B}{\rho_0}}$ which is known as the *speed of sound*. Due to the simple proportionality from the equation of state we also immediately know that

$$\frac{\partial^2\rho'}{\partial t^2} - c_{\mathrm{S}}^2\nabla^2\rho' = 0 \tag{6.101}$$

so the same wave equation also holds for the density.

It remains to derive a differential equation for the velocity. Taking the curl of Eq. (6.95) yields

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{v}) = -\frac{1}{\rho_0}\nabla \times \nabla \mathcal{P}' = 0. \tag{6.102}$$

This means $\nabla \times \mathbf{v}$ does not explicitly depend on time, so we can conveniently take $\nabla \times \mathbf{v} = 0$ initially everywhere, and hence for all times. Using Eq. (6.95) and Eq. (6.98) now gives

$$\begin{aligned}\frac{\partial^2 \mathbf{v}}{\partial t^2} &= -\frac{1}{\rho_0}\nabla\left(\frac{\partial \mathcal{P}'}{\partial t}\right) = \frac{B}{\rho_0}\nabla(\nabla \cdot \mathbf{v}) \\ &= \frac{B}{\rho_0}\Big(\nabla^2 \mathbf{v} - \nabla \times (\nabla \times \mathbf{v})\Big) \\ &= \frac{B}{\rho_0}\nabla^2 \mathbf{v}\,,\end{aligned} \tag{6.103}$$

since $(\nabla \times \mathbf{v}) = 0$. This yields

$$\frac{\partial^2 \mathbf{v}}{\partial t^2} - c_{\mathrm{S}}^2 \nabla^2 \mathbf{v} = 0 \tag{6.104}$$

which means that the velocity of the fluid disturbance also satisfies the same wave equation. The solutions are thus *sound waves* in pressure, density, and velocity, with speed c_{S}. An example of a solution is a plane wave (here written for pressure) which looks like

$$\mathcal{P}' = \mathcal{P}'(\mathbf{r} \cdot \hat{n} - c_{\mathrm{S}} t) \tag{6.105}$$

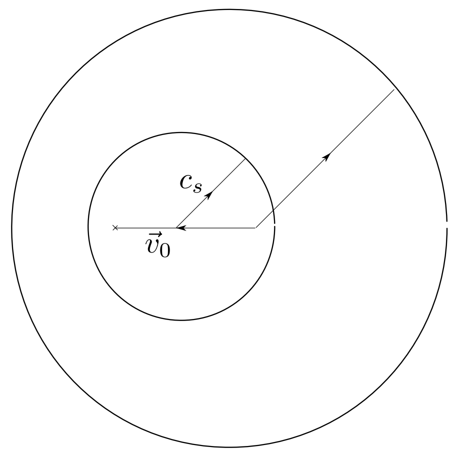
when traveling in the direction of $\hat{n}$.

Note that $\nabla \cdot \mathbf{v} \neq 0$ was important for this derivation. It is reasonable to ask if there is a way that we can determine when the approximation $\nabla \cdot \mathbf{v} = 0$ may be justified. For a flow with characteristic velocity v_0 this can be done by defining the *Mach number*

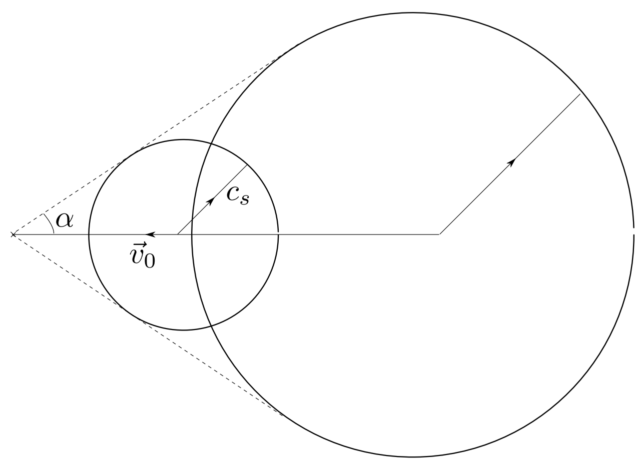
$$M = \frac{v_0}{c_{\mathrm{S}}}\,, \tag{6.106}$$

since the scaling of terms involving $\nabla \cdot \mathbf{v}$ will be determined by this ratio. If $M \ll 1$ then we can treat flow as approximately incompressible, with $\nabla \cdot \mathbf{v} = 0$.

Considering flows with large values of M leads to the concept of shock waves. Consider a flow with initial velocity v_0 in which there is a disturbance. If $M < 1$ then the flow is said to be subsonic, and the perturbation spreads everywhere, because the speed of the perturbation is larger than that of the flow.



On the other hand if $M > 1$, then the disturbance is swept downstream to the right by the flow, and actually propagates downstream within a cone of angle γ defined by $\sin(\gamma) = c_s/v_0 = \frac{1}{M}$, as shown below.



If we consider a supersonic plane, then we should view this picture the other way around, where the fluid is static and the disturbance (plane) moves through it, traveling to the left at faster than the speed of sound. This causes a sonic boom, which is the air pressure wave given by the dashed lines trailing the plane, which moves at speed c_S. (Another example is thunder, where the rapid increase in temperature of plasma of ions causes rapid air expansion, making a shockwave.)

6.6 Viscous Fluid Equations

Internal friction occurs in a fluid when neighboring fluid particles move with different velocities, which means that $\frac{\partial v_i}{\partial x_j} \neq 0$. Viscous stresses oppose such relative motion. Adding these friction terms changes some of our fluid equations. In particular, the continuity equation remains unchanged, while the Euler equations along with the conservation laws for momentum, energy, and entropy must be modified.

To consider this friction we will work to first order in the $\frac{\partial v_i}{\partial x_j}$ partial derivatives, treating these as the most important terms. These derivatives can be arranged into

$$\sigma_{ij} = \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\nabla\cdot\mathbf{v}\right) + \zeta\delta_{ij}\nabla\cdot\mathbf{v} \tag{6.107}$$

to define the viscous stress tensor with elements σ_{ij}. The constant coefficients of the two terms are the shear viscosity η and the bulk viscosity ζ, where $\eta > 0$ & $\zeta > 0$. The form of the viscous stress tensor σ_{ij} is dictated by the fact that it must vanish for constant $\mathbf{v}$ and for uniform rotation $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$ where there is no friction. Writing out $\mathbf{v} = \omega_x(y\hat{z} - z\hat{y}) + \ldots$ we see that $\nabla\cdot\mathbf{v} = 0$ and $\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} = 0$, etc., for the uniform rotation, dictating the symmetric form of the terms in σ_{ij}. The remaining organizational choice is to let η multiply a traceless tensor.

Momentum conservation still comes from

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla\cdot\mathbb{T} = \mathbf{f} \tag{6.108}$$

where now we include a friction term in the stress tensor to account for the viscous transfer of momentum. Thus

$$T_{ij} = \mathcal{P}\delta_{ij} + \rho v_i v_j - \sigma_{ij} \tag{6.109}$$

is the new total stress tensor.

A simple rule for incorporating σ_{ij} is simply to replace $\mathcal{P}\delta_{ij} \to \mathcal{P}\delta_{ij} - \sigma_{ij}$. With this we can add friction to the Euler equation. In particular we have

$$(\nabla\mathcal{P})_i = \frac{\partial}{\partial x_k}\delta_{ki}\mathcal{P} \to \frac{\partial}{\partial x_k}\big(\delta_{ki}\mathcal{P} - \sigma_{ki}\big)\,, \tag{6.110}$$

where we can compute that

$$\begin{aligned}\frac{\partial}{\partial x_k}\sigma_{ki} &= \eta\Big(\frac{\partial^2 v_i}{\partial x_k\partial x_k} + \frac{\partial}{\partial x_i}\nabla\cdot\mathbf{v} - \frac{2}{3}\frac{\partial}{\partial x_i}\nabla\cdot\mathbf{v}\Big) + \zeta\frac{\partial}{\partial x_i}\nabla\cdot\mathbf{v} \\ &= \eta\nabla^2 v_i + \left(\zeta + \frac{\eta}{3}\right)\frac{\partial}{\partial x_i}\nabla\cdot\mathbf{v}\,. \end{aligned} \tag{6.111}$$

Plugging this into the Euler equation yields

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{\nabla\mathcal{P}}{\rho} - \frac{\eta}{\rho}\nabla^2\mathbf{v} - \frac{1}{\rho}\left(\zeta + \frac{\eta}{3}\right)\nabla(\nabla\cdot\mathbf{v}) = \frac{\mathbf{f}}{\rho} \tag{6.112}$$

which is the *Navier-Stokes equation.*

A common case we will study is when $\mathbf{f} = 0$ & $\nabla \cdot \mathbf{v} = 0$, which reduces the Navier-Stokes equation to

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla \mathcal{P}}{\rho} + \nu \nabla^2 \mathbf{v} \tag{6.113}$$

where

$$\nu \equiv \frac{\eta}{\rho} \tag{6.114}$$

is the *kinematic viscosity*, and the bulk viscosity term has dropped out. The dimensions of the kinematic viscosity are $[\nu] = m^2/s$, which is simpler than $[\eta] = kg/(ms)$.

Other useful equations can be derived for the situation where $\nabla \cdot \mathbf{v} = 0$ & ρ is constant. Taking the divergence of the Navier-Stokes equation removes both the $\partial \mathbf{v}/\partial t$ term, and the $\nu \nabla^2 \mathbf{v}$ term, leaving $\rho \nabla \cdot [(\mathbf{v} \cdot \nabla)\mathbf{v}] = -\nabla^2 \mathcal{P}$. Writing this out in components we find

$$\nabla^2 \mathcal{P} = -\rho \frac{\partial}{\partial x_j} v_i \frac{\partial}{\partial x_i} v_j = -\rho \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i}\,, \tag{6.115}$$

since $\partial v_j/\partial x_j = \nabla \cdot \mathbf{v} = 0$. This equation can be used to compute the pressure if the velocity is determined, since it simply acts like a source term. Taking the curl of the Navier-Stokes equation, and recalling that $\mathbf{\Omega} = \nabla \times \mathbf{v}$ we find

$$\frac{\partial \mathbf{\Omega}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{\Omega}) = \nu \nabla^2 \mathbf{\Omega}\,, \tag{6.116}$$

where the algebra to arrive at the terms on the LHS was consider earlier in our discussion of the Euler equation, and the new pieces is the term on the RHS.

In the presence of viscosity the boundary conditions change from what we had previously. Molecular forces between the viscous fluid & any surface mean that the fluid adheres to the surface, and hence that the velocity of the fluid and surface must be the same for both the tangential and longitudinal components. Therefore the boundary condition for a moving wall is $\mathbf{v} = \mathbf{v}_{\rm wall}$, which also covers the case $\mathbf{v} = 0$ for a wall at rest.

Another important concept is the force exerted by the fluid on a surface. This has a contribution both from the pressure as well as from the friction. The force per unit area $\mathcal{F}_i$ is given by

$$\mathcal{F}_i = -n_j(\mathcal{P}\delta_{ji} - \sigma_{ji}) = -\mathcal{P} n_i + \sigma_{ij} n_j\,, \tag{6.117}$$

where $\hat{n}$ is the normal vector pointing out of the surface, and the first term is the pressure acting along this normal vector, while the second is the friction that has tangential components.

Starting with the Navier-Stokes equation we can also derive a modified form for energy conservation. Rather than carrying out this derivation explicitly, we will just examine the final result in integral form, which is a bit more intuitive:

$$\frac{\partial}{\partial t} \int_V \left(\frac{1}{2} \rho \mathbf{v}^2 \right) dV = -\oint_{\partial V} \left(\rho \left(\frac{\mathbf{v}^2}{2} + \frac{\mathcal{P}}{\rho} \right) v_i - v_j \sigma_{ji} \right) d\mathcal{S}_i - \int_V \sigma_{ij} \frac{\partial v_i}{\partial x_j}\, dV\,. \tag{6.118}$$

Here the term on the LHS is the change of the kinetic energy in the volume V with time. The first integral on the RHS is the energy flux through the closed bounding surface ∂V, and the second integral is the decrease in energy that is caused by dissipation. To see this even more clearly we can consider integrating over the whole fluid with $\mathbf{v} = 0$ at ∞ (or on ∂V). This removes the flux term and leaves

$$\frac{\partial E}{\partial t} = -\int_V \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \frac{\partial v_i}{\partial x_j}\, dV = -\frac{\eta}{2} \int_V \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)^2 dV < 0\,. \tag{6.119}$$

where we can check the second equality by squaring and manipulating the summed over dummy indices i and j. Thus we see that friction causes energy to dissipate just as we would expect (and this also justifies our sign choice of $\eta > 0$).

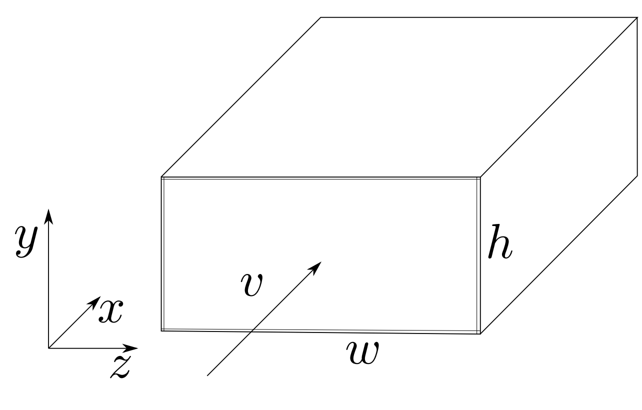
Entropy conservation is modified at temperature T to $\rho T \dot{S} = \sigma_{ij} \frac{\partial v_i}{\partial x_j}$, where the left-hand side of the equation is the heat gain per unit volume, and the right-hand side of the equation is the energy dissipated by viscosity. (If we allow thermal conduction (from temperature gradients) in the fluid, then there is another term on the right-hand side that appears as $\nabla \cdot (\kappa \nabla T)$ for conductivity κ.)

6.7 Viscous Flows in Pipes and Reynolds Number

We start our investigation of fluid flow with viscosity, by studying steady flows in pipes, typically caused by a pressure gradient. This type of steady viscous flow is also called Poiseuille flow, after Jean Poiseuille who first studied them experimentally. We take the flow to be incompressible, $\nabla \cdot \mathbf{v} = 0$, and hence the continuity equation implies that ρ is constant (just as it did in the ideal fluid case). Thus the Navier-Stokes equation for such flows reduces to

$$\mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla \mathcal{P}}{\rho} + \nu \nabla^2 \mathbf{v} \tag{6.120}$$

Example Lets start by considering flow in a long rectangular duct, aligned with the x-axis, with height h and width w. We also take it be a thin duct with $h \ll w$, and with a pressure gradient along the pipe in the x-direction.



This can be approximated by infinite parallel plates, taking $w \to \infty$ and holding h fixed. Here

$$\mathbf{v} = v_x(y)\,\hat{x}, \tag{6.121}$$

since there is no dependence on x or z by the translational symmetry in the plane, and we equally well can not develop velocity components v_y or v_z due to this symmetry. This implies that the term $\mathbf{v}\cdot\nabla\mathbf{v} = v_x\frac{\partial \mathbf{v}}{\partial x} = 0$ Taking the inner product of Eq.(6.120) with $\hat{y}$ and $\hat{z}$ then removes all the terms that depends on the velocity (which is only in the $\hat{x}$ direction), giving

$$\frac{\partial \mathcal{P}}{\partial y} = \frac{\partial \mathcal{P}}{\partial z} = 0\,. \tag{6.122}$$

Thus the pressure $\mathcal{P} = \mathcal{P}(x)$ and can have a gradient only in the $\hat{x}$ direction. Taking the inner product of Eq.(6.120) with $\hat{x}$ gives

$$\frac{\partial \mathcal{P}}{\partial x} = \rho\nu\nabla^2 v_x = \eta\frac{\partial^2 v_x}{\partial y^2} = k\,, \tag{6.123}$$

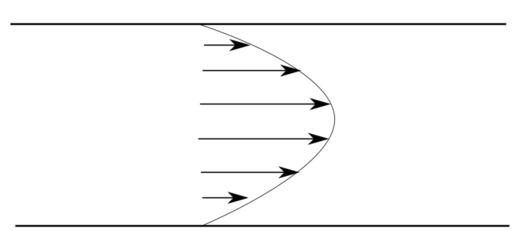
where we have introduced a constant k. Since $\frac{\partial \mathcal{P}}{\partial x}$ only depends on x, while $\eta\frac{\partial^2 v_x}{\partial y^2}$ only depends on y, they must both be equal to a constant. Let us say $\frac{\partial \mathcal{P}}{\partial x} = k < 0$, so that the pressure drops as we move in the $\hat{x}$ direction. (In the Navier-Stokes equation, this pressure drop balances the viscous stress term.) Then integrating the equation for $v_x(y)$ gives

$$v_x(y) = \frac{k}{2\eta}y^2 + ay + b\,, \tag{6.124}$$

where we have introduced two integration constants a and b. To solve for a and b we impose the boundary conditions that the velocity must vanish at $y = 0$ and $y = h$, giving

$$v_x(y) = \frac{(-k)}{2\eta}y(h-y)\,. \tag{6.125}$$

Recalling that $k < 0$ we see that $v_x(y) > 0$, flowing down the pipe. (If we had reversed k the flow would be in the opposite direction.) The velocity field we have derived flows in the pipe with a parabolic profile with its maximum in the center:

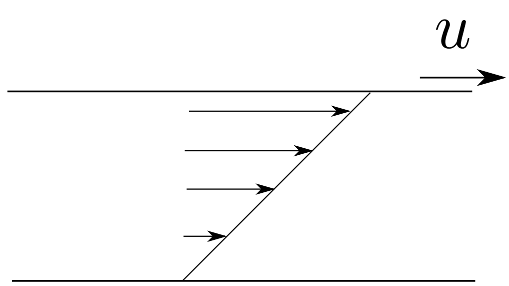


Lets also calculate the friction force per unit area that the fluid exerts on the pipe wall. The bottom plate at $y = 0$ has a unit vector $\hat{n} = \hat{y}$, so from Eq. (6.117) the force along $\hat{x}$ is

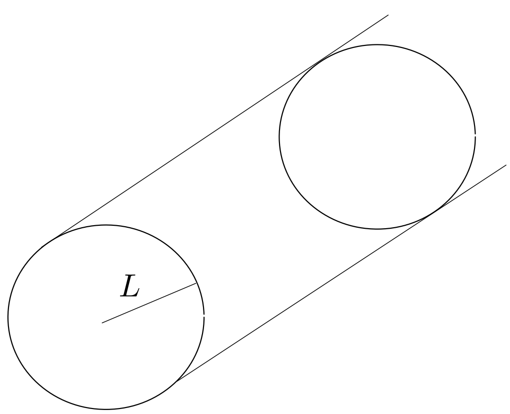
$$\mathcal{F}_x = \sigma_{yx} = \eta \frac{\partial v_x}{\partial y}|_{y=0} = -\frac{hk}{2} > 0. \tag{6.126}$$

Intuitively this is the direction we expect, the fluid tries to drag the pipe along with it.

Example Lets now consider the same example of fluid between infinite parallel plates, but now with no pressure gradient. Instead we take the top plate to move with velocity $\mathbf{u} = u\hat{x}$. Here $\mathcal{P} = \mathcal{P}_0$ is constant and uniform with $k = 0$, so the second derivative of $v_x(y)$ is zero, $\nabla^2 v_x = 0$, and the solution for $v_x(y)$ can at most be linear. The solution for this case is $v_x(y) = \frac{uy}{h}$, which satisfies the boundary conditions $v_x(0) = 0$ and $v_x(h) = u$. Thus the fluids velocity field is linear for this case:



Example Next consider a long cylindrical pipe of radius L, oriented along $\hat{x}$, again with a pressure gradient along the pipe. We will approximate the pipe as being infinitely long so there is a translational symmetry along x.



Due to the translational symmetry we know that $\mathbf{v} = \mathbf{v}(y, z)$. To fully exploit the consequences of the symmetry it is useful to use cylindrical coordinates (x, r, θ) so we can also easily impose the rotational symmetry about $\hat{x}$ to conclude $\mathbf{v} = \mathbf{v}(r)$. The fact that there is

r dependence makes sense since we know that $\mathbf{v}$ must vanish at the edge of the pipe, $r = L$, but we do not want it to vanish everywhere. Continuity and symmetry also imply that the velocity is only in the $\hat{x}$ direction, so in fact

$$\mathbf{v} = v_x(r)\hat{x}\,. \tag{6.127}$$

For example, consider an annulus shaped closed surface formed by the region between two cylinders cocentric with the pipe. The flow into and out of this surface must be balanced by continuity. The flow on the ends of the surface automatically balance each other since $\mathbf{v}$ is independent of x. But since $\mathbf{v}$ is r dependent, the only way the flow through the circular sides can balance each other is if there is no flow in the $\hat{r}$ direction.

With this setup we can again confirm that $(\mathbf{v} \cdot \nabla)\mathbf{v} = v_x(r)\frac{\partial}{\partial x} v_x(r)\hat{x} = 0$, leaving $\nabla \mathcal{P} = \eta \nabla^2 \mathbf{v}$ from the Navier-Stokes equation. Taking the inner product with $\hat{y}$ and $\hat{z}$ we see that $\frac{\partial \mathcal{P}}{\partial y} = \frac{\partial \mathcal{P}}{\partial z} = 0$, so $\mathcal{P} = \mathcal{P}(x)$. Taking the inner product with $\hat{x}$ gives

$$\frac{\partial \mathcal{P}(x)}{\partial x} = \eta \nabla^2 v_x(r) = k \tag{6.128}$$

where since $\frac{\partial \mathcal{P}}{\partial x}$ depends only on x while $\eta \nabla^2 v_x(r)$ is independent of x, the two must be equal to a constant k. Again we choose $k < 0$ to have a pressure gradient that pushes the fluid down the pipe in the $\hat{x}$ direction ($k > 0$ would simply reverse the flow). For the velocity this gives

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_x}{\partial r}\right) = \frac{k}{\eta}, \tag{6.129}$$

and integrating this gives

$$v_x(r) = \frac{k}{4\eta} r^2 + a \ln\left(\frac{r}{r_0}\right) + b \tag{6.130}$$

for some constants a and b. (The constant r_0 is introduced to make the equation dimensionfully correct, but is not independent, since any change to r_0 can be compensated by a change to b). Since $v_x(r)$ has to be finite at $r = 0$ we must have $a = 0$ (if the geometry excluded the region at the middle). The condition $v_x = 0$ at $r = L$ fixes b so that

$$v_x(r) = \frac{(-k)}{4\eta}\left(L^2 - r^2\right)\,. \tag{6.131}$$

Lets calculate the discharge rate of fluid in such a pipe, as mass per unit time. This is given by

$$\text{discharge rate} = \rho \int_0^L 2\pi r v_x(r)\, dr = -\frac{\pi k L^4}{8\nu} > 0\,. \tag{6.132}$$

Note that this rate is quite sensitive to the radius, being proportional to L^4. This is why you don't want narrow blood vessels.

Reynolds Number and Similarity

How do we quantify the importance of viscosity? Let us consider a flow and identify a characteristic length L (examples of which might be the radius of a pipe or the size of a sphere inserted in a flow) and a characteristic speed u (examples of which might be the fluid velocity in the middle of a pipe or the fluid velocity upstream from flow around a sphere). The quantity

$$R = \frac{uL}{\nu} = \frac{\rho u L}{\eta} \tag{6.133}$$

is the dimensionless *Reynolds number*. A large R means viscosity is not as important compared to the characteristic u and L of the system, while small R means that viscosity plays an important role. It is important to note that viscosity is always important near a surface where it modifies the boundary condition on $\mathbf{v}$. (The above examples of Poiseuille flow have $R \lesssim 2000$.)

The introduction of R leads nicely into a discussion of the utility of exploiting dimensional analysis in fluid dynamics problems. This is best explained by way of examples.

Example Consider a steady incompressible flow in a system that is characterized by a single length scale L and initial velocity u. What could the possible functional form be for the velocity and pressure in such a system? Considering the coordinates and fluid field variables we can only have

$$\frac{\mathbf{v}}{u} = \mathbf{h}\left(\frac{\mathbf{r}}{L}, R\right) , \qquad \frac{\mathcal{P}}{\rho u^2} = g\left(\frac{\mathbf{r}}{L}, R\right) . \tag{6.134}$$

Here $\mathbf{h}$ and g are a dimensionless vector and scalar function respectively. The ratio $\mathbf{r}/L$ is dimensionless, as is R, and the dimensions of $\mathbf{u}$ and $\mathcal{P}$ are compensated by u and ρu^2 respectively. Note that if we consider flows that have the same R, then the properties of those flows are related by changes to the scales associated with $\mathbf{v}$, $\mathbf{r}$, or $\mathcal{P}$. Such flows are called *similar*.

Example Consider a viscous flow past a sphere of radius a with initial velocity given by $\lim_{x\to-\infty} \mathbf{v} = u\hat{x}$. Here the Reynolds number is $R = \frac{u}{a}\nu$. If we double ν and double u then R is unchanged. Due to the relations in Eq. (6.134) we thus can predict that we will have the exact same form for the solutions with $\mathbf{v}$ twice as large as before, and $\mathcal{P}$ being four times as large as before.

Note that in general other dimensionless ratios, like the ratio of two length scales, or the Mach number $M = \frac{u}{c_S}$ could also appear. (For $M \ll 1$ we treat the fluid as incompressible and neglect M for the dimensional analysis.) To determine how many independent dimensionless ratios can be formed for the analysis of a general dimensional analysis problem, one counts the number of variables and subtracts the number of unrelated types of dimensions that appear in these variables. For most fluid problems this will mean subtracting three for kg, meters, and seconds.

6.8 Viscous Flow Past a Sphere (Stokes Flow)

Lets consider a steady, incompressible, viscous fluid with small R flowing past a sphere of radius a. The fluid has velocity $\mathbf{u}$ when it is far away from the sphere. This is the same problem we treated for an ideal fluid with potential flow, but now we want to consider the impact of viscosity, and resolve the puzzle we found in our solution for an ideal fluid. To make the problem solvable we work in the limit $R \ll 1$.

Here the Navier-Stokes equation is

$$\mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla \mathcal{P}}{\rho} + \frac{\eta}{\rho} \nabla^2 \mathbf{v} \,. \tag{6.135}$$

Lets determine the relative importance of the two velocity terms in the $R \ll 1$ limit. Using a to characterize spatial variations, and u to characterize velocity we find

$$\mathbf{v} \cdot \nabla \mathbf{v} = \mathcal{O}\Big(\frac{u^2}{a}\Big) \,, \qquad \frac{\eta}{\rho} \nabla^2 \mathbf{v} = \mathcal{O}\Big(\frac{\eta u}{\rho a^2}\Big) = \mathcal{O}\Big(\frac{u^2}{aR}\Big) \,. \tag{6.136}$$

Therefore the viscosity term, which is enhaced by a factor of $1/R$, dominates. Neglecting the $\mathbf{v} \cdot \nabla \mathbf{v}$ term, the Navier-Stokes equation reduces to

$$\nabla \mathcal{P} = \eta \nabla^2 \mathbf{v} \,. \tag{6.137}$$

Note that when written in terms of the shear viscosity η, that the density ρ has dropped out of this equation, and hence the constant ρ will not play a part in the solution. Taking the divergence of this equation we find $\nabla^2 \mathcal{P} = \eta \nabla^2 \nabla \cdot \mathbf{v} = 0$, so the pressure satisfies Laplace's equation. Using dimensional analysis for the pressure we expect a solution of the form

$$\mathcal{P} = \frac{\eta \mathbf{u} \cdot \mathbf{r}}{a^2} \, g\Big(\frac{\mathbf{r}}{a}\Big) \,, \tag{6.138}$$

where the dimensional analysis requirement of having only a single factor of the velocity $\mathbf{u}$ and a scalar result, leads to including the factor of $\mathbf{u} \cdot \mathbf{r}$. Note that we have not included R as a possible argument since we are expanding with $R \ll 1$. Due to the fact that $\mathcal{P}$ must satisfy Laplace's equation and is proportional to $\mathbf{u} \cdot \mathbf{r}$, we can immediately recognize the dipole solution (which we met earlier for the potential ϕ in the ideal fluid case):

$$\mathcal{P} = \mathcal{P}_0 + k \frac{\eta a \mathbf{u} \cdot \mathbf{r}}{r^3} \,. \tag{6.139}$$

Here we have included an overall constant pressure $\mathcal{P}_0$ to satisfy any boundary condition on the pressure at $r \to \infty$, and a dimensionless constant k which is still to be determined.

Next we note that the vorticity of the fluid satisfies

$$\begin{aligned} \nabla \times \mathbf{\Omega} &= \nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v} = -\nabla^2 \mathbf{v} = -\frac{1}{\eta} \nabla \mathcal{P} \\ &= -\frac{ka}{r^3} \bigg(\mathbf{u} - \frac{3(\mathbf{u} \cdot \mathbf{r}) \mathbf{r}}{r^2} \bigg) \,. \end{aligned} \tag{6.140}$$

Enforcing that $\mathbf{\Omega}$ vanishes as $r \to \infty$, the solution for this equation is

$$\mathbf{\Omega} = ka\frac{\mathbf{u}\times\mathbf{r}}{r^3}\,. \tag{6.141}$$

Thus we see that unlike the ideal fluid case, there is now a non-zero vorticity in the fluid here.

Next we turn to determining the velocity, which can only depend on $\mathbf{v} = \mathbf{v}(\mathbf{r}, \mathbf{u}, a)$. Again we do not include R as an argument since we are expanding for $R \ll 1$. By dimensional analysis the velocity must be linear in $\mathbf{u}$ so the most general possible solution takes the form

$$\mathbf{v} = \mathbf{u}\, f\Big(\frac{r}{a}\Big) + \frac{\mathbf{r}(\mathbf{u}\cdot\mathbf{r})}{a^2}\, g\Big(\frac{r}{a}\Big)\,, \tag{6.142}$$

where the functions f and g are dimensionless. The gradient of one of these dimensionless functions gives $\nabla f(r/a) = (\mathbf{r}/(ar))f'(r/a)$. Computing the divergence of the velocity with this form we have

$$0 = \nabla\cdot\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{r}}{ar}f'\Big(\frac{r}{a}\Big) + \frac{(\mathbf{u}\cdot\mathbf{r})r}{a^3}g'\Big(\frac{r}{a}\Big) + \frac{4(\mathbf{u}\cdot\mathbf{r})}{a^2}g\Big(\frac{r}{a}\Big)\,. \tag{6.143}$$

Therefore we find that the scalar functions must satisfy

$$f'\Big(\frac{r}{a}\Big) = -\frac{r^2}{a^2}g'\Big(\frac{r}{a}\Big) - \frac{4r}{a}g\Big(\frac{r}{a}\Big)\,. \tag{6.144}$$

Next we equate the Laplacian of the velocity and gradient of the pressure, which can be simplified to give

$$\begin{aligned}\nabla^2\mathbf{v} &= \Big(\nabla^2 f + \frac{2g}{a^2}\Big)\mathbf{u} + \Big(\nabla^2 g + \frac{4g'}{ar}\Big)\frac{(\mathbf{u}\cdot\mathbf{r})\mathbf{r}}{a^2} \\ &= \frac{1}{\eta}\nabla\mathcal{P} = \frac{ka}{r^3}\mathbf{u} + \Big(-\frac{3ka^3}{r^5}\Big)\frac{(\mathbf{u}\cdot\mathbf{r})\mathbf{r}}{a^2}\,.\end{aligned} \tag{6.145}$$

Note that here $\nabla^2 g(r/a) = (1/r^2)(d/dr)r^2(d/dr)g(r/a)$. Equating the coefficients of the two structures we find

$$\begin{aligned}\nabla^2 g + \frac{4g'}{ar} = \frac{g''}{a^2} + \frac{6g'}{ar} &= -\frac{3ka^3}{r^5}\,, \\ \frac{f''}{a^2} + \frac{2f'}{ar} + \frac{2g}{a^2} &= \frac{ka}{r^3}\,.\end{aligned} \tag{6.146}$$

To solve the equation for g we try a polynomial solution of the form $g(x) = C_n x^n$ giving

$$C_n\big[n(n-1) + 6n\big]x^{n-2} = -3kx^{-5}\,. \tag{6.147}$$

Here $n = -3$ is a particular solution to the full inhomogeneous equation with $C_{-3} = k/2$. Also $n = 0$ and $n = -5$ are homogeneous solutions where the LHS vanishes, and the corresponding coefficients C_0 and C_5 must be fixed by boundary conditions. Looking back at our starting point in Eq. (6.142) we see that the full set of boundary conditions are

$$\begin{aligned} &\mathbf{v}=\mathbf{u} \text{ at } r=\infty: \qquad && \lim_{r\to\infty} r^2 g(r/a)\to 0\,, \qquad && \lim_{r\to\infty} f(r/a)\to 1\,, \\ &\mathbf{v}=0 \text{ at } r=a: && g(1)=0\,, && f(1)=0\,, \end{aligned} \tag{6.148}$$

This fixes $C_0 = 0$ and $C_{-5} = -k/2$ so that

$$g(r/a) = \frac{ka^3}{2r^3}\Big(1-\frac{a^2}{r^2}\Big)\,. \tag{6.149}$$

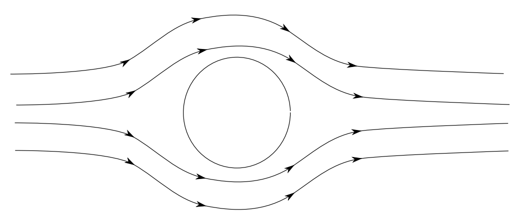
Using Eq. (6.144) and integrating once we find that

$$f(r/a) = \frac{ka}{2r} + \frac{ka^3}{6r^3}\,, \tag{6.150}$$

where we have set the integration constant to zero to satisfy the boundary condition at $r = \infty$. The final boundary condition, $f(1) = 1$ then requires us to take the constant $k = -3/2$. Note that this fixes the constant k that appeared in the vorticity $\mathbf{\Omega}$ and in the pressure $\mathcal{P}$. All together we have that the final solution for the velocity is

$$\mathbf{v} = \mathbf{u}\Big(1-\frac{3a}{4r}-\frac{a^3}{4r^3}\Big) - \frac{3a^3}{4r^3}\frac{\mathbf{r}(\mathbf{u}\cdot\mathbf{r})}{a^2}\Big(1-\frac{a^2}{r^2}\Big)\,. \tag{6.151}$$

The flow looks like:



Next we turn to determining the drag force on the sphere. In general the drag force on an object in the direction j is given by an integral of the force per unit area over the surface,

$$\mathbf{F}_{\mathrm{D}j} = \int_{\partial V} d\mathcal{S}_i\, (\mathcal{P}\delta_{ij} - \sigma_{ij})\,. \tag{6.152}$$

Lets take the inflowing velocity to be in the $\hat{x}$ direction, $\mathbf{v}(r\to\infty) = u\hat{x}$. Then $\mathbf{F}_{\mathrm{D}}\cdot\hat{x}$ will be the drag force on our sphere in the direction of the bulk fluid flow. With spherical coordinates (r,θ,ϕ) where θ is the polar angle, we have $d\mathbf{S} = \hat{r}a^2 d\cos\theta d\phi$ with $r = a$, as well as $\hat{r}\cdot\hat{x} = \cos\theta$ and $\hat{\theta}\cdot\hat{x} = -\sin\theta$. Thus

$$\hat{x}\cdot\mathbf{F_D} = a^2\int d\cos\theta\, d\phi\,\Big(-\mathcal{P}\cos\theta + \sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta\Big)\,. \tag{6.153}$$

Computing the needed components on the sphere $r = a$ we find

$$\sigma_{rr} = 2\eta \frac{\partial v_r}{\partial r}\bigg|_{r=a} = 0\,, \qquad \sigma_{r\theta} = \eta\Big(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\Big)_{r=a} = -\Big(\frac{3\eta}{2a}\Big) u \sin\theta\,,$$
$$\mathcal{P} = \mathcal{P}_0 - \Big(\frac{3\eta}{2a}\Big) u \cos\theta\,. \tag{6.154}$$

Thus the drag force on the sphere is

$$\hat{x}\cdot\mathbf{F_D} = \Big(\frac{3\eta u}{2a}\Big)\int d\cos\theta\, d\phi\, a^2 = (6\pi\eta a) u\,, \tag{6.155}$$

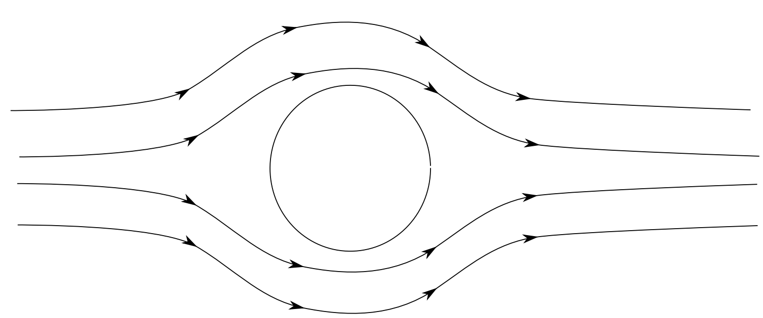
which is the famous Stoke's formula for the viscous (friction) drag force on a sphere. (Note that we could have obtained the factor of $\eta a u$ by dimensional analysis.)

In addition to drag forces like this, that point in the direction of the fluid flow, objects may also experience *lift forces* that are tangential to direction of the fluid flow. Such forces occur for wing-shaped objects and are important for many physical phenomena, including lift on airplanes.

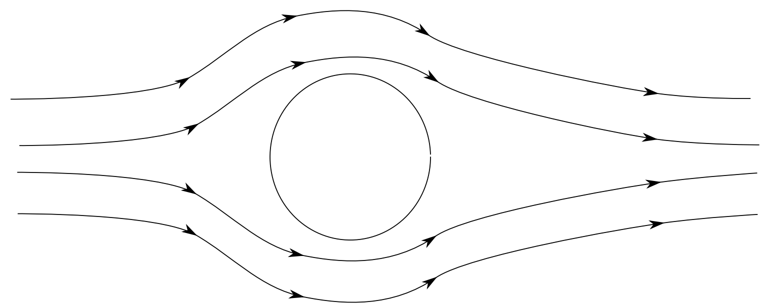
Dynamic Vortices and Turbulence

For our flow about the fixed sphere, lets consider what happens as we increase R. From our analysis above it is clear that at some point the non-linear $\mathbf{v}\cdot\nabla\mathbf{v}$ term we dropped will become important. The $\partial\mathbf{v}/\partial t$ will also become important, with flows that are more dynamical, changing with time. Lets consider how the flow appears for various values of R:

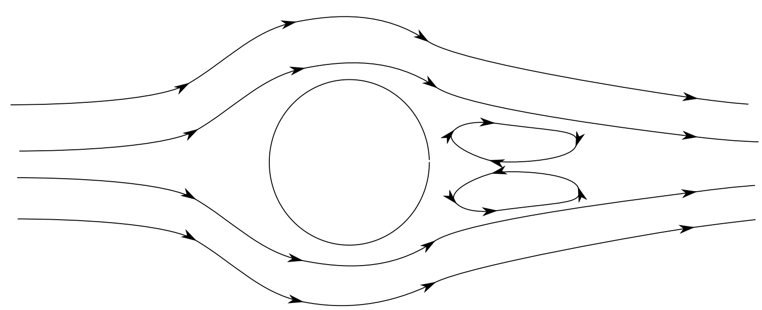
- For $R \ll 1$, the flow is symmetric and is (somewhat counter intuitively) qualitatively like the case of $\eta \to 0$. This is also called "Stokes flow" or a "laminar flow".



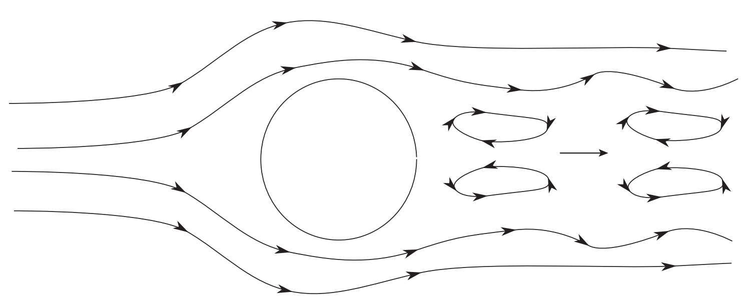
- For $R \approx 1$, the flow is still like Stokes flow, but the stream lines are no longer as symmetric, with a more clear wake developing behind the sphere.



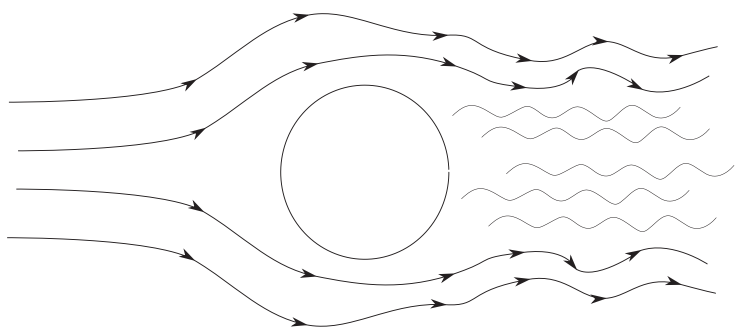
- For $10 \lesssim R \lesssim 100$, detached vortices called *eddies* form behind the sphere, though the flow is still steady. Note that directly behind the sphere between the vortices that the fluid is now flowing in the opposite direction to the asymptotic inflow $\mathbf{u}$. As R increases, the flow becomes looses its steady nature, with the time dependence emerging by having through oscillations of the vortices.



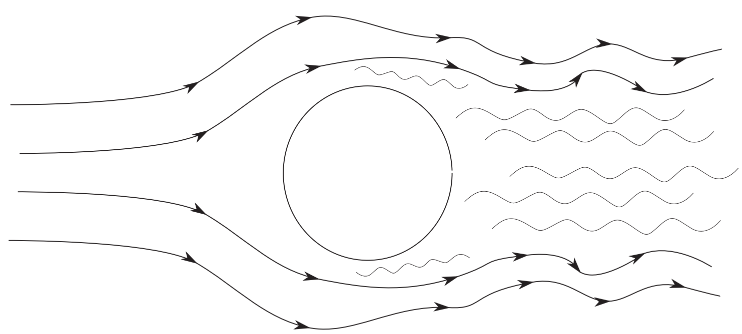
- For $150 \lesssim R \lesssim 1000$, vortices start to be cyclically shed and drift downstream in a *wake* behind the sphere. This time dependent solution appears like it has interaction between the eddies, where one pair pushes the next downstream.



- For $10^3 \lesssim R \lesssim 2 \times 10^5$, the wake becomes highly irregular, exhibiting a phenomena known as turbulence which we will discuss in more detail below. Here there are unsteady, interacting vortices at all length scales.



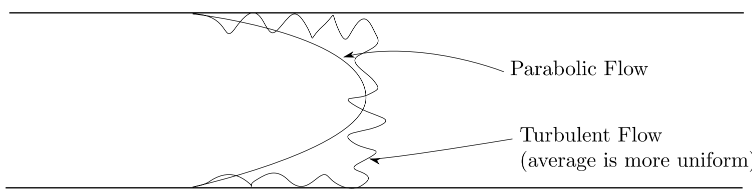
- For $R \gtrsim 2 \times 10^5$, the turbulent wake narrows and the boundary layer around the sphere is no longer laminar, also becoming turbulent.



Turbulence is characterized by a flow that: i) is highly irregular (technically chaotic) in $\mathbf{x}$ and/or t, ii) has nonzero vorticity, iii) has much greater dissipation of energy compared to more uniform laminar viscous flows, and iv) has eddies and vortices forming over many length scales with energy that in a three dimensional flow cascades from the largest eddies down to the smallest eddies where it dissipates into heat due to viscous friction. Turbulent mixing is a very effect mechanism of transport for energy, momentum, temperature, and so on. Examples of turbulence include many familiar phenomena: the circulation of air in the atmosphere, the water flow from a faucet which becomes white at a certain point when the flow rate is increased, water in rapids, dust kicked up by wind, water beside a ship moving in an otherwise smooth lake, clear air turbulence causing a drop in lift for airplanes, and so on.

In the last value of R discussed for flow around our shere, a turbulent boundary layer appeared. This causes an abrupt drop in drag in the flow over objects, and is a very useful phenomena. In particular, by introducing imperfections we can cause this turbulent boundary layer to form at smaller values R, meaning smaller velocities. This is why golf balls have dimples and baseballs and tennis balls have visible seams.

We also get turbulence in flow through pipes at large R. The viscous flow in pipes we previously considered were laminar flow at smaller R values and had velocity distributions that were parabolic, while in contrast a turbulent flow will be non-uniform at small scales, but when averaged causes a more uniform flow down the pipe at larger length scales.



In general the chaotic and irregular nature of turbulence makes it difficult to treat with analytic methods, and a complete description of turbulence remains an unsolved problem.